Mathematics GU4041
Introduction to Modern Algebra

Assignment #12

Due November 30, 2016

Throughout, p denotes a prime number.

1. How many elements of order 5 are contained in a group of order 20? Why?

2. Let G be a p-group. Show that every composition factor of G is isomorphic to $\mathbf{Z}_p$.

3. Show that a group of order 132 is never simple.

4. Determine all groups of order 33 up to isomorphism.

5. Suppose that the prime $p \neq 2$.

(a) Show that $k^2 \equiv 1 (\text{mod } p)$ has only 2 solutions mod p.

(b) Show the only automorphisms $\phi : \mathbf{Z}_p \to \mathbf{Z}_p$ with $\phi \circ \phi = \text{id}$ are $\phi(g) = g^{\pm 1}$. Hint: use (a).

(c) Show that any group of order $2p$ is isomorphic either to $\mathbf{Z}_2 \times \mathbf{Z}_p$ or to D_{2p}. Hint: use (b).

6. Let N be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $H \cap N$ is the unique Sylow p-subgroup of H.