Mathematics GU4041
Introduction to Modern Algebra

Assignment #11

Due November 23, 2016

1. Prove that the center ZG is a normal subgroup of G.

2. If the index $[G : ZG] = n$, prove that every conjugacy class in G has at most n elements.

3. Let T be a regular tetrahedron. Following the example of the cube done in class, (a) using the counting formula, determine the order of the group G of rotations of T, (b) using Burnside's lemma, compute the number of ways to color the faces of T with 3 colors, up to rotational equivalence; (c) same thing for n colors.

4. Let $m \leq n \in \mathbf{N}$. Say that a subset $S \subset \langle n \rangle$ is *fixed* by a permutation $\sigma \in \Sigma_n$ if $\sigma(S) = S$. Prove that the average, over all $\sigma \in \Sigma_n$, of the number of subsets with exactly m elements fixed by σ is 1. (Of course, for many σ the number will be 0.)

5. Using Burnside's lemma or otherwise, prove that the number of conjugacy classes in the dihedral group D_{2n} is $(n+3)/2$ if n is odd, $(n+6)/2$ if n is even.

6. Describe, with proof, all of the finite groups having exactly two conjugacy classes.