Mathematics GU4041
Introduction to Modern Algebra

Assignment #10

Due November 16, 2016

1. Let a group G act on a set S. Define a relation by $s \sim t$ if and only if there exists $g \in G$ such that $s = g \cdot t$. Prove that this is an equivalence relation. (The equivalence classes are what we call *orbits*.)

2. (a) Prove that $g \cdot h := ghg^{-1}$ defines an action of G on itself (this is called the *conjugation action*).

(b) Determine the *conjugacy classes*, that is, the orbits of the conjugation action, for $G = Q_8$, D_{10}, and $\mathbf{Z}_5$.

3. Let $H < G$, not necessarily normal.

(a) Prove that $g \cdot kH := gkH$ specifies a well-defined and transitive action of G on G/H.

(b) For $s = kH$, prove that the stabilizer $G_s = kHk^{-1}$.

4. Let $\langle n \rangle = \{1, \ldots, n\}$, as usual.

(a) Prove that Σ_n acts on the power set $P\langle n \rangle$ by $\sigma \cdot S := \sigma(S)$, where $\sigma : \langle n \rangle \to \langle n \rangle$ and $S \subset \langle n \rangle$.

(b) How many orbits are there? Prove your answer correct.

5. Show that the group G of rotations of a regular tetrahedron is isomorphic to the alternating group A_4. Hints: How does G act on the set of vertices? What is its order? Which 3-cycles does it contain?

6. Show that the group H of rotations of a cube is isomorphic to the symmetric group Σ_4. Hints: Consider the action of H on the set of four pairs of opposite vertices. Use this to define a homomorphism $H \to \Sigma_4$. Is it injective?

In the last two problems, it's acceptable to be somewhat loose when defining and describing specific rotations of the tetrahedron and cube. Imitate the example of the dodecahedron explained in class. Drawing pictures is adequate — don't feel that you have to write down explicit coordinates or rotation matrices.