Mathematics GU4041
Introduction to Modern Algebra

Assignment #1

Due September 12, 2016

Before doing anything else, read the syllabus carefully and thoroughly.

In problems **1**–**4**, write careful proofs of the statements given. Follow the style used in class consisting of complete sentences in the English language. Every statement should have a reason, such as a definition or a proposition stated in class. Do not use truth tables except where asked to do so. Hint: to prove $\Leftrightarrow$, it suffices to prove first $\Rightarrow$ and then $\Leftarrow$.

1. Prove that $P \vee \sim P$ (regardless of the truth value of P).

2. Prove that $\sim (P \wedge Q) \Leftrightarrow (\sim P \vee \sim Q)$ (regardless of the truth values of P and Q). Also make a truth table showing all possibilities for $P, Q, \sim P, \sim Q, P \wedge Q$, and the two sides of the biconditional above.

3. If $P \Rightarrow Q$ is any conditional statement, there are three other related conditional statements: (i) the *converse* $Q \Rightarrow P$, (ii) the *inverse* $\sim P \Rightarrow \sim Q$, and (iii) the *contrapositive* $\sim Q \Rightarrow \sim P$.

(a) Prove that $(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$ (regardless...).
That is, an implication has the same truth value as its contrapositive.

(b) Give examples of P and Q such that $P \Rightarrow Q$ is true but $Q \Rightarrow P$ is false.
That is, an implication need *not* have the same truth value as its converse.

(c) Give examples of P and Q such that $P \Rightarrow Q$ is true but $\sim P \Rightarrow \sim Q$ is false.
That is, an implication need *not* have the same truth value as its inverse.

4. Proof by cases: if $P \Rightarrow R$, $Q \Rightarrow R$, and $P \vee Q$ are all true, then so is R.

5. Write the negation of the statement $\forall x \in S\ \exists y \in T \,|\, (P(x,y) \wedge Q(x,y))$ in terms of $\sim P(x,y)$ and $\sim Q(x,y)$. Make up a sentence in natural language (i.e. everyday speech) that has the form above. Also write its negation in natural language.

Hint for problems **7**–**9**: use the definition of set equality.

7. Suppose S is any set.

(a) Prove that $\varnothing \subset S$.

(b) Is it necessarily true that $\varnothing \in S$? Give either a proof or a counterexample (i.e. a specific S where you prove $\varnothing \notin S$).

(c) Is it necessarily true that $\varnothing \notin S$? Again, give either a proof or a counterexample.

(d) Prove that if $S \subset \varnothing$, then $S = \varnothing$.

8. For a set A, prove that $A \cup \varnothing = A$.

9. For sets A, B, prove that $A = (A \setminus B) \cup (A \cap B)$. Also illustrate this with a Venn diagram.