Mathematics GU4041
Introduction to Modern Algebra

Final Examination
December 20, 2016

READ AND FOLLOW CAREFULLY ALL INSTRUCTIONS BELOW

Turn off all electronic devices.
Brief visits to the men's or women's room (on the 2nd or 4th floors) are OK, but one at a time only, and leave your phone at the front.
Write your name, "Modern Algebra, Prof. Thaddeus," and the number of blue books on the cover of each blue book.
Write your name on the attendance sheet when it comes around.
Write all answers and work in your blue books. Do not hand in this sheet.
On each page you use, write the number of the problem *in a circle in the margin.*
You may do more than one problem on a single page. Just put all numbers by their problems.
You may do the problems out of order, but this is discouraged as it can lead to misgrading.
When there is any doubt, state briefly but clearly what statements from the text, lecture, or assignments you are using.
In grading the exams, I will emphasize accuracy, brevity, and clarity. Aim for all three.
Attempt all 12 problems. Each is worth 10 points. Good luck!

1. As best you can, state both parts (existence and uniqueness) of the Jordan-Hölder theorem.

2. Prove that $\phi : G \to G$ given by $\phi(g) = g^{-1}$ is a homomorphism if and only if G is abelian.

3. Let G be a group, $S \subset G$ any subset such that $S = gSg^{-1}$ for every $g \in G$. Prove that $\langle S \rangle$ is normal in G.

4. Suppose $n \geq 5$. For $2 \leq k \leq n$, let $S_k \subset \Sigma_n$ be the set of all k-cycles, and let G_k be the subgroup generated by S_k.

(a) Prove that $G_k \lhd \Sigma_n$.

(b) If k is odd, prove that $G_k = A_n$.

(c) If k is even, prove that $G_k = \Sigma_n$.

5. Give a composition series for Σ_4, and prove that it is a composition series.

6. (a) Describe the group Aut $\mathbb{Z}$. Justify your answer.

(b) Describe the group Aut $\mathbb{Z}_2$. Justify your answer.

(c) Is every semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$ abelian? Why or why not?

(d) Is every semidirect product $\mathbb{Z}_2 \rtimes \mathbb{Z}$ abelian? Why or why not?

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7. In a finite group G, for p prime, show that the intersection of all Sylow p-subgroups is normal.

8. For $n > 1$, prove that $\mathbb{Z}_n$ is generated by any nonzero element if and only if n is prime.

9. (a) If p is prime, how many elements of Σ_p have order p? Why?

(b) If p is prime, how many Sylow p-subgroups does Σ_p have? Why?

10. Classify the groups of order 15. How many are there?

11. Classify the *abelian* groups of order 36. How many are there?

12. How many different necklaces can be made by stringing together seven black or white beads as shown? If the necklace is rotated or flipped over, it remains the same necklace, of course.