Mathematics W4041x
Introduction to Modern Algebra

Answers to Practice Midterm #2

November 18, 2010

1. Let G be a group, $K \subset H$ both normal subgroups of G. Then H/K is normal in G/K and $(G/K)/(H/K) \cong G/H$.

2. Let $\#G = p$ be prime. Since $p > 1$, there exists $g \in G$ such that $g \neq e$. By Lagrange's theorem, the order of g divides p, hence is 1 or p; since $g \neq e$, it is p. Define a map $\phi : \mathbb{Z}_p \to G$ by $\phi([i]) := g^i$. Then $i \equiv j \pmod p$ implies $i = j + kp$ for some $k \in \mathbb{Z}$, which implies $g^i = g^j g^{kp} = g^j (g^p)^k = g^j e^k = g^j$, so ϕ is well-defined. And $\phi([i] + [j]) = \phi([i + j]) = g^{i+j} = g^i g^j = \phi([i])\phi([j])$, so ϕ is a homomorphism. Now $[i] \in \ker \phi$ if and only if $g^i = e$ if and only if $p|i$ if and only if $[i] = [0]$, so ϕ is injective. Finally, an injective function between finite sets of the same size must also be surjective. Hence ϕ is an isomorphism.

3. Let $\phi : GL(n, \mathbf{Q}) \to \{\pm 1\}$ be $\phi(A) := \det A/|\det A|$. Then $\phi(AB) = \det(AB)/|\det(AB)| = (\det A \det B)/(|\det A|\,|\det B|) = \phi(A)\phi(B)$, so ϕ is a homomorphism. And $A \in \ker \phi$ if and only if $1 = \det A/|\det A|$ if and only if $\det A > 0$, so $\ker \phi = H$, and hence H is a normal subgroup. Furthermore, by the first isomorphism theorem, $G/H \cong \phi(G) = \{\pm 1\} \cong \mathbb{Z}_2$.

4. We know that any group of prime order is simple (and cyclic). And subgroups of abelian groups are normal, as are subgroups of index 2. Hence $1 \lhd \{R^3, R^6\} \lhd \{R, R^2, R^3, R^4, R^5, R^6\} \lhd D_{12}$ with composition factors $\cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2$. Alternative: $1 \lhd \{R^2, R^4, R^6\} \lhd \{R, R^2, R^3, R^4, R^5, R^6\} \lhd D_{12}$ with composition factors $\cong \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2$.

5. (a) Since $\#Q = 8$, this follows immediately from Cayley's theorem.

(b) Let $n < 8$ and $\phi : Q \to \Sigma_n$ be a homomorphism. Then Q acts on $\{1, \ldots, n\}$ by $q \cdot i := \phi(q)(i)$. Each orbit has < 8 elements, so by the counting formula (aka orbit-stabilizer theorem) each stabilizer has > 1 element. But every subgroup of Q besides 1 contains -1, for every non-identity element of Q has some power equal to -1. Hence -1 acts trivially on $\{1, \ldots, n\}$, so $-1 \in \ker \phi$. Hence ϕ can never be injective.

6. Recall that $A_n = \ker f$ where $f : \Sigma_n \to \{\pm 1\}$ is given by the determinant of the permutation matrix. Let $K := \ker(f \circ \phi)$. Then $K \lhd G$, so $K = 1$ or $K = G$. In the first case, by the first isomorphism theorem, $G = G/K$ is isomorphic to a subgroup of $\{\pm 1\} \cong \mathbb{Z}_2$, hence to $\mathbb{Z}_2$ itself (since the only subgroups are $\mathbb{Z}_2$ and 1, and a simple group is not 1 by definition). In the second case, for any $g \in G$, $f(\phi(g)) = 1$, so $\phi(g) \in A_n$.

7. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, and let $S = \{\text{functions } f : G \to \{A, B\}\}$. Then G acts on S by $(g \cdot f)(h) = f(g + h)$, and the problem is to count $\#S/G$. If $g \neq e$, then g has order exactly 3, so $G/\langle g \rangle$ contains 3 cosets, and $f \in S^g$ if and only if f is constant on each coset. So $\#S^g = 2^3$. On the other hand, clearly $S^e = S$, and $\#S = 2^9$. By Burnside's lemma,

$$\#S/G = \frac{1}{\#G} \sum_{g \in G} \#S^g = \frac{1}{9}(2^9 + 8 \cdot 2^3) = \frac{1}{9}(2^9 + 2^6) = 2^6(2^3 + 1)/9 = 64.$$