1. State the third isomorphism theorem.

2. Prove that every group of prime order is cyclic.

3. Prove that the subset $H$ of matrices in $G = GL(n, \mathbb{Q})$ whose determinant is positive is a normal subgroup, and describe the quotient group $G/H$.

4. Give a composition series for $D_{12}$, the symmetry group of a regular hexagon (and prove that it is a composition series).

5. (a) Prove that the quaternion group $Q$ is isomorphic to a subgroup of the symmetric group $\Sigma_8$.
    (b) Prove, however, that $Q$ is not isomorphic to a subgroup of $\Sigma_n$ for any $n < 8$. (Hint: show that if $Q$ acts on any set with $< 8$ elements, then $-1$ must act trivially.)

6. Prove that if $G$ is simple and $\phi : G \to \Sigma_n$ is a homomorphism, then either $G \cong \mathbb{Z}_2$ or $\phi(G) \subset A_n$. Here $\Sigma_n$ is the symmetric group, $A_n$ the alternating group.

7. A company manufactures $3 \times 3$ tiles marked with the letters A and B. They want all periodic patterns (with period 3) to be constructible from their tiles. (See the example below.) However, they do not have to make all $2^9$ possibilities: the seams between the tiles are barely visible, so the two configurations shown are equivalent. How many different types of tile do they have to keep in stock?

\[
\begin{array}{ccccccc}
B & B & B & B & B & B & B \\
A & B & B & A & B & A & B \\
B & B & B & B & B & B & B \\
A & B & B & A & B & A & B \\
B & B & B & B & B & B & B \\
A & B & B & A & B & A & B \\
B & B & B & B & B & B & B \\
\end{array}
\]