Mathematics W4041x
Introduction to Modern Algebra

Answers to Midterm Exam #2

November 18, 2010

1. If G is a group, H and N subgroups with $N \lhd G$, then HN is a subgroup, $N \lhd HN$, $H \cap N \lhd H$, and $H/(H \cap N) \cong HN/N$.

2. If $G \not\cong 1$, then there exists $g \in G$ such that $g \neq e$. The subgroup $\langle g \rangle$ generated by g is not 1, so it must be G. We know that every cyclic subgroup (that is, every subgroup generated by a single element) is isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}_n$ for some n (Assignment 7 # 1). But $\mathbb{Z}$ has the nontrivial proper subgroup $2\mathbb{Z}$, while $\mathbb{Z}_n$ for $n = ab$ composite has the nontrivial proper subgroup $a\mathbb{Z}_n$. The only remaining possibility is $G \cong \mathbb{Z}_p$ for p prime.

3. Suppose $n \in N_1 \cap N_2$ and $g \in G$. Since $N_1 \lhd G$, $gng^{-1} \in N_1$. Likewise, since $N_2 \lhd G$, $gng^{-1} \in N_2$. Hence $gng^{-1} \in N_1 \cap N_2$. So $N_1 \cap N_2 \lhd G$.

Alternative: Let $\pi_1 : G \to G/N_1$ and $\pi_2 : G \to G/N_2$ be the projection homomorphisms. Then $N_1 \cap N_2 = \ker(\pi_1 \times \pi_2) : G \to G/N_1 \times G/N_2$, and all kernels are normal.

4. By the first isomorphism theorem, $\phi(G) \cong G/\ker\phi$, so $\#G = \#\phi(G)\#\ker\phi$ and hence $\#\phi(G) \mid \#G$. And $\phi(G)$ is a subgroup of H, so by Lagrange's theorem $\#\phi(G) \mid \#H$ also. But $(\#G, \#H) = 1$, so $\#\phi(G) = 1$ and $\phi(G) = \{e\}$.

5. Since x and y are in the same orbit, there exists $g \in G$ such that $y = g \cdot x$. Then $h \in G_y \Leftrightarrow h \cdot y = y \Leftrightarrow h \cdot (g \cdot x) = g \cdot x \Leftrightarrow g^{-1} \cdot (h \cdot (g \cdot x)) = x \Leftrightarrow (g^{-1}hg) \cdot x = x \Leftrightarrow g^{-1}hg \in G_x \Leftrightarrow h \in gG_xg^{-1}$.

6. Let $\phi_1(g) := \phi(g, e)$ and $\phi_2(g) := \phi(e, g)$. Then $\phi_1(gh) = \phi(gh, e) = \phi((g, e)(h, e)) = \phi(g, e)\phi(h, e) = \phi_1(g)\phi_2(g)$, so ϕ_1 is a homomorphism, and similarly for ϕ_2. And $\phi(g_1, g_2) = \phi((g_1, e)(e, g_2)) = \phi(g_1, e)\phi(e, g_2) = \phi_1(g_1)\phi_2(g_2)$, but also $\phi(g_1, g_2) = \phi((e, g_2)(g_1, e)) = \phi(e, g_2)\phi(g_1, e) = \phi_2(g_2)\phi_1(g_1)$.

7. Let S be the set of all functions $\{1, 2, 3, 4, 5, 6, 7\} \to \{B, W\}$. The colorings of the necklace correspond to elements of this set. It is acted upon by the dihedral group D_{14}, and we need to count the number of orbits. By Burnside's lemma, this is given by

$$\frac{1}{\#D_{14}} \sum_{g \in D_{14}} \#S^g.$$

The identity fixes all 2^7 functions. Since 7 is prime, any other rotation generates $\mathbb{Z}_7$, which acts transitively on the set $\{1, 2, 3, 4, 5, 6, 7\}$. Hence the rotation fixes only 2 necklaces, the all-black and all-white ones. The other 7 elements are all reflections in an axis through one bead, as shown. They fix the 2^4 necklaces having the same colors on opposite pairs of beads. The grand total is therefore $(2^7 + 6 \cdot 2 + 7 \cdot 2^4)/14 = (64 + 6 + 56)/7 = 18$.