If \( S \) is a finite set, let \( \#S \) be the number of elements in the set, that is, the number \( n \in \mathbb{N} \) for which \( S \) is bijective to \( \{1, \ldots, n\} \). (Convention: \( \#\emptyset = 0 \).) You may use without proof the following facts:

(a) \( \#S \) is well-defined, that is, if \( \{1, \ldots, m\} \) is bijective to \( \{1, \ldots, n\} \), then \( m = n \); (b) if \( T \subset S \), then \( \#(S \setminus T) = \#S - \#T \). You may prove these things for extra credit if you like; submit them directly to me.

1. (Exercise 3 from class) Let \( m, n \in \mathbb{Z} \). Prove that \( m \mid n \) and \( n \neq 0 \) imply \( m \leq |n| \) (where \( |n| = n \) if \( n \geq 0 \), \( -n \) if \( n < 0 \)). Hint: trichotomy.

2. (Exercise 4 from class) Let \( d, a, b \in \mathbb{Z} \). Prove that \( d \mid a \) and \( d \mid b \) imply that for all \( x, y \in \mathbb{Z} \), \( d \mid (ax + by) \).

3. Read the paragraph on the Euler \( \phi \)-function (also known as the totient function) on p. 7 of Dummit & Foote. Then:

(a) Determine \( \phi(n) \) for \( n = 7, 8, 9, 10, 11 \), showing (a little of!) your work.

(b) If \( p \) is prime, prove that \( \phi(p^a) = p^a - p^{a-1} \) as stated in the text.

The other statement asserted there, that \( (a, b) = 1 \) implies \( \phi(ab) = \phi(a)\phi(b) \), is harder to prove: it will follow from the Chinese Remainder Theorem (see p. 265).

Now read §0.3 of Dummit & Foote and do exercises 10, 11, 12, 13, 14 there. Observe, by the way, that \( \mathbb{Z}_n^\times \) is a group under multiplication having \( \phi(n) \) elements.

Note: I write \([n]\) where D & F write \(\bar{n}\); I write \(\mathbb{Z}_n\) where they write \(\mathbb{Z}/n\mathbb{Z}\).