In what follows, you may assume anything stated in class except what is being asked: notably, the associative, commutative, and distributive properties of multiplication.

1. Recall that the three Peano axioms, as stated in class, were roughly: (P1) 1 is not a successor; (P2) the successor function is injective; (P3) the axiom of induction. For each \( i = 1, 2, 3 \), define a set \( N_i \), an element \( 1_i \), and a function \( f_i : N_i \to N_i \), and prove that it does not satisfy the \( i \)th Peano axiom but does satisfy the other two. Are such sets unique in the same way that \( N \) is?

2. The well-ordering principle says that every nonempty subset of \( N \) contains a least element. Prove that this element is unique.

3. Prove that \( N \) has no greatest element, that is, no element greater than or equal to every element of \( N \).

4. Prove that for all \( x \in N \), \( x \geq 1 \).

5. Prove that for all \( x, y \in N \), \( x > y \) implies \( x \geq y + 1 \).

6. (a) Prove that for all \( x, y, z \in N \), \( y < z \) implies \( xy < xz \).
   (b) Prove that for all \( x, y, z \in N \), \( y \leq z \) implies \( xy \leq xz \).

7. For \( x, y \in N \), propose a rule to define \( x^y \) similar to those for \( x + y \) and \( x \cdot y \) given in class. (You don’t have to prove that it exists, though it would follow the argument for \( x + y \) straightforwardly.)

8. A sequence of natural numbers is just a function \( a : N \to N \). Its values \( a(i) \) are frequently denoted \( a_i \). For example, \( a_i = i^2 + 1 \). For such a sequence, we may define \( \sum_{i=1}^{n} a_i \) by the rule

\[
\begin{cases}
\sum_{i=1}^{1} a_i = a_1, \\
\sum_{i=1}^{n+1} a_i = (\sum_{i=1}^{n} a_i) + a_{n+1}
\end{cases}
\]

and then use the axiom of induction to prove that this assigns a unique value to \( \sum_{i=1}^{n} a_i \).
Assuming this, use the axiom of induction to prove that \( 2 \sum_{i=1}^{n} i = n^2 + n \). Here \( 2 = 1 + 1 \).