1. Let Bij $\mathbb{R}^3$ be the group of all bijections $\mathbb{R}^3 \rightarrow \mathbb{R}^3$.
   (a) Prove that the subset of Bij $\mathbb{R}^3$ consisting of translations, that is, bijections $f(\vec{v}) = \vec{w} + \vec{v}$ for a constant $\vec{w}$, is a subgroup isomorphic to the additive group $\mathbb{R}^3$.
   Hint: realize it as the image of a homomorphism.
   (b) Prove that the subset $SO(3) \subset$ Bij $\mathbb{R}^3$ consisting of rotations, that is, bijections $f(\vec{v}) = A\vec{v}$ for a $3 \times 3$ matrix $A$ such that $A^T A = I$ and $\det A = 1$, is also a subgroup.
   (c) Define an action of $SO(3)$ on $\mathbb{R}^3$ by group automorphisms and prove that the subset $\text{Aff} \mathbb{R}^3 \subset$ Bij $\mathbb{R}^3$ consisting of bijections $f(\vec{v}) = \vec{w} + A\vec{v}$ is a subgroup isomorphic to a semidirect product $\mathbb{R}^3 \rtimes SO(3)$.

2. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Prove that any permutation of the 3 non-identity elements of $G$ defines an automorphism, so that $\text{Aut } G \cong \Sigma_3$.

3. Prove that $\mathbb{Z}_5^\times \cong \mathbb{Z}_4$. (In fact, we’ll prove next semester that $\mathbb{Z}_n^\times$ is cyclic whenever $n$ is a prime power.)

4. Give a proof of the following theorem, which was stated in class:
   Let $K$ act on $N$ and $K'$ act on $N'$ by group automorphisms. If $\psi : N \rightarrow N'$ and $\phi : K \rightarrow K'$ are isomorphisms satisfying $\psi(k \cdot n) = \phi(k) \cdot \psi(n)$ for all $k \in K$ and $n \in N$, then $\beta : N \rtimes K \rightarrow N' \rtimes K'$ defined by $\beta(n, k) := (\psi(n), \phi(k))$ is an isomorphism.

5. Let $p < q$ be two primes. Show that any group of order $pq$ is isomorphic to a semidirect product $\mathbb{Z}_q \rtimes \mathbb{Z}_p$ for some action of $\mathbb{Z}_p$ on $\mathbb{Z}_q$.

6. Classify all of the groups of order 55 up to isomorphism. How many are there? Which are abelian?

7. Classify all of the groups of order 20 up to isomorphism. How many are there? Which are abelian?