1. Any finite group is isomorphic to a subgroup of a symmetric group.

2. (a) \( \angle A \triangle \angle B \triangle \angle C \); (b) \( \angle \{ \pm 1 \} \triangle \{ \pm 1, \pm i \} \triangle \{ Q \} \); (c) \( \angle A_6 \times 1 \times 1 < A_6 \times A_7 \times 1 < A_6 \times A_7 \times A_8 \); (d) \( 1 < 3 \angle Z_{15} \triangle \angle Z_{15} < D_{30} \).

3. Yes, \((12345)\); yes, \((12345)(67)\); no, since the order is the least common multiple of the cycle lengths in a disjoint cycle decomposition, and cycle types \(5+3=8\) or \(15\) are too long for \(\Sigma_7\); \((1234)(567)\) has order 12.

4. \# \(A_5 = 5!/2 = 60 = 2^2 \cdot 3 \cdot 5\). The Sylow 3-subgroups and 5-subgroups have prime orders 3 and 5, hence are isomorphic to \(Z_3\) and \(Z_5\). In fact \((123)\) and \((12345)\) generate them. The Sylow 2-subgroups have order 4. One of them is \(\{e, (12)(34), (13)(24), (14)(23)\}\), which is \(\cong Z_2 \times Z_2\); any map taking \(e\) to \(e\) is an isomorphism.

5. By the Chinese remainder theorem, all but \(Z_{15} \times Z_6\) are isomorphic to \(Z_3 \times Z_5 \times Z_2\). But \(Z_{15} \times Z_6 \cong Z_3 \times Z_3 \times Z_5 \times Z_2\) is not isomorphic to the others, as it contains no elements of order 9.

6. True. Let \(\sigma = (123 \cdots \ell)\) and \(\tau = (\ell + 1 \cdots n)\). Then \(\sigma \tau = \tau \sigma\), \(|\sigma| = \ell\), \(|\tau| = m\), and \(\tau^{-1}(1, \ldots, \ell) = id_{(1, \ldots, \ell)}\) as well as \(\sigma^{-1}(1, \ldots, \ell) = id_{(1, \ldots, \ell)}\). Let \(N = \angle Z_{15} \times Z_m \rightarrow \Sigma_n\) be given by \(N(i, j) = \sigma^i \tau^j\).

7. False: \(Z_3 \times Z_5 \triangle Z_6\), as in the previous problem. Or \(Z_2 \times Z_2 \cong \{e, (12)(34), (13)(24), (14)(23)\} \triangle A_5\).

8. Say \(N \triangle G\) but \(N = \{e, n\} \triangle n \neq e\). Since normal, for all \(g \in G\), we have \(gng^{-1} \in N\). But \(gng^{-1} \neq e\), for otherwise \(n = g^{-1}eg = e\). So \(gng^{-1} = n\). Hence \(gn = ng\) and \(n \in ZG\). So \(N \triangle ZG\).

9. \(\Delta < G \times G \Rightarrow\) for all \(g \in G\) and \((h_1, h_2) \in G \times G\), \((h_1, h_2)(g, g)(h_1, h_2)^{-1} \in \Delta \Rightarrow\) for all \((h_1, h_2) \in G \times G\), \(h_1g h_1^{-1} = g h_1 h_1^{-1}\). Taking \(h_1 = h\) and \(h_2 = e\), the latter implies \(ghg^{-1} = g\), hence \(h = gh = gh\), so \(G\) is abelian. Conversely, if \(G\) is abelian, then \(h_1g h_1^{-1} = g = h_1 h_1^{-1}\).

10. Note \(99 = 3^2 \cdot 11\). By Sylow Thm 3, the no. of Sylow 3’s divides 11 and is \(\equiv 1 \mod 3\), hence 1. Also, the no. of Sylow 11’s divides 9 and is \(\equiv 1 \mod 11\), hence 1. So both are normal, and by the Main Thm on Direct Products, \(G \cong S_3 \times S_11\).

11. Let \(\phi : N \times K \rightarrow K\) be \(\phi(n, k) = k\). Then \(\phi((n_1, k_1)(n_2, k_2)) = \phi(n_1(k_1 \cdot n_2), k_1k_2) = \phi(n_1, k_1)\phi(n_2, k_2)\), so \(\phi\) is a homomorphism. It is clearly surjective since \(\phi(e, k) = k\), and its kernel is exactly \(N \times 1\), so the result follows from the first isomorphism theorem.

12. Let \(S\) be a Sylow \(p\)-subgroup; if \(p\) divides \# \(G\), then \(S\) is nontrivial, say of order \(p^n\). Let \(g\) be any nontrivial element; by Lagrange, \(g\) has order \(p^n\) where \(0 < n \leq n\). Let \(h = g^{p^n-1}\); then \(h\) has order \(p\).

13. Because the arrows point clockwise, there is no reflection symmetry, so the symmetry group is only rotations \(Z_p\). It acts on the set \(\{n\}\) of colorings, that is, functions \(Z_p \rightarrow \{1, \ldots, n\}\). The number of different types of necklace is the number of orbits. By Burnside’s lemma, this is \(\frac{1}{p^p} \sum S^p\). For \(g = e\), we have \(S^{e} = S\) and \# \(S^{g} = p^n\). Any other \(g\) acts transitively on the beads, so the only fixed necklaces are those in a solid color, and then \# \(S^g\) = \(n\). Hence the total number is \(\frac{n}{p}(n^p + \binom{n}{p} - 1)\). (Notice, by the way, that we already know \(p | (n^p - 1)\) by Fermat’s little theorem.)

14. (a) Proof 1: If \(g_1, g_2 \in N(H)\), then \(g_1Hg_2^{-1} = H\), so \((g_1g_2)H(g_1g_2)^{-1} = g_1Hg_2^{-1}g_1^{-1} = g_1Hg_1^{-1} = H\), so \(N(H)\) is closed under multiplication. Also, multiplying \(g_1Hg_1^{-1} = H\) on the left by \(g_1^{-1}\) and on the right by \(g_1\) yields \(H = g_1^{-1}Hg_1\), so \(g_1^{-1} \in N(H)\), so \(N(H)\) is closed under inversion. Finally, \(eHe^{-1} = H\), so \(e \in N(H)\).

(b) This is trivial, since for all \(g \in N(H)\), \(gHg^{-1} = H\), which we know implies \(N \triangle N(H)\).

(c) Let \(G\) act by conjugation on the set of all subgroups of \(G\). Then \(c\) is the size of the orbit through \(H\). On the other hand, since \(g \in N(H) \Leftrightarrow gHg^{-1} = H\), it follows that \(N(H)\) is the stabilizer group of \(H\). By the counting formula (aka orbit-stabilizer theorem), \(c = [G : N(H)]\).

(d) The orbit of \(H\) consists of all subgroups generated by a 3-cycle \((abc)\). Since \((abc)\) and \((cba)\) (and all recylings) generate the same subgroup, such subgroups are in bijective correspondence with 3-element subsets of \(\{1, \ldots, n\}\), which number \(c = \frac{n!}{3!(n-3)!}\). Hence \# \(N(H)\) = \# \(\Sigma_n/c\), that is, \# \(N(H)\) = \(3!(n-3)!\). (One can also see directly that \(N(H) \cong \Sigma_3 \times \Sigma_{n-3}\), but this takes more effort.)