Write all answers and work in your blue books. Do not hand in this sheet. 

On each page you use, write the number of the problem in a circle in the margin.

You may do more than one problem on a single page. Just put all numbers by their problems. You may do the problems out of order, but this is discouraged as it can lead to misgrading.

When there is any doubt, state briefly but clearly what statements from the text, lecture, or assignments you are using.

In grading the exams, I will emphasize accuracy, brevity, and clarity. Aim for all three.

Attempt all 14 problems. Each is worth 10 points, except the last, which is worth 20 points.

Good luck and happy holidays!

1. State Cayley’s theorem.

2. Give composition series for the following groups: (a) the symmetric group $\Sigma_7$; (b) the quaternion group $Q$; (c) $A_6 \times A_7 \times A_8$; (d) the dihedral group $D_{30}$.

3. Does $\Sigma_7$ have an element of order 5? 10? 15? If so, what is it? If not, why not? What is an element of $\Sigma_7$ having the largest possible order?

4. For each prime $p$ dividing $\#A_5$, describe the Sylow $p$-subgroups of the alternating group $A_5$ in terms of familiar groups.

5. Which of the following groups are isomorphic: $\mathbb{Z}_9 \times \mathbb{Z}_{10}$, $\mathbb{Z}_{15} \times \mathbb{Z}_6$, $\mathbb{Z}_{18} \times \mathbb{Z}_5$, $\mathbb{Z}_{45} \times \mathbb{Z}_2$? Why?

6. Give a proof or counterexample: if $\ell + m = n$, then $\Sigma_n$ has a subgroup isomorphic to $\mathbb{Z}_\ell \times \mathbb{Z}_m$.

7. Give a proof or counterexample: a subgroup of a simple group must be simple.

8. Prove that any normal subgroup of $G$ of order 2 is contained in the center $ZG$.

9. Let $\Delta = \{(g, g) \mid g \in G\} < G \times G$. Prove that $\Delta \vartriangleleft G \times G$ if and only if $G$ is abelian.

10. Classify (with proof) the groups of order 99 up to isomorphism. How many are there?

11. Use the first isomorphism theorem to prove that if $N \rtimes K$ is a semidirect product, then $(N \rtimes K)/(N \rtimes 1) \cong K$.

12. Let $G$ be a finite group. If a prime $p$ divides $\#G$, prove that $G$ contains an element of order $p$.

13. A circular necklace is made of arrow-shaped beads strung end to end so that every arrow points clockwise. If the necklace contains a prime number $p$ of beads, and the number of available colors is $n$, how many different types of necklace can be made?

14. The normalizer $N(H)$ of $H < G$ is defined to be $\{g \in G \mid gHg^{-1} = H\}$.

(a) Prove that $N(H) < G$.

(b) Prove that $H \vartriangleleft N(H)$.

(c) If $G$ is finite, let $c$ be the number of subgroups conjugate to $H$. Prove that $c = [G : N(H)]$.

(d) For $n \geq 3$, let $H = \langle (123) \rangle < \Sigma_n$. What is $\#N(H)$? Why?