K-theoretic computations in enumerative geometry

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1 Introduction

Recently, I’m reading Prof. Okounkov’s ‘Lectures on K-theoretic computations in enumerative geometry, so this notes is just a reading notes....

2 The localization theorem

Let $X$ be a smooth projective scheme with an smooth action of the smooth diagonal group $D$. We want to consider the Grothendieck group $K_D(X)$ of $D$-linearized coherent sheaf on $X$.

2.1 Basic notations and constructions

First recall the following notations and constructions

- $K_D(X)$
  the Grothendieck group of the category of $D$-equivariant locally free sheaves on $X$.
- $K^D(X)$
  the Grothendieck group of the category of $D$-equivariant coherent sheaves on $X$.
- $K_D(X^D)$
  the Grothendieck group of $D$-equivariant locally free sheaves on the fixed locus $X^D$.
- $K^D(X^D)$
  the Grothendieck group of category of $D$-equivariant locally free sheaves on the fixed locus $X^D$.
- $\Delta, X(D)$:
  character group of $D$
- $R(D) = \mathbb{Z}[\Delta]$.
  $\mathbb{Z}$-group algebra associated to the character group $\Delta = X(T)$, or simply the representation ring of $D$.
  It’s an integral domain. For $\chi \in \Delta$, we usually use $e^\chi$ to denote the corresponding element in $R(D)$. And let $S \subset R(D)$ be the multiplicative subset generated by $(1 - e^\chi)$ for all non-trivial $\chi$, then we know $0 \notin S$.
- $cl(\mathcal{F}), [\mathcal{F}]$
  The class represented by $\mathcal{F}$ in $K_D(X)$.
• $K_D(Spec(k)) \cong R(D)$.
  From this point of view $K_D(Spec(k))$ contains more information than the ordinary $K(Spec(k)) \cong \mathbb{Z}$.

• pull-back $f^!$.
  For $f$ a $D$-equivariant morphism, $f^! = K_D(f)$, which is induced by the ordinary pull-back operation of sheaves.

• push-forward $f_!$.
  For $f$ a $D$-equivariant morphism, $f_! = K_D(f)$, which is induced by the ordinary push-forward operation of sheaves, or we can say, it’s induced by the direct image functor $f_*$.

• $ch, tr$
  Let $E$ be a $k$-linear representation of $D$, then
  $$tr(E) := \sum (dim_k E_{\chi}) e^{\chi} \in R(D).$$
  It’s just the decomposition of a representation into the direct sum of 1-dimensional irreducibles.

• $tr_X$.
  If $D$ acts trivially on $X$, then $\mathcal{F} = \oplus \mathcal{F}_\chi$. We forget the $D$-equivariant structure of $\mathcal{F}_\chi$, and just view it as an element in $K(X)$, then we can define a map
  $$tr_X : K_D(X) \to K(X) \otimes \mathbb{Z} R(D)$$
  $$E \to \sum \mathcal{F}_\chi \otimes e^{\chi}$$

• $tr_x$ or $tr$?
  If $x$ is a (closed) fixed point of the $D$-action on $X$, which means
  $$tr_x : K_D(x) \to K(x) \otimes \mathbb{Z} R(D)$$
is the isomorphism we just defined, however, it’s just the map $tr$, since a class in $K(X)$ is represented by its dimension.
  $$tr : K_D(x) \to R(D); E \to \sum_{\chi} m_\chi E_{\chi}.$$  

• $\lambda$-operation.
  $$\lambda^i : K_D(X) \to K_D(X); [\mathcal{F}] \mapsto [\lambda^i \mathcal{F}].$$
  $$\lambda^i : K_D(X) \to 1 + t K_D[[t]]; [\mathcal{F}] \mapsto 1 + \sum_{i=1}^{+\infty} \lambda^i [\mathcal{F}] t^i.$$  
Note that $\lambda$ is a group homomorphism. $tr_X$ is an isomorphism (we already know the case when $X = Spec(k)$.)

• $A = \text{Sym}_k E$.
  The symmetric algebra associated to the representation $E$ over $k$, it’s also an integral domain. It has a natural grading, which we can secretly take as $k[x_0, \ldots, x_n]$.

• $D - A$-graded modules.
  A $D - A$-graded module $M$ is a $A$-graded module with a $D$ action on each graded piece subjected to
  $$d(am) = d(a)d(m); d \in D, a \in A, m \in M.$$  
The morphisms in the category of $D - A$-graded modules are those graded of degree 0, $A$-linear, $D$-linear.
• twisted $D - A$-graded modules.
  Given a graded $D - A$-graded module, the twisted $D - A$-graded module $M_\chi$ is isomorphic to $M$ as $A$-modules, but with a twisted action by $D$:
  \[
  d \cdot m := \chi(d)dm.
  \]
  Note that we have
  \[
  \text{Hom}_{D-A-graded}(A_\chi(-n), N) \cong (N_n)_\chi.
  \]
  $(N_n)_\chi$ means the submodule of $N_n$ with the $D$-action of weight $\chi$.

• $H^i(X, \mathcal{F})$ is a representation of $D$.
  Consider Čech cohomology and a theorem by Hideyasu Sumihiro which says that we can find a $D$-invariant affine open cover of $X$. There’s another way to define it, namely we have
  \[
  \mathcal{F} \xrightarrow{\sigma^*} d^* \mathcal{F} \longrightarrow \mathcal{F}
  \]
  where the second map is given by the linearization of $\mathcal{F}$, thus we have
  \[
  H^i(X, \mathcal{F}) \rightarrow H^i(X, \sigma^* \mathcal{F}) \rightarrow H^i(X, \mathcal{F}).
  \]

• The Lefschetz trace $\chi_D(X, \mathcal{F})$.
  Since the functor
  \[
  \mathcal{F} \rightarrow \sum (-1)^i \text{tr} H^i(X, \mathcal{F})
  \]
  is additive, so it induces the Lefschetz trace
  \[
  \chi_D(X, \mathcal{F}) : K_D(X) \rightarrow R(D).
  \]

• $Ch(\mathcal{F})$

• $Todd(\mathcal{F})$

• $Todd(X) := Todd(T_X)$.
  The Todd class of the tangent sheaf of $X$.

• $ct_D$ is the composition
  \[
  K_D(X) \xrightarrow{\text{tr}_X} K(X) \otimes_{\mathbb{Z}} R(D) \xrightarrow{Ch \otimes id_{R(D)}} A(X) \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} R(D)
  \]

• $Todd_D$ is the composition
  \[
  K(X) \otimes_{\mathbb{Z}} R(D) \xrightarrow{Todd \otimes id_{R(D)}} A(X) \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} R(D)
  \]

2.2 $K_D(X)$ and $K^D(X)$; $i^!$ and $i_!$

Proposition 2.1 (coherent sheaves v.s vector bundles, $K_D(X) = K^D(X)$). The natural of $K_D(X)$ into the Grothendick group of the category of $D$-linearized coherent sheaves on $X$ is an isomorphism.
Remark (tensor product is not the multiplication in $K^D(X)$). Since tensor product is only right exact, this is not the multiplication in $K^D(X)$, i.e. if we have a short exact sequence of coherent sheaves

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0.$$ 

Then in $K^D(X)$, we have $[\mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_3]$; however, if $\mathcal{F}$, a coherent sheaf, but not flat, then $[\mathcal{E}_2 \otimes \mathcal{F}] \neq [(\mathcal{E}_1 \oplus \mathcal{E}) \otimes \mathcal{F}]$. On the other hand, by the theorem above, we know $K^D(X) = K^D(X)$ for a smooth projective scheme $X$, tensor product is the multiplication in $K_D(X)$, so we know $K^D(X)$ does have a ring structure. That is by Hilbert syzygy theorem, we can always find a free resolution

$$0 \to \mathcal{V}_{n+1} \to \cdots \to \mathcal{V}_0 \to \mathcal{E} \to 0$$

then we also have

$$0 \to \mathcal{V}_{n+1} \otimes \mathcal{F} \to \cdots \to \mathcal{V}_0 \otimes \mathcal{F} \to \mathcal{E} \otimes \mathcal{F} \to 0.$$ 

By the definition of $K^D(X)$, we know

$$[\mathcal{E}][\mathcal{F}] = \sum (-1)^i [\mathcal{V}_i][\mathcal{F}].$$

Then by some general fact from homological algebra, we also get

$$[\mathcal{E}][\mathcal{F}] = \sum (-1)^i \text{Tor}_i^{O_X}(\mathcal{E}, \mathcal{F}).$$

Example 2.2 (transverse intersection). Let $Y_1, Y_2$ be two closed subschemes of $X$ in "general position", then we have

$$[O_{Y_1}][O_{Y_2}] = [O_{Y_1 \cap Y_2}].$$

Let $i : Y \to X$ be a $D$-equivariant closed immersion of smooth projective schemes with $D$-action.

Proposition 2.3 (the projection formula, push-pull). For any $x \in K_D(X)$, $y \in K_D(y)$, we have

$$i_!(i^!x \otimes y) = x \otimes i_!y$$

Proposition 2.4 (the self-intersection formula).

$$i_!i^!y = y_{\lambda - 1}N_{Y/X}^\vee.$$ 

Proposition 2.5 (the Cartesian formula). Let

$$
\begin{array}{ccc}
T & \xrightarrow{j'} & Y \\
\downarrow{i'} & & \downarrow{i} \\
Z & \xrightarrow{j} & X
\end{array}
$$

be a Cartesian square of $D$-equivariant immersions between smooth projective schemes with $D$-action. Then there exists $\gamma_T \in K_D(T)$, such that for $\forall y \in K_D(y)$

$$i_!(\gamma_T j^!y) = j^!(i_!y).$$

2.3 basic properties

Proposition 2.6 (Hideyasu Sumihiro). A normal variety over an algebraically closed field with an action of a torus is covered by invariant affine open subsets.

Remark (normality is necessary). Consider $\mathbb{P}^1/\{0, \infty\}$, or the projective nodal curve.
Remark (Torus action is necessary). Białynicki-Birula and Swiecicka’s paper "On complete orbit spaces of $SL(2)$ actions II".

Proposition 2.7 ($X^D$ is smooth).

Proposition 2.8 (Yuri, Manin). Let $Z$ be a smooth projective variety(?), $z \in Z$ a closed point, $j_z : z \to Z$ the inclusion, then we have
\[ K(Z) = \mathbb{Z} \oplus \ker(j_z^*) \]
and $\ker(J^*)$ is nilpotent. If $Z$ is an irreducible component of $X^D$, consider the isomorphism $tr_Z$, we actually have
\[ K(Z) = (\mathbb{Z} \oplus \ker(j^*)) \otimes \mathbb{Z} R(D) \cong R(D) \oplus \ker(j^*) \]
and $\ker(j^*)$ is nilpotent. The last identity is due to the fact that $K(Z)$ is a natural $R(D)$-algebra, $\ker(j^*) \otimes \mathbb{Z} R(D) = \ker(j^*)$.

Proof. See Manin, "Lectures on $K$-functors in algebraic geometry."

2.4 the localization theorem

Theorem 2.9 (the localization theorem).

The inclusion $i : X^D \to X$ induces an $R(D)$-linear map
\[ i^! : K_D(X) \to K_D(X^D) \]
which is an isomorphism after localization w.r.t $S$, its inverse is given by
\[ S^{-1}K_D(X^D) \to S^{-1}K_D(X) \]
\[ y \mapsto S^{-1}i!(y(\lambda^{-1}N^\vee_{X^D/X})^{-1}). \]

Before proving this theorem, we might ask

• why $\lambda^{-1}N^\vee$ is invertible in $S^{-1}K_D(X^D)$, in other words, why $\lambda^{-1}N^\vee \in S$?

2.5 compute $\chi_D(X, \mathcal{F})$ via $\chi(X^D, i^!*\mathcal{F})$

Proposition 2.10 (just like $H^i(Y, \mathcal{F}) = H^i(X, i_*\mathcal{F})$). Let $i : Y \to X$ be a $D$-equivariant closed immersion, then
\[ \chi_D(Y, y) = \chi_D(X, i!(y)). \]

Remark (when $H^i(Y, \mathcal{F}) = H^i(X, i_*\mathcal{F})$)?

Proposition 2.11 (compute $\chi_D(X^D, -)$). If $D$ acts trivially on $X$, then the following diagram commutes:

The first proposition tells us if a coherent sheaf is the push-forward of some coherent sheaf on $Y$, then their characters are the same. The second proposition tells us if you want to compute the character on the fixed locus, you can just use the ordinary, unlinearized $\chi$. And the localization theorem tells us proposition is always true, if we consider the localization w.r.t $S$, that is

Proposition 2.12 (local-global). Let $i : X^D \to X$, $\forall x \in X$
\[ X^D(X, x) = S^{-1}\chi_D(X^D, i^!(X) \bullet (\lambda^{-1}N^\vee)^{-1}) \text{in } S^{-1}R(D). \]
Proof. By the localization theorem, after taking the localization, we have

\[ x = i_!(i^!(x) \cdot (\lambda^{-1}N^\vee)^{-1}) \]

then apply the first proposition above, we get

\[ \chi_D(X, x) = S^{-1}\chi_D(X^D, i^!(x) \cdot (\lambda^{-1}N^\vee)^{-1}) \]

Example 2.13 (\(\chi(X, \lambda^{-1}\Omega_X) = \chi(X^T, \lambda^{-1}\Omega_T)\)). Consider the cotangent sequence

\[ 0 \to N^\vee_{X^D/X} \to i^*\Omega_X \to \Omega_{X^D} \to 0. \]

Since \(\lambda\)-operation is a group homomorphism, we get

\[ \lambda^{-1}[i^*\Omega_X] = \lambda^{-1}N^\vee \cdot \lambda^{-1}[\Omega_{X^D}] \]

Note that

- \(\lambda^{-1}[i^*\Omega_X] = i^![\lambda^{-1}\Omega_X]\) by definition. In general, we don’t have this kind of identity, but since \(X\) is smooth, \(\Omega_X\) is a vector bundle, thus flat, so is its restriction on \(X^D\), thus \(i^![\Omega_X]\) contains only the first term, that is \(i^*\Omega_X = f^{-1}\Omega_X \otimes f^{-1}\mathcal{O}_Y \mathcal{O}_{X^D}\).

- \(\lambda^{-1}\Omega_{X^D} = i^![\lambda^{-1}\Omega_X] \cdot (\lambda^{-1}N^\vee)^{-1}\)

Thus by the local-global proposition above, we get

\[ \chi_D(X, \lambda^{-1}\Omega_X) = \chi_D(X^D, \lambda^{-1}\Omega_{X^D}) \text{ in } S^{-1}R(D). \]

Specially, if \(D = T\) is an algebraic torus, then the identity holds in \(R(D)\). So no need to worry about the denominator, let \(e^x = 1\), for all \(\chi\), then we get

\[ \chi(X, \lambda^{-1}\Omega_X) = \chi(X^D, \lambda^{-1}\Omega_{X^D}). \]

Remark (what’s the difference between \(D\) and an algebraic torus?).

2.6 Lefschetz fixed point theorem

Theorem 2.14 (Lefschetz fixed-point theorem, isolated, finite). If \(X^D\) is isolated and finite, for a \(D\)-equivariant coherent sheaf \(\mathcal{F}\) on \(X\), we have

\[ \sum_i (-1)^i \text{tr}^i(X, \mathcal{F}) = \sum_{z \in \mathcal{D}} \frac{\text{tr}\mathcal{F}_z}{\sum_i (-1)^i \text{tr} \wedge T_zX^\vee}. \]

Proof. Because

- \(i_!^!\mathcal{F} = [\mathcal{F}_z].\)
- \(N_{z/X} = T_{z/X}.\)

Remark \((i^!\mathcal{F} = i^*\mathcal{F} = \mathcal{F}_z?)\). This is for sure true if \(\mathcal{F}\) is a vector bundle, here because \(z\) is an isolated point, \(K_D(z) \cong \mathbb{Z}\), so it’s true in general.

Example 2.15 (Weyl character formula).
Theorem 2.16 (the cohomological formula).

$$\chi_D(X, F) = \int_{X^D} \frac{ct_D(i^* F) Todd_D(X^D)}{ct_D(\lambda_1 N^X_{X^D/\chi})}.$$  

Theorem 2.17 (the Woods-Hole formula). Let $\sigma \in D(k)$. The evaluation map

$$ev_\sigma : R(D) \to k; \chi \mapsto \chi(\sigma)$$

gives us the ordinary trace, i.e $ev_\sigma(Tr(E)) = Tr(\sigma, E)$, the trace of the $\sigma$ on $E$. If $\sigma$ is a dense(regualr) element(i.e $\chi(\sigma) \neq 1$ for all non-trivial character $\chi$), then $ev_\sigma$ can be extended to be a map

$$ev_\sigma : S^{-1}R(D) \to k.$$  

We have

$$\sum_i (-1)^i Tr(\sigma, H^i(X, F)) = \sum_{z \in X^D} \frac{Tr(\sigma, F_z)}{Det(1 - d_z \sigma)}$$

where $d_z$ is the differential at $z$.

Theorem 2.18 (Specialization to the Witt ring). Assume that char($k$) = $p \neq 0$. For an element $\sigma \in D(k)$, the composite of the evaluation map $ev_\sigma$ and the Teichmuller lifting $w : k^* \to Witt(k)$ gives a map

$$b_\sigma : R(D) \to Witt(k)$$

such that we have $b_\sigma(Tr(E)) = BTr(\sigma, E)$, the Brauer trace for the operation of $\sigma$ on $E$. If we assume further that $D$ is finite cyclic with generator $d \in D(k)$, $d$ is regular(dense), then $b_\sigma$ can be extended to be

$$R(D) \to S^{-1}R(D) \to Witt(k)$$

then we can get a formula of P.Donovan(Thm5.3, The Lefschetz-Riemann-Roch formula).

2.7 Some comparisons

Example 2.19 ($K(X)$ and $Pic(X)$).

Example 2.20 (Lefschetz fixed-point theorem and Lefschetz hyperplane theorem).

Example 2.21 ($K(X)$ and $CH(X) = A(X)$).

3 Equivariant K-theory of Grassmannians

4 Equivariant K-theory of Flag varieties

5 K-theoretic proof of the Weyl Character formula

6 Comparisons between intersection theory and K-theory

7 Exercises in Andrei’s notes, Chapter 2

Example 7.1 (Ex2.1.5).

$$\sum(-s)^k \chi_V^k(t) = \Pi(1 - st^n) = exp(-\frac{1}{n}s^n \chi_V(t^n)).$$

Choose a basis $\{e_1, \ldots, e_k\}$ for the weight decomposition of $V$, $\{e_i_1 \wedge \ldots e_i_r | 1 \leq i_1 < i_2 \ldots < i_r \leq r\}$ is a basis for the weight decomposition of $\wedge^r V$, and the second identity comes from $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$.  

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Example 7.2 (Ex2.1.7, Koszul complex). Construct an \( GL(V) \)-equivariant exact sequence
\[
\cdots \rightarrow \wedge^2 V \otimes S^* V \rightarrow \wedge^1 V \otimes S^* V \rightarrow S^* V \rightarrow \mathbb{C} \rightarrow 0.
\]

Example 7.3 (Ex2.1.9). Consider \( V \cong \text{Spec}(\mathbb{C}[x_1, \ldots, x_n]) \) as an algebraic variety on which \( GL(V) \) acts. Construct a \( GL(V) \)-equivariant resolution of \( O_0 \), the structure sheaf of \( 0 \in V \) by vector bundles on \( V \).
\[
0 \rightarrow O_{\mathbb{C}^n}(\frac{\langle n \rangle}{n!}) \rightarrow \cdots \rightarrow O_{\mathbb{C}^n}(\frac{\langle x_1, \ldots, x_n \rangle}{n!}) \rightarrow O_{\mathbb{C}^n} \rightarrow O_0 \rightarrow 0.
\]

But this is the same as given by the Koszul complex. We need something else.

Example 7.4 (Ex 2.1.12, Ex 2.1.14 \( \mu = 0 \) is not a weight of \( V \)).
\[
S^* V = (-1)^{rk V} \det V^\vee \otimes S^* V^\vee.
\]

Since \( \mu = 0 \) is not a weight of \( V \), we have
\[
\sum_{k \geq 0} \chi_{S^k V}(t) = \frac{1}{1 - t^\mu}
\]
\[
\sum_{k \geq 0} \chi_{S^k V^\vee}(t) = \frac{1}{1 - t^{-\mu}}
\]
\[
\chi_{\det V}(t) = t^{-\sum \mu}.
\]

Then \( S^* V = (-1)^{rk V} \det V^\vee \otimes S^* V^\vee \) is nothing but \( \frac{1}{1 - t^{-1}} = (-1)^{\frac{1}{1 - t}} \). With this description, we naturally get
\[
S^* (V_1 \oplus V_2) = S^* V_1 \oplus S^* V_2,
\]
specialy
\[
S^* (-V) = \wedge^* V = \sum_i (-1)^i \wedge^i V.
\]

Example 7.5 (Ex2.1.17, 2.1.18, the map \( S^* \)).
\[
K'_T \rightarrow K_{T, \text{localized}}
\]
\[
S^*(a - b) = \frac{1 - b}{1 - a}, S^* a - b = \Pi_{n \geq 0} \frac{1 - q^n b}{1 - q^n a}
\]
\[
S^* \frac{a}{(1 - q)^{k+1}} = \Pi_{n \geq 0} (1 - q^n a)^{-\binom{n+k}{n}}
\]

Here, I think \( a, b \) represent some virtual 1-dimensional \( T \)-representations with non-trivial character. Then
\[
S^* a = \frac{1}{1 - a}, S^* b = 1 - b.
\]

Together with the taylor expansion of \( \frac{1}{1 - q} \) and \( \frac{1}{(1 - q)^{k+1}} \), we get the formula above (note that \( S^* \) turns direct sums into tensor products).

Example 7.6 (inverse of \( S^* \)). Prove that the inverse to \( S^* \) is given by
\[
\chi_V(t) = \sum_{n > 0} \frac{\mu(n)}{n} \ln \chi_{S^* V}(t^n)
\]
where $\mu$ is the Mobius function

$$
\mu(n) = \begin{cases} 
(-1)^{\text{#prime factors, } n \text{ square free}} \\
0, \text{ otherwise.}
\end{cases} \quad (1)
$$

Let’s prove a special case first, the 1-dimensional representation $\chi_V(t) = t$. We need to prove

$$
t = \sum_{n \geq 1} \frac{\mu(n)}{n} \ln\left( \frac{1}{1-t^n} \right)
$$

plug in $t = 0$, they are the same. Then we compute the derivatives,

$$
1 = \sum_{n \geq 1} \frac{\mu(n)t^{n-1}}{1-t^n}.
$$

In the RHS, the coefficient of $t^k, k \geq 1$ is given by

$$
\sum_{(n-1)+rn=k} \mu(n) = \sum_{n|(k+1)} \mu(n) = 0.
$$

The last equality follows from the prime factorization of $k+1$ and the definition of $\mu(n)$. Actually, this does give us a proof. By changes of the variable, we have

$$
t^\mu = \sum_{n \geq 1} \frac{\mu(n)}{n} \ln\left( \frac{1}{1-t^{n\mu_i}} \right)
$$

thus

$$
\chi_V(t) = \sum_i t^\mu_i = \sum_i \sum_{n \geq 1} \frac{\mu(n)}{n} \ln\left( \frac{1}{1-t^{n\mu_i}} \right)
= \sum_{n \geq 1} \frac{\mu(n)}{n} \ln(\Pi_i \frac{1}{1-t^{n\mu_i}}) = \sum_{n>0} \frac{\mu(n)}{n} \ln \chi_{S^*V}(t^n).
$$

**Example 7.7** (Bialynicki-Birula decomposition, page 23). We use this method (instead of Morse theory), to compute the Poincaré polynomial of several Hilbert schemes of points.

**Example 7.8** (Ex 2.2.3, $K_G^X(X) \nleq K_G^X(X)$). Consider $X = \text{Spec}(\mathbb{C}[x_1,x_2]/(x_1x_2)) \subset \mathbb{C}^2$ with the natural action of the maximal torus $T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \subset GL(2)$. Let $\mathcal{F} = \mathcal{O}_0$ be the structure sheaf of the origin $0 \in X$.

We want to compute the minimal $T$-equivariant resolution

$$
\cdots \to \mathcal{R}^{-2} \xrightarrow{d_{-2}} \mathcal{R}^{-1} \xrightarrow{d_{-1}} \mathcal{R}^0 \xrightarrow{d} \mathcal{F} \to 0
$$

of $\mathcal{F}$ by sheaves (not necessarily vector bundles!) of the form

$$
\mathcal{R}^i = \mathcal{O}_X \otimes R^i
$$

where $R^i$ is a finite dimensional $T$-module. To do this let

$$
\mathcal{R}^0 = \mathcal{O}_X, H^0(\mathcal{O}_X) = \text{span}\{1, x_1^k, x_2^k | k \geq 1\}
$$

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\[ R^{-1} = \mathcal{O}_X \otimes (\mathbb{C} e_{-1} \oplus \mathbb{C} f_{-1}), \text{wt}(e_{-1}) = t_1, \text{wt}(f_{-1}) = t_2 \]
\[ d_{-1} : t^n_{i} \otimes e_{-1} \mapsto t^n_{i} t_1, t^n_{i} \otimes f_{-1} \mapsto t^n_{i} t_2. \]

Similarly, we define
\[ R^{-k} = \mathcal{O}_X \otimes (\mathbb{C} e_{-k} \oplus \mathbb{C} f_{-k}), \text{wt}(e_{-k}) = t_2 t_{1}^{k-1}, \text{wt}(f_{-k}) = t_1 t_{2}^{k-1} \]
\[ d_{-k} : t^n_{i} \otimes e_{-k} \mapsto (t^n_{i} t_1) \otimes f_{-(k-1)}, t^n_{i} \otimes f_{-k} \mapsto (t_2 t^n_{i}) \otimes e_{-(k-1)}. \]

It’s straightforward to check that
\[
\ker(d_{-k}) = \text{span}\{t^n_{i} \otimes e_{-k}, t^n_{i} \otimes f_{-k} \mid n \geq 1\} = \text{im}(d_{-(k+1)}).
\]

Now we know
\[ R^{-k} \otimes \mathcal{F} = (\mathcal{O}_X \otimes \mathcal{F}) \otimes R^i, \chi_{R_{i}} = t_2 t_{1}^{k-1} + t_1 t_{2}^{k-1}. \]
\[ \text{Tor}_{i}(\mathcal{F}, \mathcal{F}) := H^{-i}(R^\bullet \otimes \mathcal{F}) = R^i. \]

The last equality is because of Schur’s lemma and the fact that different $R^i$’s have different weights. This already tells us $\mathcal{F}$ is not in the image of $K^*_T(X) \rightarrow K_T(X)$. We also know
\[
\sum_i (-1)^i \frac{\chi(\mathcal{F})^2}{\chi(\mathcal{O}_X)} = \frac{1}{1+t_1^{-1}} + \frac{1}{1+t_2^{-1}} - 1 = \frac{(1-t_1^{-1})(1-t_2^{-1})}{1-t_1^{-1}t_2^{-1}}.
\]

Remark (\(\sum_i (-1)^i \chi_{\text{Tor}^i(\mathcal{F}, \mathcal{F})} = \frac{\chi(\mathcal{F})^2}{\chi(\mathcal{O}_X)}\)). In the derived category of coherent sheaves, we have
\[ \mathcal{F} \otimes L \mathcal{G} \sim \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet \]

If we assume the free resolutions are $T$-equivariant and of the form $\mathcal{O}_X \otimes F_i, \mathcal{O}_X \otimes G_i$, for some $T$-modules $F_i$ and $G_i$. We get
\[
\chi(\mathcal{F} \otimes L \mathcal{G}) = \sum_i (-1)^i \chi_{\text{Tor}^i(\mathcal{F}, \mathcal{G})} = \sum_i (-1)^i \chi(\oplus_{m+n=i}(\mathcal{O}_X \otimes F_m) \otimes (\mathcal{O}_X \otimes G_n))
\]
\[
= \frac{1}{\chi(\mathcal{O}_X)} \sum_i (-1)^i (\sum_{m+n=i} \chi(\mathcal{O}_X \otimes F_m) \chi(\mathcal{O}_X \otimes G_n))
\]
\[
= \frac{\chi(\mathcal{F})\chi(\mathcal{G})}{\chi(\mathcal{O}_X)},
\]

thus in this situation, we have
\[
\chi(\mathcal{F} \otimes L \mathcal{G}) = \frac{\chi(\mathcal{F})\chi(\mathcal{G})}{\chi(\mathcal{O}_X)},
\]
what we want in the example is above is just a special case.

Remark (every coherent sheaf on a smooth variety is perfect).

Example 7.9 (Ex 2.2.6). Generalize the last identity above to the case
\[
X = \text{Spec}(\mathbb{C}[x_1, \ldots, x_d]/I)
\]
\[ \mathcal{F} = \mathbb{C}[x_1, \ldots, x_n]/I' \]
where $I \subset I'$ are monomial ideals.
Example 7.10 (Ex 2.2.10, 2.3.14, compute $\chi(\mathbb{P}^n, \mathcal{O}(k))$ by localization). Take $X = \mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ and $GL(n+1)$ naturally acts on $X$. \{D_+(x_i)\}_{i=1}^{n+1}$ is a $T$-invariant Cech covering of $\mathbb{P}^n$. We denote $\mathcal{O}(k)$ by $\mathcal{F}$.

- $\chi(\mathbb{P}^n, \mathcal{O}(k)) = \begin{cases} \sum_{i_1+\cdots+i_{n+1}=k} t_1^{-i_1} t_2^{-i_2} \cdots t_{n+1}^{-i_{n+1}} & k \geq 0 \\ 0 & -n-1 < k < 0 \\ (t_1 t_2 \cdots t_n) \sum_{i_1+\cdots+i_{n+1}=-k-n-1} t_1^{i_1} t_2^{i_2} \cdots t_{n+1}^{i_{n+1}} & k \leq -n-1 \end{cases}$ \hspace{1cm} (2)

- **fixed points**

  $p_i = [0, \ldots, 0, x_i = 1, 0, \ldots, 0]$

  $wt(\mathcal{F}_{p_i}) = t_i^{-k}$

- **weight of the stalk**

  $wt(\mathcal{O}(p)) = \sum_{j \neq i} \frac{t_j}{t_i}$

  $(\lambda_1 N_{p_i/\mathbb{P}^n})^{-1} = \prod_{j \neq i} \frac{1}{1 - \frac{t_i}{t_j}}$

- **the localization theorem tells us that**

  $\chi(\mathbb{P}^n, \mathcal{O}(k)) = \frac{n+1}{\prod_{j \neq i} (1 - \frac{t_i}{t_j})} \sum_{i=1}^{n+1} \frac{t_i^{-k}}{(1 - \frac{t_i}{t_j})}$.

- **to convince yourself this is is the correct answer, check for example**

  $\chi(\mathbb{P}^1, \mathcal{O}(k)) = \frac{t_1^{-k}}{1 - \frac{t_1}{t_2}} + \frac{t_2^{-k}}{1 - \frac{t_2}{t_1}}$

  $= \frac{t_2^{-k} t_1 - t_1^{-k} t_2}{t_1 - t_2} = \frac{t_2^{-k} t_1 (1 - \frac{t_2}{t_1})^{k+1}}{t_1 - t_2}$

  $= \frac{t_2^{-k} (t_1 - t_2)(1 + \frac{t_2}{t_1} + \cdots + \frac{(t_2)^k}{t_1^k})}{t_1 - t_2}$

  $= \sum_{i+j=k} t_1^{-i} t_2^{-j}$.

  

  We note that this only checks the $k \geq 0$ cases, because we need the factorization of $1 - x^{k+1}$, but it’s not a problem at all, use the factorization of $x^{-k} - y^{-k}$ instead. This also gives us a way to check the two expression of $\chi(\mathbb{P}^n, \mathcal{O}(k))$, we leave it as an exercise.

Remark (Euler sequence, tangent sequence are not $T$-equivariant). We know $N_{p_i/\mathbb{P}^n} \cong \mathcal{O}(1)^n$, but we can not use this isomorphism to compute the character of the normal bundle, because it’s not a $T$-equivariant isomorphism. To be more precise, as vector bundles they’re the same, but the $T$-action on the normal bundle in this exercise doesn’t agree with the natural $T$-action on $\mathcal{O}(1)^n$.

Remark ($S^1 \mathbb{C}^n$ is irreducible). We can use the computation of the character to prove that $S^1 \mathbb{C}^n$ is irreducible as a representation of $GL(n)$.

Example 7.11 (Ex 2.3.15, 2.3.16 $\chi(G/B, \mathcal{L}_\lambda)$).
**Example 7.12** ($\chi(X_{a,b}, O(k))$, character of sheaves on a weighted projective space). For $a, b > 0$, consider the weighted projective line

$$X_{a,b} := \mathbb{C}^2 \setminus \{0\}/\left(\begin{array}{cc} z^a & 0 \\ 0 & z^b \end{array}\right), \ z \in \mathbb{C}^\times.$$ 

Then $D_+(x), D_+(y)$ are two orbifold charts. Like any $\mathbb{C}^\times$-quotient, it inherits an orbifold line bundle $O(k)$ whose sections are functions $\phi$ on $\mathbb{C}^2 \setminus \{0\}$ such that $\phi(z \cdot x) = z^k \phi(x)$. These sections are just the vector space spanned by monomials of the form $x^i y^j, ai + bj = k$.

Then we easily get the character formula

$$\sum_{k \geq 0} \chi(X_{a,b}, O(k)) s^k = \frac{1}{(1-t_1^{-1}s^a)(1-t_2^{-1}s^b)}.$$ 

We can also get this result by applying the localization theorem (for orbifolds), quite similar like the $\mathbb{P}^1$ case, we have

$$\chi(X_{a,b}, O(k)) = \frac{t_1^{-k}}{1-t_1^{-1}} + \frac{t_2^{-k}}{1-t_2^{-1}}.$$

**Example 7.13** (Ex 2.2.15 Projection formula).

$$f_*(\mathcal{F} \otimes f^* \mathcal{E}) \cong f_*(\mathcal{F}) \otimes \mathcal{E}.$$ 

**Example 7.14** (fractional power of the canonical bundle). Let $X$ be a proper nonsingular variety with a nontrivial action of $T \cong \mathbb{C}^\times$. Assume that a fractional power $\mathcal{K}^p, 0 < p < 1$ of the canonical bundle $\mathcal{K}_X$ exists in $\text{Pic}(X)$. Replacing $T$ by a finite cover, we can make it act on $\mathcal{K}^p$. Show that $\chi(X, \mathcal{K}^p) = 0$.

What does this say about projective spaces? Concretely, which are the bundles $\mathcal{K}^p, 0 < p < 1$, for $X = \mathbb{P}^n$ and what do we know about their cohomology?

Serre duality, $\mathcal{K}^\vee = \mathcal{O}_{\mathbb{P}^n}(\frac{-s(n+1)}{t})$, if we assume $(s, t) = 1$, we then need $t|s(n+1)$.

**Example 7.15** ($\chi(X, \lambda^{-1}\Omega_X) = \chi(X^T, \lambda^{-1}\Omega_T)$). Consider the cotangent sequence

$$0 \to N_{X^D/X} \to i^*\Omega_X \to \Omega_{X^D} \to 0.$$ 

Since $\lambda$-operation is a group homomorphism, we get

$$\lambda^{-1}[i^*\Omega_X] = \lambda^{-1} N^V \cdot \lambda^{-1} [\Omega_{X^D}]$$

Note that

- $\lambda^{-1}[i^*\Omega_X] = i^*[\lambda^{-1}\Omega_X]$ by definition. In general, we don’t have this kind of identity, but since $X$ is smooth, $\Omega_X$ is a vector bundle, thus flat, so is its restriction on $X^D$, thus $i^*\Omega_X$ contains only the first term, that is $i^*\Omega_X = f^{-1}\Omega_X \otimes f^{-1}O_Y\, O_{X^D}$.
\[ \lambda \Omega X \mathcal{D} = \lambda! \left[ \chi_{\mathcal{D}}(X, \lambda - 1 \Omega X) \right] \cdot (\lambda - 1 N^\vee)^{-1} \]

Thus by the local-global proposition above, we get

\[ \chi_{\mathcal{D}}(X, \lambda - 1 \Omega X) = \chi_{\mathcal{D}}(X^D, \lambda - 1 \Omega X^D) \text{ in } S^{-1}R(D). \]

Specially, if \( D = T \) is an algebraic torus, then the identity golds in \( R(D) \). So no need to worry about the denominator, let \( e^\chi = 1 \), for all \( \chi \), then we get

\[ \chi(X, \lambda - 1 \Omega X) = \chi(X^D, \lambda - 1 \Omega X^D). \]

**Example 7.16** (Ex 2.4.2, localization formula for \( \chi(X, \lambda - 1 \Omega X) \)). Let \( X \) be proper and smooth with an action of a connected reductive group \( G \). Write a localization formula for the torus action of \( T \subset G \) on

\[ \sum_p (-z)^p \chi(X, \Omega^p) \]

and conclude that every term in this sum is a trivial \( G \)-module. Without losing of generality, we may assume \( X^T \) is an irreducible subvariety (otherwise, the localization formula is just the summation over all the components, which makes no essential difference). From the example above, we know \( \chi(X, \lambda - 1 \Omega X) = \chi(X^T, \lambda - 1 \Omega X^T) \), and this is essentially what we need, we can write down the localization theorem

\[ \sum_p (-z)^p \chi(X, \Omega^p) = \chi(X, \lambda - z \Omega X) = \frac{\text{tr}(\lambda - z \Omega X | X^T)}{\text{tr}(\lambda - 1 N^\vee_{X^T / X})}. \]

Note that the torus action on \( X^T \) is just the identity action, thus the formula

\[ \frac{\text{tr}(\lambda - z N^\vee_{X^T / X}) (1 - z)^{\text{dim}X^T}}{\text{tr}(\lambda - 1 N^\vee_{X^T / X})} \]

Let \( \text{tr}(N^\vee_{X^T / X}) = w_1 + \cdots + w_k \), where \( w_i \) are Laurent polynomials w.r.t \( t_1, \ldots, t_{\text{dim}T} \), then we finally have

\[ \sum_p (-z)^p \chi(X, \Omega^p) = \chi(X, \lambda - z \Omega X) = \frac{(1 - z)^{\text{dim}X^T} \Pi_{i=1}^k (1 - zw_i)}{\Pi_{i=1}^k (1 - w_i)}. \]

No matter what kind of limit we take w.r.t \( t_i \), the formula above is always well-defined, this tells us that it’s actually of the form

\[ \sum_p (-z)^p \chi(X, \Omega^p) = \sum_p a_p z^p, a_p \in \mathbb{Z}_{\geq 0}. \]

This tells us exactly every term \( \Omega^p \) is a trivial \( G \)-module.

**Remark.** compact Kahler + Hodge theory gives triviality of \( G \)-action on each

\[ H^q(X, \Omega^p) \subset H^{p+q}(X, \mathbb{C}). \]

**8 Chapter 3**

**Example 8.1** (Ex3.3.13, 3.3.14, 3.3.15, 3.3.16 Spin representations of \( SO(V) \)).