ABSTRACT. We study the relative Gromov-Witten theory on $T^* \mathbb{P}^1 \times \mathbb{P}^1$ and show that certain equivariant limits give us the relative invariants on $\mathbb{P}^1 \times \mathbb{P}^1$. By formulating the quantum multiplications on $\text{Hilb}(T^* \mathbb{P}^1)$ computed by Devash Maulik and Alexei Oblomkov as vertex operators and computing the product expansion, we demonstrate how to get the insertion and tangency operators computed by Yaim Cooper and Rahul Pandharipande in the equivariant limits.

CONTENTS

1. Introduction 1
2. Relative stable morphisms 1
3. Moduli space of relative stable maps 6
4. Classical multiplication 7
5. Basic representations of affine Lie algebras and Vertex operators 8
6. Quantum multiplication 8
7. Diagonalization 11
8. Conclusions 14
References 14

1. INTRODUCTION

2. RELATIVE STABLE MORPHISMS

In this part, we give a brief review of the relative Gromov-Witten theory following [LLZ04]. By comparing the virtual fundamental classes of $\mathbb{A}_1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ we give the change of variables that is needed to get the relative invariants of $\mathbb{P}^1 \times \mathbb{P}^1$ from those of $\mathbb{A}_1 \times \mathbb{P}^1$.

Given a smooth projective variety $X$, and $D_1, \ldots, D_k$, smooth divisors on $X$, the relative Gromov-Witten invariants essentially count the number of stable morphisms from curves to $X$ with certain intersection conditions on the divisors and we allow degeneration of $X$ along $D_i$, a subtle point is that the degeneration is a ‘rubber’, in the sense that morphisms into the degeneration are viewed equal up to a $\mathbb{C}^*$-action. We demonstrate the definition in the case of one smooth divisor $D \hookrightarrow X$, multiple divisors only use more indices. We first introduce some notations.

- $\Delta(D) = \mathbb{P}(\mathcal{O}_D \oplus N_{D/X}) \rightarrow D$, the projective completion of the normal bundle of $D$.
- $\Delta(D)(m)$, the union of $m$ copies of $\Delta(D)$ by identifying the zero section $D_0 = \mathbb{P}(\mathcal{O}_D \oplus 0) \hookrightarrow \Delta(D)$ of the $i+1$-th copy with the $\infty$-section $D_\infty = \mathbb{P}(0 \oplus N_D/Y)$ of the $i$-th copy. Denote the $k$-th section in this degeneration by $D_k$.
- $X[m] = X \cup \Delta(D)(m)$. The $m$-fold degeneration of $X$ along the divisor.
- $(\mathbb{C}^*)^m$ acts on $\Delta(D)(m)$, the action is trivially on the divisor.
\[\begin{align*}
\pi[m] : X[m] &\to X, \text{ the natural projection which is equivariant under the torus action.} \\
\beta &\in H_2(X, \mathbb{Z}), \text{ an effective curve class.} \\
d &\equiv \int_\beta c_1(\mathcal{O}_D) \geq 0, \text{ the intersection number of } \beta \text{ with the divisor.} \\
\mu &\text{, a partition of } d \text{ and let } \ell(\mu) \text{ be the length of the partition. This basically records the intersection type of } \beta \text{ with the divisor.} \\
C &\text{, the source curve, let } (x_i)_{i=1}^{\ell(\mu)} \text{ be the marked points on } C \text{ that are mapped to be the intersection points of } \beta \text{ and } D, \text{ let } y \text{ be another free marked point.}
\end{align*}\]

\[\begin{array}{c}
\xymatrix{ 
C \\
X & & \xrightarrow{\pi} & X[m] \\
\mathbb{Z}^n & \xrightarrow{\ell} & \xrightarrow{\pi} & X[m] \\
\end{array}\]

\[\text{Definition 2.1.} \quad \overline{\mathcal{M}}_{g,1}(X; D|\mu) \text{ is the moduli space of morphisms}
\]

\[f : (C; (x_i)_{i=1}^{\ell(\mu)}; y) \to X[m]\]

with the conditions that

\[\begin{align*}
(C; (x_i)_{i=1}^{\ell(\mu)}; y) &\text{ is a connected prestable curve of arithmetic genus } g \text{ with } 1 + \ell(\mu) \text{ marked point.} \\
(\pi[m] \circ f)_\ast [C] &\equiv \beta. \\
f^{-1}(D_m) &\equiv \sum_{i=1}^{\ell(\mu)} \mu_i x_i \text{ as a Cartier divisor. In other words, the partition actually denotes the intersection type of } f \text{ with the last copy of the divisor } D \text{ in the degeneration.} \\
f^{-1}(D_i) &\text{ are nodes of } C \text{ for } 0 \leq i \leq m - 1, \text{ that is, except the last divisor in the degeneration, the intersection points of } f \text{ and } D_i \text{ are all nodes. Moreover, if } x \in f^{-1}(D_i) (i \neq m) \text{ is the intersection of two irreducible components } C_1, C_2 \text{ of } C, \text{ then } f|_{C_1} \text{ and } f|_{C_2} \text{ have the same contact order to } D_i. \\
\text{Two morphisms are identified up to the torus action on the target.} \\
|\text{Aut}(f)| &\text{ is finite, which takes into consideration of the torus action above.}
\end{align*}\]

It’s shown in [LLZ04] that \(\overline{\mathcal{M}}_{g,1}(X; D|\mu)\) is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of dimension

\[\int_\beta c_1(\mathcal{O}_X) + (1 - g)(\dim X - 3) + 1 + (\ell(u) - |\mu|).\]

For example, if \(X = \mathbb{P}^1 \times \mathbb{P}^1\), \(\beta = aH + bV\), then \(\dim \overline{\mathcal{M}}_{g,1}(\mathbb{P}^1 \times \mathbb{P}^1; \mu, v)^\text{vir} = -2a - 2b + g + \ell(\mu) + \ell(v) - |\mu| - |v|\).

\[\text{2.1. Virtual fundamental classes.} \quad \text{We explain the tangent-obstruction spaces at a point in the moduli space } \overline{\mathcal{M}}_{g,1}(\mathbb{P}^1 \times \mathbb{P}^1; \mu, v) \text{ following [LLZ04]. First we need come notations.} \\
\begin{align*}
R &\equiv x + \sum_{i=1}^{\ell(\mu)} x_i \text{ is the divisor on } C \text{ formed by those marked points.} \\
n_k^a &\equiv |\{q| q \in f^{-1}(D_k^a)\}|, \text{ the number of nodes in the fibre over the intersection divisors in the degeneration.} \\
H^0_{\text{et}}(\mathbb{R}^a_k) &\equiv \bigoplus_{q \in f^{-1}(\Delta(D^a)k)} T_q(f^{-1}(\Delta(D^a)_k)) \otimes T^a_q(f^{-1}(\Delta(D^a)_k)) \equiv \mathbb{C}^{a n_k^a}.
\end{align*}\]
• \( L^a_k \) are line bundles on \( D^a_k \) defined by
\[
L^a_k = \begin{cases} 
N_{D^a/X} \otimes N_{D^a_0/(\Delta D^a)_1} & \text{if } k = 0 \\
N_{D^a_k/(\Delta D^a)_k} \otimes N_{D^a_{k+1}/(\Delta D^a)_{k+1}} & \text{if } 1 \leq k \leq m^a - 1.
\end{cases}
\]

• \( H^1(\mathbf{R}^{a\ast}_k) \cong H^0(\mathcal{D}^a_k, L^a_k) \otimes H^0(\mathcal{D}^a_k, L^a_k) \), where \( H^0(\mathcal{D}^a_k, L^a_k) \) is viewed as a subgroup of \( H^0(\mathcal{D}^a_k, L^a_k) \) via the diagonal embedding.

Let \( f : (C, \{z_i\}_{i=1}^l, (y_i)_{i=1}^l) \to X[m_1, m_2] \) be a relative stable morphism, the tangent space \( T^1 \) and the obstruction space \( T^2 \) is given by the exact sequence
\[
\begin{array}{c}
\text{Ext}^0(\Omega_C(R), \mathcal{O}_C) \to H^0(\mathcal{D}^\ast) \to T^1, \\
\text{Ext}^1(\Omega_C(R), \mathcal{O}_C) \to H^1(\mathcal{D}^\ast) \to T^2,
\end{array}
\]
where the terms \( H^0(\mathcal{D}^\ast) \) and \( H^1(\mathcal{D}^\ast) \) can be computed from the following long exact sequence:
\[
\begin{array}{c}
H^0(C, f^*(\Omega_X[m_1, m_2](\sum_{a=1}^2 \log D^a_{m_a}))) \to H^0(\mathcal{D}^\ast) \to \bigoplus_{a=1}^2 \bigoplus_{k=0}^{m_a-1} H^0(\mathbf{R}^{a\ast}_k) \\
H^1(C, f^*(\Omega_X[m_1, m_2](\sum_{a=1}^2 \log D^a_{m_a}))) \to H^1(\mathcal{D}^\ast) \to \bigoplus_{a=1}^2 \bigoplus_{k=0}^{m_a-1} H^1(\mathbf{R}^{a\ast}_k).
\end{array}
\]
All terms out of the range shown above are zeros. Specially, in either exact sequence, the first arrow is an injection, the last arrow is a surjection.

**Example 2.2.** If \( X = \mathbb{P}^1 \times \mathbb{P}^1, D = 0 \times \mathbb{P}^1, D^2 = \infty \times \mathbb{P}^1 \) we have
\[
\Omega_X(2 \sum_{a=1}^2 \log D^a_{m_a}) \cong \Omega_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes p^* \Omega_{\mathbb{P}^1}(\log 0 + \log \infty) = \Omega_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1},
\]
where \( p \) is the projection to the first factor. Similarly, if \( X = \mathcal{A}_1 \times \mathbb{P}^1, D_1 = \mathcal{A}_1 \times 0, D_2 = \mathcal{A}_1 \times \infty \)
\[
\Omega_X(2 \sum_{a=1}^2 \log D^a_{m_a}) \cong \Omega_{\mathcal{A}_1 \times \mathbb{P}^1} \otimes p^* \Omega_{\mathbb{P}^1}(\log 0 + \log \infty) = \Omega_{\mathcal{A}_1 \times \mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1},
\]
where \( p \) is the projection to the second factor.

**Example 2.3.** Let \( \beta = dV + mH \in H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) \) be an effective curve class. The Euler characteristics of the restriction to \( \beta \) of \( \Omega_{\mathcal{A}_1 \times \mathbb{P}^1} \) and \( \Omega_{\mathbb{P}^1 \times \mathbb{P}^1} \) are well defined. Let \( \pi : \mathcal{A}_1 \to \mathbb{P}^1 \) be the natural projection. Since we have the exact sequence
\[
0 \to \pi^* \Omega_{\mathbb{P}^1} \to \Omega_{\mathcal{A}_1} \to \Omega_{\mathcal{A}_1/\mathbb{P}^1} \to 0,
\]
\( \chi(\Omega_{\mathcal{A}_1}) = \chi(\Omega_{\mathcal{A}_1}) - \chi(\pi^* \Omega_{\mathbb{P}^1}) \) can be computed in the following way: \( \mathcal{A}_1 \) can be constructed by gluing two copies of \( \mathbb{C}^2 \), let \((x, u), (y, v)\) be the coordinates respectively. The transition function is given by \((x, u) \leftrightarrow (y, v) = (\frac{1}{x}, x^2 u)\). Thus \( dy \wedge dv = -\frac{1}{x} dx \wedge (2x udx + x^2 du) = -dx \wedge du \). We also know a general differential on \( \mathbb{P}^1 \) is given by \( \frac{1}{x} dx \). Restrict everything to \( \beta \), we have \( c_1(\Omega_{\mathcal{A}_1}) = 0 \) \( c_1(\mathcal{A}_1 \times \mathbb{P}^1) = (2H)(dV + mH) = -2d \). Thus the Euler characteristic of \( \Omega_{\mathcal{A}_1} \otimes \pi^* \Omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1} \) is given by the Grothendieck-Riemann-Roch formula \((1 - g) rk + (0 - (-2d)) = 2d - g + 1 \).
Example 2.4. Let $X = \mathcal{A}_1 \times \mathbb{P}^1$, $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and denote the natural projections as $\mathcal{A}_1 \times \mathbb{P}^1 \stackrel{\pi}{\longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1$. In the case of degeneration, we have

$$\Omega_{X[m_1,m_2]}(\sum_{a=1}^{2} \log D_{(m_{a})}) = \Omega_X \otimes p^* \Omega_{\mathbb{P}^1[m_1,m_2]}(\log 0 + \log \infty)$$

$$\Omega_{Y[m_1,m_2]}(\sum_{a=1}^{2} \log D_{(m_{a})}) = \Omega_Y \otimes p^* \Omega_{\mathbb{P}^1[m_1,m_2]}(\log 0 + \log \infty).$$

Therefore, tensoring the relative differential sequence above with $\pi^* p^* \Omega_{\mathbb{P}^1[m_1,m_2]}(\log 0 + \log \infty)$, we get

$$0 \rightarrow \pi^* \Omega_{Y[m_1,m_2]}(\sum_{a=1}^{2} \log D_{(m_{a})}) \rightarrow \Omega_{X[m_1,m_2]}(\sum_{a=1}^{2} \log D_{(m_{a})}) \rightarrow \Omega_{\mathcal{A}_1 \times \mathbb{P}^1} \otimes \pi^* p^* \Omega_{\mathbb{P}^1[m_1,m_2]}(\log 0 + \log \infty) \rightarrow 0.$$

The invariants we’ll care about later is the equivariant Euler characteristic of the last term. Let $\beta = dV + mH$, it’s straightforward to check that $\langle c_1(\pi^* p^* \Omega_{\mathbb{P}^1[m_1,m_2]}(\log 0 + \log \infty)), \beta \rangle = 0$. It’s a line bundle, so it doesn’t affect the rank, in other words, the Euler characteristic of the last term in the short exact sequence is exactly the same as in the previous example, that is $2d - g + 1$.

2.2. Compare relative invariants via localization. In this subsection, by comparing all terms appear in the localization formula, we make precise the intuitively obvious observation that the only difference between the relative Gromov-Witten theory of $\mathcal{A}_1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ come from the deformation in the fibre direction of the natural projection $\mathcal{A}_1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

$$B_1 = \text{Ext}^0(\Omega_C(R), \mathcal{O}_C), \quad B_2 = \text{H}^0(C, f^* (\Omega_{\mathcal{A}_1} \boxtimes \mathcal{O}_{\mathbb{P}^1}))$$

$$B_3 = \bigoplus_{a=1}^{2} \bigoplus_{\ell=0}^{m_a-1} \text{H}^0_{\text{et}}(\mathbf{R}^1\ell^*), \quad B_4 = \text{Ext}^1(\Omega_C(D), \Omega_C)$$

$$B_5 = \text{H}^1(C, f^* (\Omega^\vee_{\mathcal{A}_1} \boxtimes \mathcal{O}_{\mathbb{P}^1})), \quad B_6 = \bigoplus_{a=1}^{2} \bigoplus_{\ell=0}^{m_a-1} \text{H}^1_{\text{et}}(\mathbf{R}^1\ell^*)$$

Let $T = (t_1,t_2,h)$ acts on $X = \mathcal{A}_1 \times \mathbb{P}^1$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$ by the following weights. Shuai: [needs a graph here]. The relative divisors are naturally $T$-equivariant. Thus $T$ acts on $\Delta(D^a)$ and $X[m_1,m_2], Y[m_1,m_2]$ naturally. $T$ acts on the moduli spaces $\mathcal{M}_{g,1}(X,D;\beta|\mu,\nu)$ and $\mathcal{M}_{g,1}(X,D;\beta|\mu,\nu)$ by moving the image. From now on, we denote two moduli spaces by $\mathcal{M}_X$ and $\mathcal{M}_Y$ respectively. The $T$-fixed point of both moduli spaces are the same—a disjoint union of combinatorial configurations parametrized by certain types of graphs, we denote one such type of fixed points by $\mathcal{F}_T$. Shuai: [In the appendix, we can analyze the fixed point loci more carefully]. Let $p \in \mathcal{F}_T$ and consider the two exact sequences defining the tangent-obstruction spaces $T^1, T^2$ at this point. Then every term in the exact sequences can be view as a $T$-module. Let $T^{i,f}$ and $T^{i,m}$ be the submodules of trivial $T$-weight and nontrivial $T$-weights respectively. Then $T^{i,f} - T^{2,f}$ defines a perfect obstruction theory on $\mathcal{F}_T$ and $T^{1,m} - T^{2,m}$ defines the virtual normal bundle $N^{\text{vir}}_T$ of $\mathcal{F}_T$ in the corresponding moduli space of relative stable morphisms. More precisely, we have

$$\frac{1}{e_T(N^{\text{vir}}_T)} = e_T(T^{2,m}) e_T(B^m_1) e_T(B^m_5)$$

$$e_T(T^{1,m}) e_T(B^m_4) e_T(B^m_6)$$

T. Graber and R. Pandharipande prove a localization theorem in [GP99] which is applicable in our case. That is, the $T$-equivariant virtual fundamental class is a summation
of pushforward of virtual fundamental classes on the fixed loci after localization. In the
case of $Y = \mathbb{P}^1 \times \mathbb{P}^1$, it reads

$$[\overline{\mathcal{M}}_Y]_{P}^{vir} = \sum_{\Gamma \in G_{g,1}} (i_{\mathcal{F}_T})_* \left( \frac{[\mathcal{F}_T]_{T}^{vir}}{e_T(N_{\mathcal{F}_T}^{vir})} \right).$$

In the case of $X = \mathcal{A}_1 \times \mathbb{P}^1$, the ordinary virtual fundamental class vanishes, we have
to use the reduced virtual fundamental class which shows up as the $h$-coefficient of the
construction of the $T$-equivariant virtual fundamental class. The localization theorem in
this case is given by

$$h \cdot [\overline{\mathcal{M}}_X]_{P}^{vir} = \sum_{\Gamma \in G_{g,1}} (i_{\mathcal{F}_T})_* \left( \frac{[\mathcal{F}_T]_{T}^{vir}}{e_T(N_{\mathcal{F}_T}^{vir})} \right).$$

As we mention above, $[\mathcal{F}_T]_{T}^{vir}$ is the virtual fundamental class from the perfect obstruction theory $T^{1,f} - T^{2,f}$, the equivariant Euler class of the virtual normal bundle $e_T(N_{\mathcal{F}_T}^{vir})$
is given by $T_{1,m} - T_{2,m}$. The relative Gromov-Witten invariants on $\mathbb{P}^1 \times \mathbb{P}^1$ needed for the
computation of Severi degree in $[CP17a]$ is the point insertion $\int_{\overline{\mathcal{M}}_Y} \ev_Y^*(pt)$. Consider the
following diagram

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_X & \xrightarrow{ev_X} & X \\
\downarrow \pi \downarrow & & \downarrow \pi \\
\overline{\mathcal{M}}_Y & \xrightarrow{ev_Y} & Y \\
\end{array}
$$

Apply the localization theorem we have

$$\int_{\overline{\mathcal{M}}_Y} \ev_Y^*(pt) = \sum_{\Gamma \in G_{g,1}} \int_{\overline{\mathcal{M}}_Y} \frac{i_Y^* \ev_Y^*(pt)}{e_T(N_{\Gamma/Y}^{vir})}$$

$$= \sum_{\Gamma \in G_{g,1}} \int_{\overline{\mathcal{M}}_Y} \frac{i_Y^* \ev_X^* \pi^*(pt)}{e_T(N_{\Gamma/Y}^{vir})}$$

$$= \sum_{\Gamma \in G_{g,1}} \int_{\overline{\mathcal{M}}_Y} \frac{i_Y^* \ev_X^* \pi^*(pt) e_T(N_{\Gamma/Y}^{vir})}{e_T(N_{\Gamma/Y}^{vir})}.$$

To compare the Euler class of the two virtual normal bundles, we go back to the exact
sequences defining them. $B_1 = \text{Ext}^0(\Omega_C(R), \mathcal{O}_C)$ and $B_4 = \text{Ext}^1(\Omega_C(R), \mathcal{O}_C)$ only depend
on the source curve, they’re the same in both cases. For $B_6 = \bigoplus_{a=1}^{2m_-} \bigoplus_{\ell=0}^{m_+} H^1_{\text{et}}(R_{\ell}^{a \ast})$, where

$$H^1_{\text{et}}(R_{\ell}^{a \ast}) = H^0(D_{(k)^{\ast}}^{a \ast} \ell_p^{m} / H^0(D_{k}^{a \ast} \ell_p^{m}).$$

In our case, $L_k^{a \ast}$ viewed as a line bundle on $D_k^{a \ast}$ is just the trivial bundle, the global functions are constants, which are not affected by
the torus action, thus $B_6^{m} = 0$ and $e_T(B_6^{\ell} = 1$ in both cases. By the formula for the
virtual normal bundle above, the difference boils down to $\frac{e_T(B_6^{\ell})}{e_T(B_6^{\ell})}$, which is just the $T$-
equivariant Euler characteristics of $f^*(\Omega_X[m_1, m_2])$. Now, we can apply
Grothendieck-Riemann-Roch theorem to compute $\frac{e_T(N^\text{vir}_{\Gamma/X})}{e_T(N^\text{vir}_{\Gamma/Y})}$.

$$e_T(N^\text{vir}_{\Gamma/X}) = e_T(C, f^*(\Omega_{\mathcal{X}[m_1,m_2]}(\sum_{a=1}^2 \log D^a_{(m_a)})))^\nu$$

$$e_T(N^\text{vir}_{\Gamma/Y}) = e_T(C, f^*(\Omega_{\mathcal{Y}[m_1,m_2]}(\sum_{a=1}^2 \log D^a_{(m_a)})))^\nu$$

$$= h^{-2d-g+1}$$

$\mathbb{P}^1 \times \mathbb{P}^1$ is compact, thus all the relative invariants are numbers. By the localization theorem, we also have $(\mu|pt|\nu)_Y = h^{-2d-g+1}(\mu|\pi^*(pt))\nu_X$. Therefore, all relative invariants of $\mathbb{P}^1 \times \mathbb{P}^1$ appear as the coefficients of $[h^{2d+g-1}]$-terms of the equivariant relative invariants of $\mathcal{A}_1 \times \mathbb{P}^1$.

$$\int_{\mathcal{H}_X} \prod \text{ev}_{X,i}^*(\gamma_i) = \sum_{\text{fixed loci}} \int_{\mathcal{H}_T} \prod \text{ev}_{T,i}^*(\gamma_i)$$

If $\gamma_i = \pi^*(\gamma'_i)$, we basically get that ‘ratio’ of the two invariants are given by $\frac{N_{\Gamma/X}}{N_{\Gamma/Y}}$. ^▲^\text{Shuai: [later on, we’ll explain this actually means only certain terms in the $\mathcal{A}_1 \times \mathbb{P}^1$ generating function come from the relative invariants of $\mathbb{P}^1 \times \mathbb{P}^1$. Next, have to}

- argue the thing above more carefully
- use Riemann-Roch theorem to compute the degree of $\frac{N_{\Gamma/X}}{N_{\Gamma/Y}} = e(\pi^*(\Theta(-2)))$.

^▲^\text{Shuai: [Tomorrow: differentiate the two tori, check the details of the reduced fundamental class, compare the localization formula regoriously, modify the text accordingly to match later computation]}

3. MODULI SPACE OF RELATIVE STABLE MAPS

**Proposition 3.1** ([MO09]). If $\mu, \nu$ are partitions of $m > 0$ and the cohomology classes labelling $\mu, \nu$ are divisors, then we have

$$u^{\ell(\mu)+\ell(\nu)}Z^\omega_{\mu \neq 0}(\mathcal{A}_n \times \mathbb{P}^1)_{\mu, (2)}; \nu = \frac{d}{du} \Theta^\omega(\mu, \nu)$$

$$u^{\ell(\mu)+\ell(\nu)}Z^\omega_{\nu \neq 0}(\mathcal{A}_n \times \mathbb{P}^1)_{\nu, (1, 0, 1)}; \mu = s_k \frac{d}{ds_k} \Theta^\omega(\mu, \nu)$$

**Otherwise, these invariants vanish.**

The $\Theta^\omega(\mu, \nu)$ above is given by

$$\Theta^\omega(\mu, \nu) = \frac{t_1 + t_2}{|\text{Aut}(\mu)||\text{Aut}(\nu)|} \sum_{1 \leq i < j \leq n} \sum_{d=1}^\infty (d \mu)^{\ell(\mu)+\ell(\nu)-2} \frac{\prod_{k=1}^\infty (\alpha_{i,j} \cdot \gamma_k) S(d \mu_k u) \prod_{k=1}^\infty (\alpha_{i,j} \cdot \eta_k) S(d \nu_k u)}{dS(du)^2} (s_1 \ldots s_{j-1})^d.$$ 

Where $S(u) = \frac{\sin(u/2)}{u/2}$ and $S(du)^2$ in the denominator means $(S(du)^2)$. Now we compute examples of the relative Gromov-Witten invariants of $\mathbb{P}^1 \times \mathbb{P}^1$ by taking the described limits of the formulas in the case of $\mathcal{A}_1 \times \mathbb{P}^1$.

**Example 3.2** ($\mu = \nu = (1)$). When $\mu, \nu$ are given by the trivial partition of 1, we have $Z_{dE \neq 0, H} = (t_1 + t_2)u^{-2} \sum_{d=1}^\infty s^d$. By our discussion above, only the $u^{2g-2}s^d(t_1 + t_2)^{2d+g-1}$ coefficient contributes to the $\mathbb{P}^1 \times \mathbb{P}^1$ invariants. Since the exponent of $t_1 + t_2$ must be 1, it forces $g = 0$ and $d = 1$. That is $Z^\omega(\mathbb{P}^1 \times \mathbb{P}^1)_{d \nu \neq 0, H} = [u^{-2}s^1(t_1 + t_2)]Z^\omega_{dE \neq 0, H} = 1.$
Example 3.3 \((\mu = \nu = (n))\). This relates to the Type-A curve counting in [CP17a]. In these cases, we have \(|\text{Aut}(\mu)| = |\text{Aut}(\nu)| = 1, \ell(\mu) + \ell(\nu) - 2 = 0\). We thus have

\[
\Theta(n), (n) = (t_1 + t_2) \sum_{d=1}^\infty \frac{S(dnu)s(dnu)}{dS^2(du)} s^d = (t_1 + t_2) \sum_{d=1}^\infty \frac{s^d}{d} \left(\frac{S(dnu)}{S(du)}\right)^2.
\]

Proposition 3.1 specializes to

\[
Z_{dE \neq 0,nH}(\mathcal{O}_X \times \mathbb{P}^1)_{\nu,\nu,(n)} = u^{-2}(t_1 + t_2) \sum_{d=1}^\infty s^d \left(\frac{S(dnu)}{S(du)}\right)^2.
\]

Apply the same argument as above and notice that \(\sin(nu) \approx \frac{(nu)^2}{6} + \frac{(nu)^4}{120} + \text{higher order terms}\). We recover the Type-A computation in [CP17a], that is

\[
Z^\circ(\mathbb{P}^1 \times \mathbb{P}^1)_{dV \neq 0,nH} = [u^{-2}s^1(t_1 + t_2)] Z_{dE \neq 0,nH}^\circ = 1.
\]

Example 3.4 (Vanishing results). We can also understand the vanishing results in [CP17a] in terms of the generating function of \(\mathcal{O}_X \times \mathbb{P}^1\) directly. That is if \(\mu\) or \(\nu\) is not a simple partition of the form \((d, 1)\), the relative Gromov-Witten invariants for \(\mathbb{P}^1 \times \mathbb{P}^1\) vanish because for given \(\mu\) and \(\nu\), the corresponding invariant is given by

\[
Z_{dE \neq 0,mH} = [(t_1 + t_2)u^{-2}s]u^{-\ell(\mu) - \ell(\nu)} d^\ell(\mu + \nu - 2)[u^0 s]\frac{\prod_{\ell(k)}(E \cdot \gamma_k)S(d\mu_k u)\prod_{\ell(k')} (E \cdot \eta_k)S(d\nu_k u)}{dS^2(du)} s^d.
\]

The leading term of the last expression is \(u^{\ell(u) + \ell(\nu) - 2}\), so we need \(\ell(u) + \ell(\nu) = 2\). Since we’re considering connected invariants, we have \(|\mu| = |\nu|\), this forces \(\mu\) and \(\nu\) to be the same simple one-row partition \((|\mu|\))

4. Classical multiplication

Proposition 4.1. The divisor \(M\) is the difference of the first Chern classes of the tautological bundle \(\mathcal{O}_X(1)^{[n]}\) and \(\mathcal{O}_X^n\) and the cup product acts diagonally with eigenvalues given by the size of the partition over \(\infty\). Namely,

\[
M = c_1(\mathcal{O}_X(1)^{[n]}) - c_1(\mathcal{O}_X^n)
\]

\[
M = a|\mu|.
\]

Shuai: [Here’s something we have to fix] Shuai: [Here’s something we have to fix] Reword: [what does reword actually mean]

Proof: By Lehn’s formula [Leh99], for a line bundle \(L\) on \(X\) the Chern classes of the tautological bundle \(L^{[n]}\) is given by

\[
\sum_{n \geq 0} c(L^{[n]})z^n = \exp(\sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \alpha_{-k}(c(L))z^m) |\text{vac}|.
\]

We have \(c_1(\mathcal{O}_X) = 1, c_1(\mathcal{O}_X(1)^{[n]}) = 1 + u\). Compute the expansion in Lehn’s formula and note that

\[
\text{codim}(\alpha_{-\mu}(|\gamma|)|\text{vac}) = \sum_i \text{codim}(\gamma_i) + \sum_i (\mu_i - 1) = \sum_i \text{codim}(\gamma_i) + |\mu| - \ell(\mu).
\]
That is to get a codimension 2 cycle, we can either let one part of the partition to be 2 or we let one the the part to have the $u$ insertion.

\[
c_1(\sigma^{[n]}_X) = \frac{1}{(n-2)!} \alpha_{-2}(1)\alpha_{-1}(-1)\ldots\alpha_{-1}(1)|\text{vac}\rangle
\]
\[
c_1(\sigma^1_X) = \frac{1}{(n-1)!} \alpha_{-1}(u)\alpha_{-1}(1)\ldots\alpha_{-1}(1)|\text{vac}\rangle + c_1(\sigma^{[n]}_X).
\]

This tells us $M = c_1(\sigma^1_X) - c_1(\sigma^{[n]}_X)$. Now by Nakajima’s result\cite{Nak99}, we have

\[
c_1(\sigma^{[n]}_X) \cup \lambda, \mu = \prod_{(i,j) \in \lambda} ((i-1)(-a) + (j-1)(a + h)) \prod_{(i,j) \in \mu} ((i-1)a + (j-1)(-a + h))|\lambda, \mu\rangle
\]
\[
c_1(\sigma^1_X) \cup \lambda, \mu = \prod_{(i,j) \in \lambda} ((i-1)(-a) + (j-1)(a + h)) \prod_{(i,j) \in \mu} (a + (i-1)a + (j-1)(-a + h))|\lambda, \mu\rangle
\]

Take the corresponding equivariant classes and compute the difference we get

\[
M \cup |\lambda, \mu\rangle = \sum_{(i,j) \in \mu} a|\lambda, \mu\rangle = a|\mu||\lambda, \mu\rangle.
\]

5. Basic representations of affine Lie algebras and vertex operators

We first recall the definition of Kac-Moody algebras associated with classical semisimple Lie algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra.

6. Quantum multiplication

We match the purely quantum multiplication part. First, we recall the constructions of the affine Lie algebra $\widehat{\mathfrak{g}}$ associated with a complex semisimple Lie algebra $\mathfrak{g}$ and its basic representation. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, $\Delta$ be the root system in $\mathfrak{h}^*$ and $Q$ be the lattice generated by the root system. Let $\langle \cdot, \cdot \rangle$ be an invariant bilinear pairing on $\mathfrak{g}$ such that $\langle a, a \rangle = 2$ for a long root $a$. The affine Lie algebra $\widehat{\mathfrak{g}}$ is defined to be

\[
\widehat{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}c \oplus \mathbb{C}d
\]

\[
[x(k) \oplus a_1c \oplus b_1d, y(l) \oplus a_2c \oplus b_2d] = [x, y](k + l) + k\delta_{k+l}\langle x, y \rangle + b_1l y(l) - b_2kx(k).
\]

The basic representation $\rho$ is an irreducible representation $V$ of $\mathfrak{g}$ and there exists a vector $v \in V$ such that $\rho(g(t))v = 0$ and $\rho(c)v = v$. The basic representation plays a central role in our description of the quantum cohomology ring of $\text{Hilb}(T^* \mathbb{P}^1)$. Here we recall the Frenkel-Kac construction of the basic representation. It’s based on the notion of a Heisenberg system $(\mathfrak{s}, Q)$, where $\mathfrak{s} = h[t^{\pm 1}] \oplus \mathbb{C}c$ can be viewed as a generalization of the oscillator algebra. Let $\mathfrak{s}_- = t^{-1}h[t^{-1}]$ be the negative part of the Heisenberg algebra. We define the Fock space $\mathcal{F}$ to be the symmetric algebra $\text{Sym}^*(\mathfrak{s}_-)$ of $\mathfrak{s}_-$. Then the basis representation as a vector space is given by

\[
L = \text{Sym}^*(\mathfrak{s}_-) \otimes \mathbb{C}[Q],
\]

where $\mathbb{C}[Q]$ is the group algebra of the root lattice $Q$. The actions can be described as follows, for any $k \neq 0$, $h(k)$ acts trivially on $\mathbb{C}[Q]$ and as a multiplication by $h(k)$ if $k \leq 0$ and as a derivation of Sym$^*(\mathfrak{s}_-)$ if $k > 0$, namely $\rho(h(k))(h'(l)) = k\delta_{k+l}\langle h, h' \rangle$. On the other hand $h(0)$ acts trivially on Sym$^*(\mathfrak{s}_-)$ and as a derivation on $\mathbb{C}[Q]$, namely $\rho(h)(e^a) = a(h)e^a$, we denote it by $\delta_h$. The action of the off-diagonal element $E_a$ are given by the so-called vertex operators associated with the root $a \in Q$. To be more precise,

\[
X(a,z) := \exp\left(\sum_{k \geq 1} \frac{z^k}{k} a(-k)\right) \exp(t_a + (\ln z)\partial_a) \exp\left(-\sum_{k \geq 1} \frac{z^{-k}}{k} a(k)\right)
\]
where \( t_a \) means the \( Q \) action on \( C[Q] \) by \( t_a(e^\beta) = e(\beta, a)e^{\beta + a}, \epsilon \) is a 2-cocycle of the group \( Q \) with values in \( \{\pm 1\} \) such that \( e(a, \beta)(-a, \beta) = e(\pi(a, \beta)) \) and \( e(a, -a) = e(a, 0) = 1 \). Note that the middle term in the vertex algebra can also be written as \( z^{\frac{1}{2}(a, a)}e^a \exp(ln z)\partial_a + \exp(ln z)\partial_a = \sum_{k \in \mathbb{Z}}z^kP_k \) where \( P_k(e^\beta) = \delta_k,_{[\beta, a]}e^\beta \) [FK81, page 47]. Finally, choose dual bases \( u_i, u^i \) of \( h \), we define the action of \( d \) to be \(-D_0\), where \( D_0 = \sum_{i=1}^{\dim q} (\frac{1}{2}u_i(0)u^i(0) + \sum_{k \geq 1} u_i(-n)u_i(n)) \).

Now we specialize the Frenkel-Kac construction to the \( \hat{sl}_2 \) case to describe the basic representations of \( \hat{sl}_2 \) and \( \hat{gl}_2 \), after base change from \( C \) to \( C\langle t_1, t_2 \rangle \), we identify the equivariant cohomology of \( T^*\mathbb{P}^1 \) with certain weight spaces of the basic representation of \( \hat{gl}_2 \). The loop algebra \( sl_2[t, t^{-1}] \) is generated by

\[
\alpha(k) = t^k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},
\quad \beta(k) = t^k \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Thus the dual bases are given by \( \{\alpha(k), \beta(k), \epsilon \} \) and \( \{\epsilon, \alpha(k)^*, \beta(k)^*\} \) and \( \{\alpha(k)^* = \frac{1}{2}\alpha(k), \beta(k)^* = \epsilon(k), \epsilon(k)^* = f(-k), \epsilon^* = \epsilon \} \). The Frenkel-Kac construction above gives us the basic representation \( L_\Lambda \) of highest weight \( \Lambda \). The negative part of the Heisenberg algebra is \( s_+ = \oplus_{k \in \mathbb{N}} \alpha(-k) \). The Fock space \( \mathcal{F} \) is the space of all partitions or the ring of infinitely many variables \( C[x_1, x_2, \ldots] \) over \( \mathbb{Z} \). Then the basic representation is \( L_\Lambda = \mathcal{F} \otimes C[Q] = \text{Sym}^j(s_-) \otimes C[Q] \). In our case, \( Q = \mathbb{Z}a \). Let \( q = e^a \in C[Q] \), we have

\[
L_\Lambda = C[x_1, x_2, \ldots; q^{\pm 1}]
\]

The operators are given as follows

\[
\begin{array}{cccc}
\alpha(-k) & \alpha & \alpha(k) & c \\
nx_n & 2q \frac{\partial}{\partial q} & 2\frac{\partial}{\partial x_n} & 1 \\
\end{array}
\]

For \( \hat{gl}_2 \), we have \( \hat{gl}_2 = \hat{sl}_2 \oplus \oplus_{k \in \mathbb{Z}} h(k) \), where \( h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). That is, it’s the direct sum of \( \hat{sl}_2 \) and the Heisenberg algebra generated by \( (h(k))_{k \in \mathbb{Z}} \). The Cartan subalgebra and its dual is just the ones of \( \hat{sl}_2 \) with \( h \) or \( h^* = \frac{1}{2}h \) added. Then the basic representation of the same highest weight is

\[
V_\Lambda = \mathcal{F} \otimes L_\Lambda = C[y_1, y_2, \ldots; x_1, x_2, \ldots; q^{\pm 1}],
\]

where we have identified the Fock space of \( (h(k))_{k \in \mathbb{Z}} \) with \( C[y_1, y_2, \ldots] \). Viewed as an element in \( \hat{gl}_2 \), \( d \) acts as

\[
d \mapsto - \sum_{n \geq 1} n y_n \frac{\partial}{\partial y_n} - \sum_{n \geq 1} n x_n \frac{\partial}{\partial x_n} - (q \frac{\partial}{\partial q})^2.
\]
Proposition 6.1. The following map gives a $\mathcal{H}$-module isomorphism:

$$H_T(\text{Hilb}^\star(\mathcal{A}_1)) \to \bigoplus_{m=0}^{\infty} V_\Lambda[\Lambda - m \delta]$$

$$p_{-\mu}(1)|\text{vac}\rangle \to y_\mu$$

$$p_{-\mu}(E)|\text{vac}\rangle \to x_\mu.$$  

Proof. $\alpha$ acts as $2q^{\frac{\alpha}{\delta}}$ on $L_\Lambda$ and acts trivially on $\mathcal{F}$. $h$ also acts trivially, that is acts by 0. Thus $V[\Lambda - m \delta] = \{v \in V_\Lambda | h(v) = (\Lambda(h) - m \delta(h))(v)\}$. $\hat{h}$ acts diagonally on monomials, for a given $y_\mu x_\nu q^k$, their eigenvalues are given by

$$\begin{array}{cccc}
\alpha & h & c & d \\
2k & 0 & 1 & k^2 + |\mu| + |\nu|
\end{array}$$

That means weight spaces in the basic representations are those spanned by monomials, more precisely,$C_{y_\mu x_\nu q^k} \cong V_\Lambda[\Lambda - m \delta + 2k \alpha^\ast] = V_\Lambda[\Lambda - m \delta + \frac{1}{2}k \alpha]$. We conclude that

$$H_T(\text{Hilb}^\star(\mathcal{A}_1)) \cong q^0 C[y_1, \ldots; x_1, \ldots] = \bigoplus_{m=0}^{\infty} V_\Lambda[\Lambda - m \delta].$$

□

Now we consider the relevant vertex operators and match the purely quantum part of $M$ in [MO09] with the type-B curve counting operator in [CP17a]. For any $\gamma \in Q$ and a complex variable $z$. The 2-cocycle in the $\hat{sl}_2$ case is given by $e(m \alpha, n \alpha) = (-1)^{mn}$. Denote $e(z) = \sum_{k \in \mathbb{Z}} e(k) z^{-k-1}$, $f(z) = \sum_{k \in \mathbb{Z}} f(k) z^{-k-1}$. Then $e(z)$ and $f(z)$ acts on $L_\Lambda$ by vertex operators:

$$e(z) \mapsto \Gamma_+(z) = X(\alpha, z) = \exp \left( \sum_{k \geq 1} \frac{\alpha_k}{k} a(-k) \right) \exp \left( - \sum_{k \geq 1} \frac{z^{-k}}{k} a(k) q^{2q^{\frac{\alpha}{\delta}} c} \right)$$

$$f(z) \mapsto \Gamma_-(z) = X(-\alpha, z) = \exp \left( - \sum_{k \geq 1} \frac{\alpha_k}{k} a(-k) \right) \exp \left( \sum_{k \geq 1} \frac{z^{-k}}{k} a(k) q^{-1} q^{2q^{\frac{\alpha}{\delta}} c} \right),$$

where $c_{\pm \alpha}(f \otimes e^{n \alpha}) = (-1)^n$ is a special case of $e(\alpha, n \alpha)$, and $z^{2q^{\frac{\alpha}{\delta}}} = z^{2n} q^n$.

Proposition 6.2. The $s$-coefficient of $\Omega_+$ in [MO09] is the operator corresponding to the type-B curve counting in [CP17a]. More precisely,

$$[s] \Omega_+ = \sum_{k \neq 0} f(k) e(-k) := \sum_{|\mu| = |\nu| > 0} \alpha_{-\mu} \alpha_\nu.$$

They’re all vertex operators of $\hat{sl}_2$.

Proof. Note that $\sum_{k \neq 0} f(k) e(-k) :$ is just the $z^0$-coefficient of $: \Gamma_+(z) \Gamma_-(z) :$. We use the formula in [KR87] page 309,

$$\Gamma_+(z) \Gamma_-(w) = (1 - \frac{w}{z})^{(\alpha | \beta)} z^{(\alpha | \beta)} c(\alpha, \beta) \exp \left( \sum_{k \geq 1} \frac{\alpha_k (z^{-k} - w^{-k})}{k} \right) \exp \left( - \sum_{k \geq 1} \frac{\alpha_k (z^{-k} - w^{-k})}{k} \right) \times e^{a + \beta} z^{a_0} w^{a_0} c \alpha c \beta$$

In our case, we have $\alpha = \alpha, \beta = -\alpha, \langle \alpha, -\alpha \rangle = -2, e(\alpha, -\alpha) = -1$. By the previous identification, we only act on the $q^0$ piece of the basic representation, thus

$$e^{\alpha + \beta} z^{a(0)} w^{-a(0)} c \alpha c = 1$$
The operator product is simplified to be
\[ \Gamma_f(z) \Gamma_g(w) = \frac{-1}{(1 - \frac{w}{z})^2 z^2} \exp \left( \sum_{k \geq 1} \frac{\alpha_k(z^k - w^k)}{k} \right) \exp \left( - \sum_{k \geq 1} \frac{\alpha_k(z^{-k} - w^{-k})}{k} \right). \]

The normal ordering means taking the \( z^0 \) of the the regular part of the expansion above, thus
\[
\sum_{k \neq 0} : f(k) e(-k) : = \text{Res}_{z=0} z^{-1} \left( \frac{1}{(z-w)^2} + \Gamma_f(z) \Gamma_g(w) \right) = [z^0] \sum_{k \geq 1} \frac{\alpha(-k)}{k} (1 - \frac{w}{z})^k z^k + \left( \sum_{k \geq 1} \frac{\alpha(-k)}{k} (1 - \frac{w}{z})^k z^{-k} \right)^2 + \ldots \]
\[
\times \left( \sum_{k \geq 1} \frac{\alpha(k)}{k} (1 + \frac{w}{z})^{-k} z^{-k} + \left( \sum_{k \geq 1} \frac{\alpha(k)}{k} (1 - \frac{w}{z})^k z^{-k} \right)^2 + \ldots \right).
\]

The position of a term means the length of the partition, the product gives all possible combinations of two partitions, the \( z^0 \)-coefficient condition means the two partitions have to have the same size. Also note that
\[
\lim_{z \to 1} \frac{(1 - \frac{w}{z})^k (1 + \frac{w}{z})^{-l}}{(1 - \frac{w}{z})^2} = -kl,
\]
which kills the denominators. In conclusion, we get
\[
\sum_{|\mu| = |\nu| > 0} \alpha_{-\mu} \alpha_{\nu} = \sum_{k \neq 0} : f(k) e(-k) : .
\]

7. DIAGONALIZATION

7.1 Monad on Hirzebruch surfaces. Let \( \pi : \Sigma_n = \mathbb{P}(\mathcal{E}^\vee) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n)) \to \mathbb{P}^1 \) be the \( n \)-th Hirzebruch surfaces. Let \( H = c_1(\mathcal{O}_{\Sigma_n}(1)) \), \( E \) be the class of \( \mathbb{P}(\mathcal{O}(-n)) \), its self-intersection equals to \(-n\), \( F \) be a fiber class of \( \pi \). We have \( E = H - nF \). The Chow ring of \( \Sigma_n \) is given by \( A(\Sigma_n) = A(\mathbb{P}^1)[H]/(H^2 - nFH) = \mathbb{Z}[F,H]/(F^2,H^2 - nFH) \), naturally one has \( \text{Pic}(\Sigma_n) = \mathbb{Z}H \oplus \mathbb{Z}F \). From now on, we denote \( \mathcal{E}(p,q) = \mathcal{E} \otimes \mathcal{O}_{\Sigma_n}(pH + qF) \) for any sheaf of \( \mathcal{O}_{\Sigma_n} \)-modules. Recall that the relative Euler sequence computes the relative canonical sheaf
\[
0 \to \mathcal{O}_{\Sigma_n}(-1,0) \to \pi^*(\mathcal{E}^\vee) \to T_{\Sigma_n/\mathbb{P}^1}(-1,0) \to 0.
\]
\[
\Omega_{\Sigma_n/\mathbb{P}^1} = \mathcal{O}_{\Sigma_n}(-2,0) \otimes \pi^*(\text{det} \mathcal{E}) = \mathcal{O}_{\Sigma_n}(-2,n)
\]
\[
T_{\Sigma_n/\mathbb{P}^1} \cong \mathcal{O}_{\Sigma_n}(2,-n)
\]
\[
\omega_{\Sigma_n} = \mathcal{O}_{\Sigma_n}(-2,n-2).
\]

For later computations, we need the following lemmata.

Lemma 7.1.
\[
H^0(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \text{ if and only if } \begin{cases} p \geq 0 \\ np + q \geq 0 \end{cases}
\]
\[
H^1(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \text{ if and only if } \begin{cases} p \geq 0 \\ q \leq -2 \\ or \quad \begin{cases} p \leq -2 \\ q \geq n \end{cases} \end{cases}
\]
\[
H^2(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \text{ if and only if } \begin{cases} p \geq -2 \\ np + q \leq -(n + 2) \end{cases}
\]

Proof. See [BBR15, Lemma 3.1]
The classical Beilinson spectral sequence is a way to describe torsion-free sheaves on $\mathbb{P}^2$ as the cohomology of certain three-term complexes—the so-called monad. With the isomorphism between the Hilbert scheme of points on $\mathbb{C}^2$ and the moduli scheme of rank 1 torsion-free sheaves on $\mathbb{P}^2$ which are trivial over infinity, the Hilbert scheme can be realized as the quiver variety with one vertex and one loop, see [Nak99] Theorem 2.1.

The essential part of the construction is a resolution of the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$. For Hirzebruch surfaces, the diagonal can also be resolved (for references to the details of the construction, we refer to [Buc87] or [AB09]). We first briefly recall the Beilinson-type spectral sequence on $\Sigma_n$, following the approach in [Nak99] we give a monad description of Hilb($T^* \mathbb{P}^1$) as a Nakajima quiver variety of type $A_1^{(1)}$, and we show that the two tautological bundles corresponding to the two vertices are exactly $\mathcal{O}_X(1)^{[n]}$ and $\mathcal{O}_{[n]}$. Shuai: [Should we recall the elementary description of the construction as in [Buc87]?]

We denote $p_i \colon \Sigma_n \times \Sigma_n$ be the projections to the two factors, $p : X \times X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the product of the ruling $\pi$. Let $\Delta_{\mathbb{P}^1} \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\Delta$ be the diagonal divisor on $\Sigma_n \times \Sigma_n$, $L = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\Delta_{\mathbb{P}^1})$. Consider the line bundle

$$\mathcal{F} = p_1^*(T_{\Sigma_n} \mathcal{P}(1,-0)) \oplus p_2^*(\mathcal{O}_{\Sigma_n}(1,0)) = \mathcal{O}_{\Sigma_n}(1,-n) \otimes \mathcal{O}_{\Sigma_n}(1,0).$$

A rank 2 locally free sheaf $\mathcal{G}$ is defined by an extension

$$0 \to \mathcal{F} \to \mathcal{G} \to p^*(L) \to 0.$$

Note that $p^*(L) = \mathcal{O}_{\Sigma_n}(0,1) \otimes \mathcal{O}_{\Sigma_n}(0,1)$. Buchdahl proved that [Buc87] the diagonal $\Delta$ can be realized as the zero locus of a global section $s$ of $\mathcal{G}$, then the Koszul complex of $s$ gives us the resolution of the diagonal $C^* \to \mathcal{O}_\Delta$:

$$0 \to \wedge^2 p^*\mathcal{G}^\vee \to \mathcal{G}^\vee \to \mathcal{O}_{\Sigma_n} \to 0.$$
If \( p = 0 \), the first sheaf in the sequence above vanishes, \( \wedge^{-p} G \otimes p_1^* \mathcal{E} = p_1^* \mathcal{E} \), if \( p = -2 \), the last sheaf in the sequence above vanishes, \( \wedge^{-2} G \otimes p_1^* \mathcal{E} = \mathcal{F} \otimes p^*(L^v) \otimes p_1^* \mathcal{E} = \mathcal{E}(-1, n - 1) \otimes \mathcal{O}_{\Sigma_n}(-1, -1) \). Take the associated long exact sequence one has

\[
E_1^{0, q} = H^q(\mathcal{E}) \otimes \mathcal{O}_{\Sigma_n}(0, 0) \tag{7.1.1}
\]

\[
\cdots \to H^q(\mathcal{E}(0, -1)) \otimes \mathcal{O}_{\Sigma_n}(0, -1) \to E_1^{-1, q} \to H^q(\mathcal{E}(-1, n)) \otimes \mathcal{O}_{\Sigma_n}(-1, 0) \to \cdots \tag{7.1.2}
\]

\[
E_1^{-2, q} = H^q(\mathcal{E}(-1, n - 1)) \otimes \mathcal{O}_{\Sigma_n}(-1, -1). \tag{7.1.3}
\]

Lemma 7.4. Let \( \mathcal{E} \) be a torsion-free sheaf on \( \Sigma_n \), trivial at infinity. We have

\[
H^0(\mathcal{E}(p, q)) = 0 \text{ for } np + q \leq -1
\]

\[
H^2(\mathcal{E}(p, q)) = 0 \text{ for } np + q \geq -(n + 1).
\]

Proposition 7.5. For any torsion free sheaf \( \mathcal{E} \) on \( \Sigma_n \) can be realized as a monad:

\[
0 \to H^1(\mathcal{E}(-2, n - 1)) \otimes \mathcal{O}_{\Sigma_n}(0, -1) \to E_1^{-1, 1} \otimes \mathcal{O}_{\Sigma_n}(1, 0) \to H^1(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n}(1, 0) \to 0.
\]

where the \( E_1^{-1, 1} \otimes \mathcal{O}_{\Sigma_n}(1, 0) \)-term can be computed from

\[
0 \to H^1(\mathcal{E}(-1, -1)) \otimes \mathcal{O}_{\Sigma}(1, -1) \to E_1^{-1, 1} \otimes \mathcal{O}_{\Sigma_n}(1, 0) \to H^1(\mathcal{E}(-2, n)) \otimes \mathcal{O}_{\Sigma_n}(0, 0) \to 0.
\]

Moreover, the second sequence splits.

Proof. The trick here is that \( \mathcal{E} \) can’t be realized as a monad just from the spectral sequence. However, for \( \mathcal{E}(-1, 0) \), the Beilinson spectral sequence becomes

<table>
<thead>
<tr>
<th>( H^2(\mathcal{E}(-2, n - 1)) \otimes \mathcal{O}_{\Sigma_n}(-1, -1) )</th>
<th>( E_1^{-1, 2} )</th>
<th>( H^2(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H^1(\mathcal{E}(-2, n - 1)) \otimes \mathcal{O}_{\Sigma_n}(-1, -1) )</td>
<td>( E_1^{-1, 1} )</td>
<td>( H^1(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n} )</td>
</tr>
<tr>
<td>( H^0(\mathcal{E}(-2, n - 1)) \otimes \mathcal{O}_{\Sigma_n}(-1, -1) )</td>
<td>( E_1^{-1, 0} )</td>
<td>( H^0(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n} )</td>
</tr>
</tbody>
</table>

By Lemma 7.4, all the four corner terms vanish. Then Theorem 7.2 forces \( E_1^{-1, 2} \) and \( E_1^{-1, 0} \) to be zeros. Thus only the \( q = 1 \) terms in the spectral sequence survive and the spectral sequence degenerates at the \( E_1 \)-page. This proves that \( \mathcal{E}(-1, 0) \) is the cohomology of

\[
0 \to H^1(\mathcal{E}(-2, n - 1)) \otimes \mathcal{O}_{\Sigma_n}(-1, -1) \to E_1^{-1, 1} \to H^1(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n} \to 0.
\]

Tensoring it with \( \mathcal{O}_{\Sigma_n}(1, 0) \) gives the first statement. To compute \( E_1^{-1, 1} \otimes \mathcal{O}_{\Sigma_n}(1, 0) \), Lemma 7.4 shows that \( H^0(\mathcal{E}(-1, -1)) = 0 \) and \( H^2(\mathcal{E}(-1, -1)) = 0 \) for any \( \Sigma_n \). Sequence 7.1.2 degenerates to the second statement in the proposition. It splits because \( \text{Ext}^1(\mathcal{O}_{\Sigma_n}(1, -1), \mathcal{O}_{\Sigma_n}) = H^1(\mathcal{O}_{\Sigma_n}(-1, 1)) = 0 \), the last equality comes from Lemma 7.1.

Now the relative Gromov-Witten theory of \( P^1 \times P^1 \) and \( A_1 \times P^1 \) can be viewed as shades of the quantum cohomology theory of the Nakajima quiver variety \( \text{Hilb}(A_1) \), which is well-understood as in \[MO12\]. Moreover, the K-theoretic version of the story is developed in \[AO17\]. Eigenvalues of quantum multiplications of any tautological class are encoded in certain Bethe equations. To be more precise, if \( X \) is a Nakajima quiver variety, the equivariant K-theoretic class of the tangent bundle is given by

\[
TX = T(T^* \text{Rep}(v, w)) - \sum_i (1 + h^{-1}) \text{End}(V_i).
\]

As we shall see in the \( \text{Hilb}(T^* \times P^1) \) example, it’s a Laurent polynomial in terms of the Chern roots \( x_{i,k} \) of the tautological bundles and the equivariant parameters. The elliptic genus \( \hat{a} \) is defined to be \( \hat{a}(\sum n_i \chi_i) = \prod (\chi_i^{\frac{1}{2}} - \chi_i^{-\frac{1}{2}})^{n_i} \), where \( \chi_i \) are weights of \( T \times \prod GL(V_i) \). In general, we have

**Proposition 7.6 ([AO17]).** The eigenvalues of \( M \) are \( \sum_{i,k} (-1)^k x_{i,k} \), where \( x_{i,k} \) are the roots of the Bethe equations

\[
\hat{a}(x_{i,k}) \frac{\partial}{\partial x_{i,k}} TX = z_i.
\]
The cohomological Bethe equations can be extracted from the K-theoretic version by taking the linear term. We work this computation out in the Hilb\((T^*\mathbb{P}^1)\) case. To simplify the notation a little bit, we denote the Chern roots of \(V^*_1\) by \(x_i\) and those of \(V^*_2\) by \(y_j\). Then in \(K_{\mathbb{C}^*} \otimes c_h^*(\text{Hilb}(T^*\mathbb{P}^1))\), we have

\[
TX = \sum_i ax_i + h^{-1} \sum_i \frac{1}{ax_i} + \sum_{i,j} \frac{y_j}{x_i} + h^{-1} \sum_i x_i + \sum_j y_j + h^{-1} \sum_i \frac{y_j}{x_i} - (1 + h^{-1}) \left( \sum_i \frac{x_i}{x_j} + \frac{y_j}{y_i} \right).
\]

The partial derivative w.r.t. \(x_i\) is

\[
x_i \frac{\partial}{\partial x_i} TX = ax_i - h^{-1} \frac{1}{ax_i} - \sum_j \frac{y_j}{x_i} + h^{-1} \sum_j \frac{x_i}{y_j} + \sum_j \frac{y_j}{y_j} - h^{-1} \sum_j \frac{y_j}{x_i} - (1 + h^{-1}) \left( \frac{x_i}{x_j} - \frac{x_j}{x_i} \right).
\]

Take the Euler class of the expression above, we get the Bethe equation

\[
q = \frac{a + x_i}{a + x_i + h} \prod_j (x_i - y_j)(x_i - y_j - h) \prod_j x_j - x_i x_j - x_i - h
\]

Similar for \(y_i\) and \(s\) (the curve degree corresponding to \((1, \omega)\)). In summary, we get

**Corollary 7.7.** The eigenvalues of \(M_S(1, Q)\) in \([CP17a]\) are given by \(\sum_i (x_i - y_i)\), where \(x_i, y_i\) are the roots of the Bethe equations

\[
q = \frac{a + x_i}{a + x_i + h} \prod_j \frac{x_i - y_j - h}{y_j - x_i - h} \prod_j \frac{x_j - x_i - h}{x_i - x_j - h}
\]

\[
s = \frac{a + y_i}{a + y_i + h} \prod_j \frac{y_i - x_j - h}{x_j - y_i - h} \prod_j \frac{y_j - y_i - h}{y_i - y_j - h}
\]

### 8. Conclusions

**References**

- [Coo17] Yaim Cooper, *A Fock Space approach to Severi Degrees of Hirzebruch Surfaces*. 


[MO12] Davesh Maulik and Andrei Okounkov, *Quantum Groups and Quantum Cohomology*.

