KAZHDAN-LUSZTIG POLYNOMIALS OF AFFINE WEYL GROUPS

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ABSTRACT. We follow [Mor11] to give a geometric interpretation of a non-recursive formula of the Kazhdan-Lusztig polynomials proved by Brenti in the $G/P$ and affine Weyl group case.

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1. INTRODUCTION

2. DECOMPOSITION OF SCHUBERT CELLS

Theorem 2.1 (Shuai: [reference???]). Let $\sigma \in W^J$. The sheaf of cohomology $\mathcal{H}^i(I\mathcal{C}_{\sigma})$ is zero if $i$ is odd. If $i$ is even and $B' \in X_{\sigma}$ is stable under $F^r$, the power of the Frobenius action $F$. Then the eigenvalues of $(F^r)^*$ on the fibre $\mathcal{H}^i(I\mathcal{C}_{\sigma})_{B'}$ are all equal to $q^{ir}$. Moreover, for any $\tau \leq \sigma$, we have

$$P_{v,\omega}(t) = \sum_{i \geq 0} \dim \mathcal{H}^{2i}(I\mathcal{C}_{\sigma})_{\tau \cdot P} t^i = P_{\tau,\sigma}(t).$$

Let $W_J$ be the subgroup generated by $J \subset S$. $W^J$, the set of minimal coset representatives of $W/W^J$. Let $\sigma \in W^J$. We fix a reduced expression $\sigma = s_1 \ldots s_r$. The set of $J$-expressions $\Gamma = \{(\theta_1, \ldots, \theta_{r+1}) \in (W^J)^{r+1} | \theta_{r+1} = 1, \theta_p \in \{\theta_{p+1}, s_p \theta_{p+1}\} \forall 1 \leq p \leq r\}$. Let $D^J$ denote the set of $J$-expressions such that \( \ell(s_p \theta_{p+1} \geq \ell(\theta_{p+1})) \) if $\theta_p = \theta_{p+1}$. The natural projection $\pi : D^J \rightarrow W^J$ is given by $\pi((\theta_1, \ldots, \theta_{r+1})) = \theta_1$. We also define

$$n_1(\theta) = \# \{ p | \theta_p = \theta_{p+1} \text{ and } s_p \theta_{p+1} \in W^J \}$$
$$n_2(\theta) = \# \{ p | \theta_p = \theta_{p+1} \text{ and } s_p \theta_{p+1} \notin W^J \}$$
$$m(\theta) = \# \{ p | \theta_p = s_p \theta_{p+1} \text{ and } \ell(\theta_p) \leq \ell(\theta_{p+1}) \}$$

Proposition 2.2 (Deodhar). The Schubert cell $X_\sigma = B\sigma \cdot P$ decomposes canonically into a disjoint union of locally closed subvarieties:

$$B\sigma \cdot P = \bigcup_{\theta \in D^J} D_\theta.$$

Moreover $D_\theta \cong A_{m(\theta)+n_2(\theta)} \times C_{n_1(\theta)}^{n_1(\theta)}$. For $\tau \in W^J$, we have

$$B\sigma \cdot P \cap B^{-\tau} \cdot P = \bigcup_{\theta \in D^J, m(\theta) = r} D_\theta.$$

Lemma 2.3. For $\tau \leq \sigma \in W^J$,

$$\#(X_\sigma \cap X^\tau(q^n)) = R^J_{\tau,\sigma}(q^n).$$
Lemma 2.4. Let $X$ be smooth over $\mathbb{F}_q$. For any object $K \in \mathcal{T}(X)$ and $a \in \mathbb{Z}$, we have
\[ \phi([w_{\leq a}K]) = \tau_{\leq a - \dim(X)}(\phi(K)). \]
For all $\tau, \sigma \in W^J$ and all $K \in \mathcal{T}(X_\sigma)$, the complex $i_{\tau,\sigma}^!j_!K$ is in $\mathcal{T}(X_\nu)$ and there's a $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-equivariant isomorphism
\[ i_{\tau,\sigma}^!j_!K \cong R\Gamma_c((X_\sigma \cap X^\tau)_{\overline{\mathbb{F}_q}}, K|_{X_\sigma \cap X^\tau}). \]
Moreover $\phi(i_{\tau,\sigma}^!j_!K) = \phi(K)R^\mu_{i_{\tau,\sigma}}(t)$. $i_{\tau,\sigma}^!IC_{X_\sigma}$ is also in $\mathcal{T}(X_\tau)$.

Shuai: [TO DO]
- Relations between Deodhar's notation and Morel's notation of the index sets
- $J$-expressions and expressions?
- Is it possible to modify Morel's method a little bit to check the $G/P$ case?
- Is it true that the relation between $P$ and $R$ polynomials can be generalized to other cases?

The differences among the forms of the index sets are due to different ways of denoting a subexpression of $\sigma = s_1 \ldots s_r$. For example
- $\Gamma = \{1, s_1\} \times \cdots \times \{1, s_r\}$
- $s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_r$. Sequences $1 \leq i_1 < \cdots < i_k \leq r$ of missing subscripts.
- $\{1 = \sigma_0, \sigma_1, \ldots, \sigma_r\}1_{j}\sigma_j \in \{1, s_j\}|1 \leq j \leq r\}$. $\sigma_j$'s are given by subsequent products of the subexpression, or we can view $\sigma_k$ as the truncation at the $k$-th position.
- $s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_r$. Sequences $[r] \setminus \{i_1, \ldots, i_k\}$ of occupying subscripts.
- Invariants can have different forms of definitions in different formalism, for example, if we view a subexpression as an element $(\gamma_1, \ldots, \gamma_r) \in \Gamma$, then
  - the occupying indices are exactly $I(\gamma) = \{i \in [r]|\gamma_i = s_i\}$.
  - the element represented by the subexpression is given by $\gamma_1 \ldots \gamma_r$.
On the other hand, if we view a subexpression as a sequence $(1 = \sigma_0, \sigma_1, \ldots, \sigma_r)$ of truncation (subsequent products), then
  - the missing indices are exactly $\{i \in [r]|\sigma_j^{-1}\sigma_j = 1\}$.
  - the element represented by this subexpression is exactly the last element in the sequence $\sigma_r$.
- Distinguished subexpression. An element $(1 = \sigma_0, \sigma_1, \ldots, \sigma_r)$ is called distinguished if it satisfies that $\sigma_j \leq \sigma_{j-1}s_j$. In our situation, it means $\sigma_j = \sigma_{j-1}s_j$ or $\sigma_j = \sigma_{j-1}$ but multiply it by $s_j$ makes it longer in the Bruhat order. On the other hand, In S.Morel's paper, some related concepts are given by $I(\gamma) = \{i \in [r]|\gamma_i = s_i\}$, $J(\gamma) = \{i \in [r]|\gamma_1 \ldots \gamma_i(-\alpha_i) \in \Phi^+\}$, a distinguished subexpression needs to satisfies the condition that $J(\gamma) \subset I(\gamma)$. Are those two definitions of distinguished subexpression actually the same? Yes, they are the same. $J(\gamma) \subset I(\gamma)$ says that if the $i$-th position is not occupied in the subexpression, that is $\gamma_i = 1$, it's impossible to send $-\alpha_i$ to be a positive root, in other words, $\gamma_1 \ldots \gamma_i(-\alpha_i) \in \Phi^-$. This is equivalent to say that $\gamma_1 \ldots \gamma_{i-1} \leq \gamma_1 \ldots \gamma_{i-1}s_i$, because the length of an element is exactly the number of positive roots transformed to be negative under the action given by this element. We know the simple reflection only change the parity of $\{-\alpha_i, \alpha_i\}$.
- In the case of a distinguished element, $I(\gamma) \setminus J(\gamma) = M(\gamma) := \{j \in [r]|\sigma_{j-1} > \sigma_j\}$. First we know $M(\gamma) \subset I(\gamma)$, if the $j$-th position isn't occupied, we must have $\sigma_{j-1} = \sigma_j$. $I(\gamma) \setminus M(\gamma) = \{i \in [r]|\gamma_i = s_i, \sigma_{i-1} = \sigma_i\} = \{i \in [r]|\gamma_1 \ldots \gamma_i(-\alpha_i) \in \Phi^+\}$. The answer is also yes, $j \in I(\gamma) \setminus J(\gamma)$ simply means $\gamma_1 \ldots \gamma_{i-1}s_i(-\alpha_i) \in \Phi^-$, that is $\gamma_1 \ldots \gamma_{i-1}(\alpha_i) \in \Phi^-$. Remember the criterion when $\ell(ws) < \ell(w)$? Note that $s_i$ is just a permutation on $\Phi^+ \setminus \{\alpha_i\}$, $\ell(ws) < \ell(w)$ if and only if $w$ already does the job.
of \( s_i \), in other words, \( w(\alpha_i) \in \Phi^- \). So now we have matched S.Morel’s notation and Deodhar’s notation.

♣♣♣ Shuai: [To DO]

- J-distinguished expressions
- The geometry of \( G/B \rightarrow G/P \).

Still, let \( \sigma = s_1...s_r(\sigma \in W^J) \). A J-subexpression is a sequence \( \theta = (\theta_1,...,\theta_{p+1} = 1) \) of elements in \( W^J \). Such that \( \theta_{k+1} = 1 \) and \( \theta_p = (\theta_{p+1}, s_p \theta_{p+1}) \). Note that, if we state in different ways as above, it essentially means subexpressions \( s_1...s_i...s_k...s_r \) but we require any subsequent expressions all lie in \( J \), this is a very strong restriction. Geometrically, this means we consider all the intermediate cells between \( W_\sigma \) and \( \sigma_\tau \) in \( G/P \), thus all intermediate states have to be elements of \( W_J \). A J-subexpression is called J-distinguished if it satisfies \( \ell(s_p \theta_{p+1}) \geq \ell(\theta_{p+1}) \) if \( \theta_p = \theta_{p+1} \). \( \pi(\theta) = \theta_1 \) just means the element in \( W_J \) represented by the product of the whole sequence. In [Deo87], the author defined three invariants of a J-subexpression:

\[
\begin{align*}
n_1(\theta) &= \{ p | \theta_p = \theta_{p+1} \text{ and } s_p \theta_{p+1} \in W^J \} \\
n_2(\theta) &= \{ p | \theta_p = \theta_{p+1} \text{ and } s_p \theta_{p+1} \notin W^J \} \\
m(\theta) &= \{ p | \theta_p = s_p \theta_{p+1} \text{ and } \ell(\theta_p) \leq \ell(\theta_{p+1}) \} 
\end{align*}
\]

Now we explain the relations between those invariants:

\[
m(\theta) + n_2(\theta) \rightarrow m(\sigma)
\]

\[
n_1(\theta) \rightarrow n(\theta).
\]

In the case \( J = \emptyset \), we naturally have \( n_2(\theta) = 0 \). \( m(\theta) \) and \( m(\sigma) \) both mean the (occupied, length decreasing) positions. In general, we also have \( n_1(\theta) + n_2(\theta) = n(\theta) \). We can also define \( j(\theta) = \{ p | \theta_p = s_p \theta_{p+1} \text{ and } \ell(\theta_p) \geq \ell(\theta_{p+1}) \} \). Then we have \( m(\theta) + j(\theta) = r - n(\theta) \), in other words, the number of occupied positions. But note that, the set of distinguished J-expressions is considerably smaller than the set of distinguished expressions simply because we require all subsequent element \( \theta_p \) to be an element in \( W_J \). ♣♣♣ Shuai: [Question: can we use the combinatorics here to understand the fibration \( G/B \rightarrow G/P \)? Is it possible to deduce the result for \( G/P \) just from results of \( G/B \)?]

Example 2.5 (Grassmannnian). Take a maximal parabolic subgroup \( P \) of \( GL(n) \), then we can get a relation between the Kazhdan-Lusztig polynomials of Schubert varieties in Grassmannnian and the \( R \)-polynomials. Work out the relations in the \( Gr(1,n) \) and \( Gr(2,n) \) case. For example, \( G = S L_4 \), the corresponding parabolic subgroup is given by the isotropic group of the standard flag, \( P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \). In the language of Coxeter groups, we have \( W = S_4 \), \( J = \{ \alpha_1 - \alpha_2, \alpha_3 - \alpha_4 \} \). \( W_J \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), \( |W_J| = 6 \). The Bruhat
The cosets order is given by

\[
\begin{array}{c c c c c c}
34 & (1,3,4,2) \\
24 & (1,2,4,3) \\
14 & (2,4,3) & (1,2,3) \\
13 & (2,3) & id \\
12 & \\
\end{array}
\]

\(W^J\), the set of minimal length representatives of \(W/W_J\) is on the right-hand side.

Shuai: [TO DO: open neighborhood in \(G/P\), and coset structure of \(S_4/(12),(34))\]. The cosets \(W/W_J\) are given by

\[
\begin{align*}
()W_J &= \{(0),(3,4),(1,2),(1,2)(3,4)\} \\
(2,3,4)W_J &= \{(2,3),(2,3,4),(1,3,2),(1,3,4,2)\} \\
(2,4,3)W_J &= \{(2,4,3),(2,4),(1,4,3,2),(1,4,2)\} \\
(1,2,3)W_J &= \{(1,2,3),(1,2,3,4),(1,3),(1,3,4)\} \\
(1,2,4,3)W_J &= \{(1,2,4,3),(1,2,4),(1,4,3),(1,4)\} \\
(1,3)(2,4)W_J &= \{(1,3)(2,4),(1,3,2,4),(1,4,2,3),(1,4)(2,3)\}
\end{align*}
\]

Shuai: [open neighborhood of \(X_\tau\) in \(X_\sigma\) as cells in \(G/P\)]. We list the information that we need

- \(U \cap \sigma U^{-\sigma^{-1}}\) is an affine space of dimension \(\ell(\sigma)\). Because \(U\) is an affine space viewed as a subspace of \(\mathbb{A}^{\mathbb{N}^2}\). Conjugate by a given \(\sigma\) is a linear map on \(\mathbb{A}^{\mathbb{N}^2}\). The intersection is just a intersection of two affine subspaces.
- \(B\sigma \cdot B \cong B\sigma \cdot P \cong U \cap \sigma U \sigma^{-1}\).

Shuai: [TO DO]

- Deodhar’s definition of relative \(P_{x,y}\) and \(R_{x,y}\).
- Work out the example for \(SL_3\) and \(SL_4\).
- Modify Kumar’s proof to describe the local structure of \(G/P\) in the finite case.
- Work out the relative \(P_{x,y},R_{x,y}\) in the \(\mathfrak{sl}_3\) case.
- Bruhat order on \(D_\infty, \mathfrak{sl}_2, \mathfrak{sl}_3\) case. How to transfer our hand-drawn Bruhat orders on \(G/P\) and \(G/B\) as tex code?
- Write a program to check the combinatorial relations between the \(R\)-polynomials and \(P\)-polynomials in the \(\hat{A}_1\) or maybe also some twisted version.

**Example 2.6 (SL\(_3\)(C)).** Let \(G = SL_3(C)\), \(B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\), the standard Borel subgroup. \(P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\), the standard parabolic subgroup corresponding to the simple root \(\{\alpha_2 - \alpha_3\}\). Then \(G/B\) is the complete flag variety in \(C^3\), \(G/P\) parametrizes lines in \(C^3\), in other
words, \( G/P \cong \mathbb{P}^2 \). The \( G \)-action on \( \mathbb{P}^2 \) is just the matrix multiplication on column vectors \( (x_0,x_1,x_2)^T \) from the left. \( W_J = \langle (2,3) \rangle \cong \mathbb{Z}/2\mathbb{Z} \), the right cosets and minimal length representatives are given as follows:

\[
\begin{align*}
()W_J &= \{(0,(2,3))\} \\
(1,2)W_J &= \{(1,2),(1,2,3)\} \\
(1,3,2)W_J &= \{(1,3),(1,3,2)\}.
\end{align*}
\]

Note that although \((1,3)\) is a simple permutation, it has the longest length 3 in the Bruhat order. The geometry of the fibration is encoded in the graph \( S_3 \) Shuai: [Bruhat order and the subgraph given by the partial flag variety, in our case, it’s just \( \mathbb{P}^2 \). First we lift elements in \( W_J = \{(0,(1,2),(1,3,2))\} \) to \( G \) for later computation.

\[
\begin{align*}
() &= \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}, \\
(1,2) &= \begin{bmatrix} 0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \end{bmatrix}, \\
(1,3,2) &= \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \end{bmatrix}
\end{align*}
\]

The corresponding points are given by the image of the action those elements on \([1 : 0 : 0] \in \mathbb{P}^2 \). The Bruhat decomposition is given by the \( B^- \)-orbits:

\[
X_0 = (x_1 = x_2 = 0) \cong \text{pt}, X_{(1,2)} = (x_1 \neq 0,x_2 = 0) \cong \mathbb{A}^1, X_{(1,3,2)} = (x_2 \neq 0) \cong \mathbb{A}^2.
\]

The opposite Bruhat decomposition is given by the \( B^+ \)-orbits,

\[
X^0 = \{x_0 \neq 0\} \cong \mathbb{A}^2, X^{(1,2)} = \{x_0 = 0,x_1 \neq 0\} \cong \mathbb{A}^1, X^{(1,3,2)} = \{x_0 = x_1 = 0\} \cong \text{pt}.
\]

Now we demonstrate that the local structure of \( \overline{X}_u \) along \( X_v \) has a nice product description. For example \( X_{(1,2)} \hookrightarrow \overline{X}_{(1,3,2)} = \mathbb{P}^2 \). It’s given by the following multiplication

\[
X_{(1,2)} \times X^{(1,2)} \to \mathbb{P}^2 \\
(bv \cdot P, [gP]) \mapsto [bgP].
\]

Namely, the first component is viewed as elements in the group, the second component \( X^{(1,2)} = B^-(1,2)\)-orbits of \([1 : 0 : 0]\) is viewed as points in \( G/P \cong \mathbb{P}^2 \), the map is induced from the group action on the left. \( \mathbb{A}^2 \) Shuai: [Why this map is well-defined?]. In our case, it’s given by

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x \\ y \end{bmatrix} \mapsto \begin{bmatrix} bx + cy \\ dx + ey \\ fy \end{bmatrix},
\]

where \( a,d,f,x \neq 0 \). \( dx = (dx + ey) - \frac{e}{f}(fy) \), thus we know the image is isomorphic to \( \mathbb{A}^2 \). To be more clear, the morphism defined above has some ambiguity in the first factor, actually, we have to choose representatives in \( X_{(1,2)} \cong U \cap (1,2)U_J(1,2) \), where

\[
U_J = \begin{bmatrix} 1 & 0 & 0 \\
0 & * & 1 \\
0 & 0 & 1 \end{bmatrix}. \quad \text{That is } X_{(1,2)} \equiv \begin{bmatrix} 1 & * & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}. \quad \text{Thus in the map above, we’re safe to assume that } a = d = f = 1, c = e = 0, \text{ the image is given by } [bx : dx : y], \text{ since } d,x \neq 0, \text{ this is just } \mathbb{A}^2 \equiv \{x_1 \neq 0 \} \subset \mathbb{P}^2, \text{ which is surely an open neighborhood of } X_{(1,2)} = \{x_1 \neq 0, x_2 = 0\}, \text{ even more down to earth, the local picture of } X_{(1,2)} \text{ is just the second axis embedded in a plane. This is essential for our computation of } \mathcal{I}_{i,j,u,w} \eta. \quad \text{Next, we try to compute all the relative } R \text{-polynomials and relative } F \text{-polynomials.} \mathbb{A}^2 \text{ Shuai: [In general, it’s shouldn’t be hard to write a program to do this, for this particular example, let’s just ]}
REFERENCES

