K-THEORY OF Hilb^n(K3)

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This paper is dedicated to our advisors.

Abstract. This paper is a sample prepared to illustrate the use of the American Mathematical Society’s \LaTeX document class \texttt{amsart} and publication-specific variants of that class for AMS-\LaTeX version 2.

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1. Examples and basic properties of $K3$ surfaces

In this section, we collect some well-known examples and properties of $K3$ surfaces.

Example 1.1 (Fermat quartic surface, continue). The transcendental lattice of the Fermat quartic surface in $\mathbb{P}^4$ is given by

\[
\begin{pmatrix}
8 & 0 \\
0 & 8
\end{pmatrix}
\]

And Pic($X$) has rank 20. It contains 48 lines

\[x_0^4 + x_1^4 = x_j^4 + x_k^4; \{i, j, k\} = \{1, 2, 3\}.\]

By adjunction formula, any line on a quartic surface in $\mathbb{P}^3$ has self-intersection number $-2$. Moreover the intersection matrix has rank 20 and discriminant $-64$.

Example 1.2 (Modularity of the Fermat quartic surface).

\[#X(\mathbb{F}_p) = 1 - \alpha_p + p^2.\]

and almost all $\alpha_p$ is the coefficient of the $q$-expansion of some new form with CM-structure. And the Fermat quartic(over $\mathbb{Q}$) is associated to a unique new form of weight 3 and level 16 with CM in $\mathbb{Q}(\sqrt{-1})$, and the Hecke operators gives us

\[\alpha_p = 2(x^2 - 4y^2); \] if $p^2 = x^2 + 4y^2$.

Remark 1.3. The modularity comes from the fact that $\rho(X) = 20$, and we can construct a 2 dimensional representation of the Galois group on the transcendental lattice.

\textit{Date:} January 1, 2001 and, in revised form, June 22, 2001.

2000 \textit{Mathematics Subject Classification.} Primary 54C40, 14E20; Secondary 46E25, 20C20.

\textit{Key words and phrases.} Enumerative geometry, $K3$ surfaces, algebraic geometry.

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Example 1.4 (D₄ lattice and Jacobi forms). The D₄ lattice is given by the bilinear form

\[ M := \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & -1 & -1 & 2 \end{pmatrix} \]

The corresponding Dynkin diagram is

\[ \circ \alpha_4 \] \[ \circ \alpha_2 \] \[ \circ \alpha_1 \] \[ \circ \alpha_3 \]

We consider a Jacobi form of weight 2 and index \( \frac{1}{2} \) associated to this lattice

\[ \Theta(z, \tau) := \sum_{x \in \mathbb{Z}^4} e^{-2\pi i (x + \frac{\alpha}{2}, x + \frac{\alpha}{2})} q^{(x + \frac{\alpha}{2}, x + \frac{\alpha}{2})} \]

where \( \alpha = 2e_1 + e_2 + e_3 + e_4 \). In the space of Jacobi forms, we can rewrite \( \Theta(z, \tau) \) as a combination of classical modular forms and theta functions, namely

\[ \Theta(z, \tau) = -\frac{\vartheta_1(z, \tau) \eta(2\tau)^6}{\eta(\tau)^3} \]

where

\[ t = e^{2\pi iz}, \quad q = e^{2\pi i\tau} \]

\[ \vartheta_1(z, \tau) = iq^{\frac{1}{4}}(t^{\frac{3}{4}} - t^{-\frac{3}{4}}) \prod_{m \geq 1}(1 - q^m)(1 - t q^m)(1 - t^{-1} q^m) \]

\[ \eta(\tau) = q^{\frac{1}{24}} \prod_{m \geq 1}(1 - q^m) = \Delta^{\frac{1}{24}}. \]

To prove this formula, we only need to check that both sides are Jacobi forms of the same weight and index, together with some special values such as

\[ \Theta\left(\frac{1}{2}, \tau\right) = -\sum_{x \in \mathbb{Z}^4} q^{(x + \frac{\alpha}{2}, x + \frac{\alpha}{2})} = 2 \frac{\eta(2\tau)^8}{\eta^4} \]

\[ -\frac{\vartheta_1\left(\frac{1}{2}, \tau\right) \eta(2\tau)^6}{\eta(\tau)^3} = 2 \frac{\eta(2\tau)^8}{\eta^4}. \]

Example 1.5 (Basic (computations in) Gromov-Witten theory). We consider an elliptic K3 surface and try to understand the following statements

- **Yau-Zaslow**

  \[ \sum_{h \geq 0} q^h \int_{[M^{(N)}_{x_i(k)}]} 1 = \prod_{m \geq 1} \frac{1}{1 - q^m} \]

- **F^{GW}(y, q)**

  \[ F^{GW}(y, q) = \sum_{h \geq 0} \sum_{k \geq 0} q^h y^{k - \frac{1}{2}} \int_{[M^{F}_{y_j(k, h)}]} 1. \]

- **4F^{GW}(y, q)**

  \[ 4F^{GW}(y, q)^4 = \sum_{n, k \geq 0} <1 >_{0, \beta_{n, k}} q^n y^{k}. \]

Example 1.6 (Gromov-Witten theory on a Kummer surface, Jacobi forms). We consider the geometry of a Kummer surface \( S = Km(A) \) and verify the following computation

- ****

  \[ \sum_{n, k ; \beta_{n, k} = 0} <1 >_{0, \beta_{n, k}} q^n y^{k} = \frac{4}{\Delta(2\tau)} \Theta(z, \tau)^4. \]
Example 1.7 (Beauville, K3\[^n\]). First recall that

$$H^2(K3^n, \mathbb{Z}) \cong H^2(K3, \mathbb{Z}) \oplus \mathbb{C}[E]$$

where \([E]\) is the class of the exceptional divisor of the Blow up \(\pi : K3^n \to K3^n\). Together with a quadratic bilinear form defined by Beauville, \(H^2(H^2(K3^n, \mathbb{Z})\) can be realized as a lattice

$$L_{K3,2n-2} \cong A_{K3} \oplus <-(2n-2)>$$

$$\cong 3U \oplus 2(-E_8) \oplus <-(2n-2)>.$$

Lemma 1.8 (\(T_0[\text{Def}(X)]\)).

\[T_0[\text{Def}(X)] \cong H^1(X, T_X) \cong H^1(X, \Omega_X) = b_2(X) - 2.\]

Example 1.9 (Lagrangian fibration, K3). Let \(E\) be an elliptic curve. First we prove

\(X := \{ (x_0, x_1, \ldots, x_n) \in \text{Sym}^{n+1}(E) | x_0 + x_1 + \ldots x_n = 0 \} \cong \mathbb{P}^n\).

Recall that \(\text{Pic}^0(E) \cong E\) i.e we use \(\text{Pic}^0(E)\) to describe \(E\), then \(x_i \sim \mathcal{O}(p_i - q)\), where \(q\) is the origin. Then

\[X = \{ (p_0, \ldots, p_{n+1})/\text{Sym}_{n+1}[p_0] + \ldots [p_n] \sim (n+1)[q] \}\]

This tells us that \(X\) can be identified with a complete linear system of degree \(n + 1\), but Riemann-Roch, we know that \(h^0(E, \mathcal{O}_E((n+1)[q])) = n + 1\), thus we know

\[X \cong \mathbb{P}^n.\]

As a consequence (consider \(add : \text{Sym}^{n+1}(E) \to E\), we know

\[\text{Sym}^{n+1}(E) \cong \mathbb{P}^n \times E.\]

Apply Matsushita’s result (see the remark below), we can give several examples of Lagrangian fibrations

- (Fibration over \(\mathbb{P}^1\)) If \(S \to \mathbb{P}^1\) is an elliptic K3 surface, then
  \(\pi : \text{Hilb}^n(S) \to \text{Sym}^n(S) \to \text{Sym}^n(\mathbb{P}^1) \cong \mathbb{P}^n\)
  is a Lagrangian fibration.

- (Trivial fibration over an elliptic curve \(E\)) Let \(S\) be a K3 surface associated to \(E \times E/G\), where \(G \subset \text{Aut}(E)\) is the cyclic group of order \(2, 3, 4\) or \(6\). Consider the composition

\[\text{Hilb}^n(S) \to \text{Sym}^n(S) = S \times \cdots \times S/S_n \to E \times \cdots \times E/(G^n \times S_n).\]

Note that the Lagrangian fibration is induced by the projection to the even factors (\(2^{nd}, 4^{th}, \ldots\)), thus we get the base

\[(E/G)^n/S_n \cong (\mathbb{P}^1)^n/S_n \cong \mathbb{P}^n.\]

- (Generalized Kummer varieties.) Let \(F \hookrightarrow A \to E\) be a two dimensional abelian variety fibred over an elliptic curve. Then consider \(K_n(A) := \pi^{-1}([0])\)

\[\pi : \text{Hilb}^{n+1}(A) \to \text{Sym}^{n+1}(A) \to A\]

But we also have

\[\text{Sym}^{n+1}(A) \to \text{Sym}^{n+1}(E) \cong \mathbb{P}^n \times E.\]

Then we get a fibration \(K_n(A) \to \mathbb{P}^n\), and the fibre is isomorphic to the abelian subvariety

\[\{(x_0, x_1, \ldots, x_n) \in F^{n+1} | x_0 + \ldots x_n = 0 \}.

And this construction can be easily generalized to higher dimensional abelian varieties.

**Theorem 1.10** (Fibre structures of projective irreducible symplectic variety; D.Matsushita). Let \(f : X \to B\) is a fibration of a projective irreducible symplectic variety over a projective variety. Assume further that

- \(E\) is an abelian variety up to a finite unramified covering map.
- \(\text{dim}(B) = n\) and has only \(\mathbb{Q}\)-factorial log terminal singularities.
- \(K_E \sim \mathcal{O}_E\).
- \(-K_B\) is ample and the Picard number \(\rho(B) = 1\)
Remark 1.11 (some explicit elliptic curves).

Remark 1.12 (crepant resolution).

**Example 1.13** (Some special cycles on Hilb^n(S)).

**Example 1.14** (Double plane). Let \( \mathcal{L} \) be a line bundle on \( \mathbb{P}^2 \) and suppose that \( c_1(\mathcal{L}) = 3H \), we choose a section \( s \in H^0(\mathcal{L}^\otimes 2) \) with smooth locus \( D \). Let \( \{U_i\} \) be an open covering of \( \mathbb{P}^2 \) which trivialize the line bundle with local coordinates \( \{t_0^i, t_1^i, t_2^i\} \) and

\[
X := \{(t_0^i, t_1^i, t_2^i; x_i) \in U_i \times \mathbb{C} | x_i^2 = s_i(t_1^i, t_2^i)\}.
\]

Then we can naturally define a branch covering

\[
\pi : X \rightarrow \mathbb{P}^2.
\]

By local computation, we can easily get

\[
\omega_X = \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{L}) \cong \mathcal{O}_X.
\]

**Example 1.15** (Elliptic K3 ). This is just a toy example, let \( X \) be the Kummer surface associated to the abelian surface \( E_1 \times E_2 \), where \( E_1, E_2 \) are two elliptic curves.

\[
\pi_i : X \rightarrow E_i/\pm \cong \mathbb{P}^1
\]

**Remark 1.16.** Use the multiplicity of (topological) Euler characteristic, we know if \( X \) is an elliptic surface with nodal singular fibres, since for elliptic fibre \( X_t, c(X_t) = 0 \), we have

\[
24 = c_2(T_X) = e(X) = \sum_{X_t, singular} e(X_t)
\]

So, we know a generic elliptic K3 has exactly 24 nodal fibres.

**Example 1.17** (Properties of a typical K3 surface). We list some basic properties of a typical K3 surface that we should know

- Hodge diamond, Betti numbers

\[
\begin{array}{c|cccc|c|cccc}
 & h^{0,0} & h^{0,1} & h^{1,0} & h^{2,0} & h^{2,2} & h^{2,1} & h^{1,1} & h^{0,2} & \hline
0 & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\
1 & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\
20 & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\
1 & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\
0 & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\
0 & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\
1 & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\
\end{array}
\]

And the Betti numbers are

\[
b_0 = b_4 = 1, b_1 = b_3 = 0, b_2 = 22.
\]

- Lefschetz (1, 1)-theorem

\[
\text{Pic}(X) \cong H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C})
\]

This comes from the exponential sequence

\[
0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0
\]

the observation that the image of the map \( c_1 : \text{Pic}(X) \rightarrow H^2(X, \mathbb{Z}) \)

is real, and the Hodge decomposition

\[
H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).
\]
• Picard number.

\[ \rho(X) = \text{rank}(\text{Pic}(X)) \leq h^{1,1}(X) = 20. \]

- Every number between 0 and 20 can be realized by some complex K3 surface.
- For any algebraic K3, \( \rho(X) \geq 1 \) and \( \rho(X) = 21 \) is impossible over an algebraically closed field.
- \( \rho(X) \) is always even over \( \overline{F}_p \).

• Symmetric bilinear pairing

\[ (\cdot, \cdot) : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z} \]

\[ (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta. \]

which is just the intersection paring under Poincare duality. And we have

\[ \Lambda_{K3} := H^2(X, \mathbb{Z}) \cong (-E_8)^2 \oplus U^3. \]

\[ U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

and

\[ E_8 = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \end{pmatrix} \]

corresponding to the Dynkin diagram

\[ E_8 : \circ \circ \circ \circ \circ \circ \circ \circ \]

\[ \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \]

• The period space. Motivated by the Hodge decomposition and \( \forall 0 \neq \omega \in H^0(X, \Omega_X), \gamma \in H^{1,1}(X), \)

we have

\[ (\omega, \bar{\omega}) \in \mathbb{R}_+, (\omega, \gamma) = 0, (\omega, \omega) = (\bar{\omega}, \bar{\omega}) = 0 \]

, we define

\[ \Omega = \{ x \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) | (x, x) = 0, (x, \bar{x}) \geq 0 \}. \]

The importance of \( \Omega \) is that it’s a fine moduli space of marked K3 surfaces. Note that

\[ \dim_{\mathbb{C}}(\Omega) = 20. \]

• Signature. \((\cdot, \cdot)\) has signature

- \((3, 19)\) on \( H^2(X, \mathbb{R}) \).
- (Hodge’s index theory) \((1, 19)\) when restricted to \( H^{1,1}(X) \cap H^2(X, \mathbb{R}); \)
- (On \( \text{Pic}(X) \)) \((1, \rho(X) - 1)\) when \( X \) is projective (\( \exists \) ample line bundle), \((0, \rho(X))\) if it’s not projective.
2. Moduli spaces

2.1. Examples of moduli spaces. To get some feelings of moduli spaces, let’s list some basic examples

- \( \mathcal{M}_{g,n} \); \( \mathcal{M}_{0,4} \cong \mathbb{P}^1 \).
- \( \mathcal{M}_{g,n}(\mathbb{P}^0, 0) \cong \mathcal{M}_{g,n} \).
- \( \mathcal{M}_{g,n}(X, 0) = \mathcal{M}_{g,n} \times X \).
- \( \mathcal{M}_{0,0}(\mathbb{P}^n, 1) \cong G(2, n+1) = G(1, \mathbb{P}^n) \).
- \( \mathcal{M}_{0,0}(\mathbb{P}^2, 2) \cong \text{Bl}_Z(\mathbb{P}^5) \),

where \( \mathbb{P}^5 \) is the moduli space of conics in \( \mathbb{P}^2 \), and \( Z \) is the locus of double line, which is actually the Veronese surface (degree= 4) given by the \( 2 \times 2 \) minors of

\[
\begin{pmatrix}
x & u & v \\
u & y & w \\
v & w & z
\end{pmatrix}.
\]

Recall that use this description, we can solve the five conics problem, but here I just want to mention that the locus of conics tangent to a given conic is a degree 6 hypersurface in the moduli space \( \mathbb{P}^5 \).

- \( \mathcal{M}_{1,0}(\mathbb{P}^2, 3) \)

And some basic properties of the moduli spaces above

- If \( g > 2 \), we have
  \[ \dim(\mathcal{M}_g) = 3g - 3 \]
  This can be computed from the following information, denote a genus \( g \) curve by \( X \).
  - \( \dim(\text{Pic}^d(X)) = g \).
  - Apply Riemann-Roch theorem to a degree \( d \) line bundle \( \mathcal{L} \) on \( X \), since \( h^1(X, \mathcal{L}) = 0 \) (simply for degree reasons), we know \( h^0(X, \mathcal{L}) = d - g + 1 \). take two generic sections \( s_1, s_2 \) then we get a degree \( d \) cover of \( \mathbb{P}^1 \). Thus the dimension of such covers is
    \[ \dim(\mathcal{M}_g) + (d - g + 1) + (d - g) + g. \]
  - By Riemann-Hurwitz theorem, a cover as above has ramification index \( 2d + 2g - 2 \).
  So we know
    \[ \dim(\mathcal{M}_g) + g + 2d - 2g + 1 = 2d - 2g + 2. \]
- \( (\text{Harer – Zagier}) \)
  \[ \chi(\mathcal{M}_g) = \frac{B_{2g}}{2g(2g - 2)} \]
  \[ \chi(\mathcal{M}_{g,n}) = (-1)^n \frac{(2g + 2 - 3)B_{2g}}{[2g(2g - 2)]!} \]

2.2. Constructions of some moduli spaces.

Example 2.1 (\( \mathcal{M}_g \) as a quasi-projective variety). The starting point is that by a theorem of Grothendieck, Hilbert schemes (as a fine moduli space) of subschemes in \( \mathbb{P}^n \) with fixed Hilbert polynomial exist. And every curve is canonically embedded in \( \mathbb{P}(H^0(C, \omega_{C/k}^{[3]}) = \mathbb{P}^{5g-6} \), the second identity comes from the Riemann-Roch theorem. However, this isomorphism is not Canonical. We first define the functor \( \mathcal{H}_g \)
\[
\mathcal{H}_g(S) = \{ \text{families of genus } g \text{ curves } f : C \to S \}
\]

an isomorphism
\[ \mathbb{P}(f_*\omega_{C/S}^{[3]} \cong \mathbb{P}^{5g-6} \times S) \]

Then we can show that \( \mathcal{H} \) is represented by a closed subscheme of the Hilbert scheme of \( \mathbb{P}^{5g-6} \). Then \( \mathcal{M}_g \)
can be realized as the GIT quotient of \( \mathcal{H}_g \) by the natural action of \( \text{PGL}(5g-6) \).
2.3. Sheaves and cycles on moduli spaces.

**Example 2.2** ($\psi_1$ on $\cM_{0,4} \cong \mathbb{P}^1$). Consider the forgetful morphism $\pi: \cM_{0,4} \to \cM_{0,3} \cong pt$. By the comparison theorem, and an abuse of notation we have

$$
\psi_1 = \pi^* \psi_1 + D_{0,\{1,4\}}.
$$

We know

- On $\cM_{0,3}$, $\psi_1 = 0$, because we only have a vector space at a point.
- On $\cM_{0,4}$, $D_{0,\{1,4\}}$ represents reducible stable curves with a unique node, two components of genus 0. There’s only one such curve! Thus we know

$$
D_{0,\{1,4\}} = pt \in A_1(\mathbb{P}^1) \Rightarrow \psi = O_{\mathbb{P}^1}(1).
$$

Note that $D_{0,\{1,4\}}, D_{0,\{2,4\}}, D_{0,\{3,4\}}$ are the three points not in $\cM_{0,4} \cong \mathbb{P}^1 - \{0,1,\infty\}$.

![Figure 1. Boundary divisor $D_{0,\{1,4\}}$ on $\cM_{0,4}$](image)

2.4. **Virtual fundamental classes.** To understand the construction of the virtual fundamental class of a moduli space, we have to know some deformation theory.

2.5. **Examples of Gromov-Witten theory.**

**Example 2.3** (Gromov-Witten theory for a point).

**Example 2.4** (Gromov-Witten theory for a $\mathbb{P}^1$).

**Example 2.5** (Gromov-Witten theory for a $\mathbb{P}^2$).

**Example 2.6** (Gromov-Witten theory of blow-ups of $\mathbb{P}^2$).

**Example 2.7** (Gromov-Witten theory of the Kummer K3).

3. **Examples of quantum cohomology rings**

**Example 3.1** (Quantum cohomology of $\mathbb{P}^2$).

$$
QH^*(\mathbb{P}^2) = \mathbb{Q}[H,q]/(H^3 - 1).
$$

Note that $A^*(\mathbb{P}^2) = \mathbb{Q}[H]/H^3$ is not a specialization of the quantum cohomology ring directly.

**Example 3.2** (Small quantum cohomology of $X = \mathbb{P}^n$).

$$
QH^*_s(\mathbb{P}^n) = \mathbb{Q}[H,q]/(H^{n+1} - q).
$$

**Example 3.3** (Small quantum cohomology of Grassmannians).

$$
QH^*_s(G(k,n)) = \mathbb{Z}[\sigma_1, \ldots, \sigma_l; q]/(S_{k+1}(\sigma), \ldots, S_{n-1}(\sigma), S_n(\sigma) + (-1)^l q),
$$

where

- $l = n - k$.
- $\sigma_i = c_i(\mathcal{O})$, the $i^{th}$ Chern class of the universal quotient bundle.
- $S_r(\sigma) = \det(\sigma_{j+i-1})_{1 \leq j, i \leq r}$. 

7
J.Bryan and N.C. Leung’s paper

Somehow, I find the two topics I’m interested in recently relate to each other closely.

Theorem 4.1 (Yau-Zaslow). Let \( \beta_k \) be a primitive curve on a smooth projective K3 surface \( S \), and \( \beta_k^2 = 2k - 2 \). Let \( N_k \) denote the number of rational curves in the linear system \( |\beta_k| \), then

\[
\sum_{k \geq 0} N_k q^{k-1} = \frac{1}{q} \prod_{m \geq 1} \frac{1}{(1 - q^m)^{24}} = \frac{1}{\Delta}.
\]

Sketch of Yau-Zaslow’s argument. Yau-Zaslow’s line of argument is quite indirect, however beautiful at the same time.

- By the adjunction formula, we know the genus of \( \beta_k \) is \( k \).
- Apply Riemann-Roch theorem for surface, let \( \mathcal{L} = \mathcal{O}_S(\beta_k) \)
  \[
  \chi(\mathcal{L}) = \frac{c_1(\mathcal{L})^2}{2} + \frac{c_1(\mathcal{L})c_1(T_S)}{2} + \frac{c_1(T_S)^2 + c_2(T_S)}{12}.
  \]
  Since \( c_1(T_S) = 0, c_2(T_S) = 24 \) and \( h^1(S, \mathcal{L}) = 0 \), we know the complete linear system \( |\beta_k| \) is isomorphic to \( \mathbb{P}^k \). Because imposing a node will set a constrain on \( |\beta_k| \), so we expect to get a finite number of rational curves with \( n \) nodes (‘rationality’ comes from the fact that each nodes decrease the genus by 1). Denote this number by \( N_k \).
- Consider the compactified universal Jacobian constructed by Bershasky, Sadov, and Vafa
  \[
  \pi : \bar{J} \to |\beta_k| \cong \mathbb{P}^k.
  \]
  If we assume that all curves in \( |\beta_k| \) are reduced and irreducible, then \( \bar{J} \) is a smooth hyperkahler manifold of dimension \( 2k \).
- Assume further that every curve in the linear system has at most nodal singularity, then one can show the Euler characteristic of \( \pi^{-1}(C) \) is always 0 unless \( C \) is a rational curve with \( n \) nodes, in this case \( e(\pi^{-1}(C)) = 1 \). In conclusion, we have
  \[
  \chi(\bar{J}) = N_k.
  \]
- One can show \( \bar{J} \) is birational to the Hilbert scheme of \( k \) points on \( S \), \( \text{Hilb}^k(S) \) is also a smooth hyperkahler manifold. By a theorem of Batyrev, which says that compact, birational equivalent, projective, Calabi-Yau manifolds have the same Betti numbers, we get
  \[
  N_k = \chi(\text{Hilb}^k(S)).
  \]
- \( \chi(\text{Hilb}^k(S)) \) was computed by Gottsche first (by using the decomposition theorem). The result is
  \[
  \sum_{k \geq 0} \chi(\text{Hilb}^k(S)) = \prod_{m \geq 1} \frac{1}{(1 - q^m)^{24}},
  \]
  which implies
  \[
  \sum_{k \geq 0} N_k q^{k-1} = \frac{1}{q} \prod_{m \geq 1} \frac{1}{(1 - q^m)^{24}} = \frac{1}{\Delta}.
  \]

\[\square\]

References


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Figure 2. This is an example of a figure caption with text.