1 Introduction

Some very basic knowledge and examples about GIT (Geometric Invariant Theory) and maybe also Equivariant Intersection Theory.

2 "Finite flat group schemes" by John Tate

3 Linearizations of line bundles

Let, $G$ be a connected linear algebraic group, $X$ a normal $G$-variety, and $\mathcal{L}$ a line bundle over it, then $\mathcal{L}^n$ admits a $G$-linearization for some $n$. Note first that this doesn’t mean large enough powers of $\mathcal{L}$ admit $G$-linearization, just some of them. The second thing is that $X$ being normal is crucial, see the following example.

Example 3.1 ($\mathcal{L}^n$ admits no linearization for any $n$). Consider the nodal curve in $\mathbb{P}^2$:

$$C : y^2z = x^3 - x^2z$$

Then we can get $C$ by identifying $\infty$ and 0 in $\mathbb{P}^1$. Namely

$$t \mapsto (t^2 + 1, t^3)$$

Identify the fibres over $\infty$ and 0 of $\mathcal{O}_{\mathbb{P}^1}(k), (k \neq 0)$, we can get a line bundle $\mathcal{L}$ over $C$. Let $\mathbb{G}_m$ act on $\mathbb{P}^1$ by

$$[x, y] \mapsto [tx, t^{-1}y]$$

$\infty, 0$ are fixed, so we have a natural $\mathbb{G}_m$ action on $C$, if $\mathcal{L}$ admits a $\mathbb{G}_m$ linearization, then by pulling it back, we can get a $\mathbb{G}_m$ linearization of $\mathcal{O}_{\mathbb{P}^1}(k)$, and $\mathbb{G}_m$ acts on fibres over $\infty, 0$ with the same character. However, this is impossible for and $\mathbb{G}_m$ linearization of $\mathcal{O}_{\mathbb{P}^1}(k)$. To see this, we have a SES

$$0 \to X(G) \to Pic^G(X) \to Pic(X) \to Pic(G).$$
In our particular situation, it is

\[ 0 \to \mathbb{Z} \to \text{Pic}^G(\mathbb{P}^1) \to \mathbb{Z} \to 0. \]

Which simply says that if \( O_{\mathbb{P}^1}(k) \) is linearizable, then we have different linearizations parametrized by \( \mathbb{Z} \), we can construct all these linearizations explicitly

\[ \text{Tot}(O_{\mathbb{P}^1}(k)) = \mathbb{F}(1, 1, k) - [0, 0, 1] \to \mathbb{F}(1, 1, k) - [0, 0, 1] \]

\[ [x, y, z] \mapsto [tx, ty, t^k z]; \forall j \in \mathbb{Z} \]

Then the characters of \( \mathbb{G}_m \) action on fibres over \( \infty \) and \( 0 \) differs by \( t^k \). This contradiction tells us that \( \mathcal{L} \) is not \( \mathbb{G}_m \) linearizable, same obstruction holds for \( \mathcal{L}^n \).

**Remark.** The obstruction for a line bundle to be linearizable is given by the Schur multiplier \( H^2(G, k^\times) \), i.e we have a SES

\[ 0 \to \text{Hom}(G, k^\times) = X(G) \to \text{Pic}^G(X) \to \text{Pic}(X) \to \text{Pic}(G) \to H^2(G, k^\times) \]

Note that if \( G \) is a connected linear algebraic group, then the action of \( G \) on \( \text{Pic}(X) \) must be trivial, in other words \( \text{Pic}(X)^G = \text{Pic}(X) \), and for a connected affine algebraic group acting on a normal variety, we actually have

\[ 0 \to X(G) \to \text{Pic}^G(X) \to \text{Pic}(X) \to \text{Pic}(G) \]

**Remark** (Intuitively, why \( O_{\mathbb{P}^n}(k) \) is not \( \text{PGL}(n+1) \)-linearizable, if \( n+1 \mid k ? \)). My intuition comes from the following sequence is true for \( X \) affine or proper and connected over \( k \), specially, for projective smooth varieties, everything works fine.

\[ 0 \to X(G) \to \text{Pic}^G(X) \to \text{Pic}(X) \to \text{Pic}(G) \]

Now, it’s intuitively easy to understand the following statements

- every \( O_{\mathbb{P}^n}(k) \) admits \( \mathbb{Z} \) different \( \text{GL}(n+1) \)-linearization given by \( \text{det}^n \). Let me just write down the group action on the total space

\[ \text{GL}(n+1) \ni g : \mathbb{F}(1, 1, \ldots, k) - [0, 0, \ldots, 1] \to \mathbb{F}(1, 1, \ldots, k) - [0, 0, \ldots, 1] \]

\[ [x_0, \ldots, x_n; z] \mapsto [g(x_0, x_1, \ldots, x_n); \text{det}^n z]. \]

- every \( O_{\mathbb{P}^n}(k) \) admits a unique \( \text{SL}(n+1) \)-linearization

\[ \text{SL}(n+1) \ni g : \mathbb{F}(1, 1, \ldots, k) - [0, 0, \ldots, 1] \to \mathbb{F}(1, 1, \ldots, k) - [0, 0, \ldots, 1] \]

\[ [x_0, \ldots, x_n; z] \mapsto [g(x_0, x_1, \ldots, x_n); z]. \]

- \( \mathcal{L} \in \text{Pic}(\mathbb{P}^n) \) admits a \( \text{PGL}(n+1) \)-linearization if and only if its image in \( \text{Pic}(G) \) is zero. Note that the last map in the sequence is given by

\[ \delta : \text{Pic}(X) \to \text{Pic}(G); \mathcal{L} \mapsto \text{pr}_2^* \mathcal{L} \otimes \sigma^* \mathcal{L}^{-1}|_{G \times x_0}. \]
is a choose point in $X$, so actually $\delta$ is not canonical. And we know

$$pr_2^*\mathcal{O}_{\mathbb{P}^n}(k) \otimes \sigma^*\mathcal{O}_{\mathbb{P}^n}(-k)|_{G \times x_0} = \mathcal{O}_{\mathbb{P}^N}(-k)|_G.$$  

And the last group is trivial if and only if $n+1|k$. Haha, you might ask where does the last equality comes from? It’s comes from several facts

- $PGL(n + 1) = \mathbb{P}^N - \Delta$, where $\Delta$ means the determinate variety.
- From the decomposition of $\mathbb{P}^N$ above, we get $Pic(PGL(n + 1)) \cong \mathbb{Z}/(n+1)\mathbb{Z}$, with the restriction of $\mathbb{P}^N(1)$ as a generator.
- since the complement has codimension $\geq 2$, we actually have

$$Pic(\mathbb{P}^N \times \mathbb{P}^n) \cong Pic(PGL(n + 1) \times \mathbb{P}^n).$$

In other words, every line bundle on $PGL(n+1) \times \mathbb{P}^n$ is the restriction of a line bundle on $\mathbb{P}^N \times \mathbb{P}^n \cong \mathbb{Z} \oplus \mathbb{Z}$, this is how we actually write down $\sigma^*\mathcal{L}^{-1}$.

- the group action is the restriction of the birational map

$$\mathbb{P}^N \times \mathbb{P}^n \dashrightarrow \mathbb{P}^n$$

$$[(a_{ij}),(x_0,x_1,\ldots,x_n)] \mapsto [\sum_{j=0}^n a_{1j}x_j, \ldots, \sum_{j=0}^n a_{nj}x_j].$$

Computation of $\sigma^*$ is easy since the group action is just a bilinear function on every component, the pull-back just means separate this linearity.

Actually, we can really write down the $PGL(n+1)$-linearization explicitly, say for the canonical bundle $\mathcal{O}_{\mathbb{P}^n}(-(n+1))$

$$g : \mathbb{P}(1,1,\ldots,-(n+1)) - [0,0,\ldots,1] \rightarrow \mathbb{P}(1,1,\ldots,-(n+1)) - [0,0,\ldots,1]$$

$$[x_0, \ldots, x_n; z] \mapsto [g(x_0, x_1, \ldots, x_n); z \cdot det^{-1}(g)]; g \in PGL(n + 1).$$

**Remark** (fixed-point and character).

**Remark** (induced action on cohomology groups $H^i(X, \mathcal{L})$). *Instead of the the chain of linear maps*

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(G \times X, \sigma^*\mathcal{F}) \xrightarrow{I} \Gamma(G \times X, pr_2^*\mathcal{F}) \cong \mathbb{C}[G] \otimes \Gamma(X, \mathcal{F}).$$

On higher cohomology groups, we have

$$H^i(X, \mathcal{F}) \longrightarrow H^i(G \times X, \sigma^*\mathcal{F}) \xrightarrow{I} H^i(G \times X, pr_2^*\mathcal{F}) \cong \mathbb{C}[G] \otimes H^i(X, \mathcal{F}).$$
The identity
\[ H^i(G \times X, pr_2^* \mathcal{F}) \cong \mathbb{C}[G] \otimes H^i(X, \mathcal{F}) \]
comes from the fact \( pr_2 \) is a flat morphism (Fibrewise Hilbert polynomial criterion, or actually it’s just the trivial family), thus from the general fact if you have a flat morphism \( f : X' \to X \) between noetherian schemes
\[ H^i(X, \mathcal{F}) \otimes_A A' \cong H^i(X', f^* \mathcal{F}) . \]
This simply because if \( A' \) is flat over \( A \), then \( \otimes_A A' \) commutes with taking cohomology groups of the Cech complex. I like a more explicit way of constructing the group action on cohomology groups, let \( \{ U_i \} \) be an open covering of \( X \), then so is \( \{ gU_i \} \), if \( \alpha \in \Gamma(X, \mathcal{F}) \) is given by \( (\alpha|_{U_i}) \), we simply define \( g\alpha \) to be the global section defined by the image of
\[ I^{-1}_{gU_i} : \mathcal{F}(U_i) \to \mathcal{F}(gU_i) . \]
Namely let \( \beta = g\alpha \), we have \( \beta|_{gU_i} = I^{-1}_{gU_i}(\alpha|_{U_i}) \). Same construction works for \( H^i \). The equivariant structure of \( \mathcal{F} \) makes everything fit perfectly.

## 4 Picard group: special push-forward and pullback

If \( \pi : X \to Y \) is an morphism between varieties, we have
\[ \pi^* \mathcal{O}_Y = \mathcal{O}_X \otimes_{\pi^{-1} \mathcal{O}_Y} \pi^{-1} \mathcal{O}_Y = \mathcal{O}_X . \]
This follows from the construction. This is more than just ‘view \( f \in \mathcal{O}_Y \) as an element \( f \circ \pi \in \mathcal{O}_X \)’. For push-forward, we don’t have similar property in general, but in some important situations, we do have, and also by constriction, namely the global Proj. For example, of we have a projective bundle \( p : \mathbb{P} \mathcal{E} \to X \), we do have
\[ \pi_* \mathcal{O}_{\mathbb{P} \mathcal{E}} = \mathcal{O}_X . \]
For global Spec, \( \alpha : \text{Spec}(\mathcal{A}) \to X \), we have similar property
\[ \alpha_* \mathcal{O}_{\text{Spec} \mathcal{A}} = \mathcal{A} . \]
Remember \( \mathcal{A} \) is a sheaf of \( \mathcal{O}_X \)-algebra.

**Theorem 4.1** \( (\text{Pic}(\mathbb{P} \mathcal{E}) = \text{Pic}(X) \times \mathbb{Z}) \). If \( X \) is a noetherian regular scheme and \( \mathcal{E} \) is a vector bundle on \( X \) with rank at least 2, then we have \( \text{Pic}(\mathbb{P} \mathcal{E}) \cong \text{Pic}(X) \times \mathbb{Z} \).

**Proof.** We prove the following map is an isomorphism
\[ \text{Pic}(X) \times \mathbb{Z} \to \text{Pic}(\mathbb{P} \mathcal{E}) \]
\[ (\mathcal{L}, m) \to \mathcal{O}_{\mathbb{P} \mathcal{E}}(m) \otimes p^* \mathcal{L} . \]
• injectivity. If we have $p^*(\mathcal{L}(m)) = \mathcal{O}_{\mathbb{P}^e}$, then
\[ \mathcal{O}_X = p_*\mathcal{O}_{\mathbb{P}^e} = p_*(\mathcal{O}_{\mathbb{P}^e}(m) \otimes p^*\mathcal{L}) = \mathcal{L} \otimes p_*\mathcal{O}_{\mathbb{P}^e}(m). \]
The last step is by the push-full formula. Note by the cohomology and base change theorem, we know
\[ p_*\mathcal{O}_{\mathbb{P}^e}(m) = \text{Sym}^m \mathcal{E}. \]
Since $\text{rank}(\mathcal{E}) \geq 2$, the identity above is possible only if $m = 0$, consequently, we get $\mathcal{L} \cong \mathcal{O}_X$.

• surjectivity. By tensoring a suitable $\mathcal{O}_{\mathbb{P}^e}(m)$, we may assume $\mathcal{M} \in \text{Pic}(\mathbb{P}^e)$ is trivial on $p^{-1}U_i$, where $\{U_i\}$ is a suitable open covering of $X$, then $\mathcal{M}$ is given by some cocycles $p_{ij} : p^*\mathcal{O}_X|_{U_i} \to p^*\mathcal{O}_X|_{U_j}$, and since $p^*p_*\mathcal{O}_X|_{U_i} = \mathcal{O}_X|_{U_i}$, this naturally defines a line bundle $\mathcal{L}$ on $X$, and by construction, we have $p^*\mathcal{L} \cong \mathcal{M}$.

\[ \square \]

![Figure 1: The Universe](image)

**References**