1. (10 points) Solve the initial value problem
\[ y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}, \quad y(\pi) = 0, \quad t > 0. \]

Solution:
Use the method of integrating factors. Let \( \mu(t) \) be the integrating factor, then we find \( \mu(t) = e^{2\ln(t)} = t^2 \). Integrate the equation and plug in the initial value to obtain \( y(t) = \frac{\sin(t)}{t^2} \).
2. Consider the initial value problem:

\[ y'' - 3y' - 10y = 0, \quad y(0) = a > 0, \quad y'(0) = -1. \]

(a) (10 points) Solve the initial value problem.

Solution:
The characteristic equation is \( r^2 - 3r - 10 = 0 \), or \((r - 5)(r + 2) = 0\). As a result, we find \( y_1 = e^{5t} \) and \( y_2 = e^{-2t} \). Then \( y = C_1y_1 + C_2y_2 \). Plugging in the initial condition we obtain \( C_1 + C_2 = a \) and \( 5C_1 - 2C_2 = 0 \). Solve this system to obtain \( C_1 = \frac{2a-1}{7} \), and \( C_2 = \frac{5a+1}{7} \).

(b) (5 points) Find the critical value of \( a \) that separates solutions that approach \(+\infty\) as \( t \to +\infty \) from solutions that approach \(-\infty\) as \( t \to +\infty \).

Solution:
Since \( e^{-2t} \) is bounded as \( t \to +\infty \), and \( e^{5t} \to \infty \) as \( t \to +\infty \), we need to determine the sign of \( C_1 \). If \( C_1 > 0 \), then \( y \to +\infty \), and if \( C_1 < 0 \) then \( y \to +\infty \). From this we see that \( a = \frac{1}{2} \) is the critical value.
3. (a) (10 points) Consider the differential equation

$$t^2y'' - 2y = 0 \quad t > 0.$$ 

Given that $y_1(t) = t^2$ is solution, find a second, linearly independent solution.

**Solution:**
Use the method of reduction of order. We guess $y_2(t) = u(t)t^2$. Plug this into the equation to find that $y_2(t)$ is a solution provided

$$t^2u'' + 4t u' = 0.$$ 

Set $v = u'$ and solve this equation by integration to obtain $u' = v = ct^{-4}$ for some constant $c$. Integrate again and take $c = -3$ to find $u(t) = t^{-3}$. As a result we obtain $y_2(t) = t^{-1}$. 
(b) (10 points) Find the general solution of the inhomogeneous equation

\[ t^2 y'' - 2y = 3t^2 - 1 \quad t > 0. \]

Solution:

Use the method of variation of parameters to find a particular solution. We guess \( Y_p(t) = u_1 t^{-1} + u_2 t^2 \). Compute \( Y'_p = u'_1 t^{-1} - t^{-2} u_1 + 2 t u_2 + t^2 u'_2 \). We set all terms containing a derivative of \( u_1 \) or \( u_2 \) to be zero;

\[ t^2 u'_2 + t^{-1} u'_1 = 0. \]

Compute \( Y''_p \) and plug into the equation to obtain

\[-u'_1 + 2 t^3 u'_2 = 3t^2 - 1.\]

Solve these two equations for \( u'_1, u'_2 \) and integrate to find

\[ u_1 = -\frac{t^3}{3} + \frac{t}{3}, \quad u_2 = \ln(t) - \frac{1}{6 t^2}. \]

As a result, we get

\[ Y_p(t) = t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2}. \]
4. The population of a protected species of wild salmon in the Pacific Northwest is modeled by the logistic equation

\[ \frac{dP}{dt} = P(1 - P). \]

Suppose that the Department of Fisheries and Oceans authorizes fishermen to remove \( K \) salmon per unit time, as well as increasing fisheries on the smaller fish the salmon eat. As a result, the new model for the population \( P \) is given by

\[ \frac{dP}{dt} = 3P\left(1 - \frac{P}{3}\right) - K, \quad K < \frac{9}{4}. \]

(a) (5 points) Draw the phase line for this new model, and find the equilibria. Classify the equilibria as stable or unstable.

**Solution:**
The graph of \( dP/dt \) vs. \( P \) is an upside down parabola with zeros at \( P_{\pm} = \frac{3 \pm \sqrt{9 - 4K}}{2} \). Since \( K < \frac{9}{4} \), this means there are two distinct roots. Then \( P_- \) is unstable, \( P_+ \) is stable.

(b) (5 points) Suppose that when the new fisheries act was signed into law, the population \( P \) was stable. For what value of \( K \) will the new law cause the population of wild salmon to become extinct?

**Solution:**
The population at the time of signing is stable. Since the population before the law was signed was given by \( dP/dt = P(1 - P) \), we know that \( P = 1 \). The population will go extinct if \( 1 < P_- = \frac{3 - \sqrt{9 - 4K}}{2} \). Do some algebra so find this implies that \( K > 2 \).
(c) (5 points) Can you suggest some changes to the law which will ensure that the population of wild salmon never becomes extinct? Provide some evidence for your suggestions. (You don’t need to fill this page.)

Solution:
I basically took anything here, as long as it was motivated in some way by the model. A sample solution I was hoping for was the following. Rather than remove $K$ fish, the government could have legislated that the number of salmon allowed to be fished was a fixed percentage of the total population. This changes the model to

$$\frac{dP}{dt} = 3P(1 - \frac{P}{3}) - \alpha P,$$

for some $\alpha < 1$. This means that the fisherman are allowed to fish $\alpha$ percent of the total population. In this new model, no matter what the initial population is (as long as $P > 0$), the population increases towards the stable equilibrium at $P = (1 - \alpha)$. 
5. (a) (10 points) Find a fundamental set of power series solutions to the equation

\[(3 - x^2)y'' - 3xy' - y = 0,\]

about the point \(x_0 = 0\). Find the first three terms in the expansion, and provide a recursion relation to determine all the coefficients.

**Solution:**

Guess \(y = \sum_{n=0}^{\infty} a_n x^n\). Plug this into the equation and reindex to obtain the recursion relation

\[a_{n+2} = \frac{n + 1}{3(n + 2)} a_n.\]
(b) (5 points) Consider the equation
\[ 2x^2 y'' + 9 \sin(x) e^{(e^x - 1)} y' + 6 \cos(\sin(x)) y = 0. \]
Suppose that \( y \) is any solution of this equation on the set \( x > 0 \). We define
\[ A := \{ b \in \mathbb{R} : \lim_{x \to 0^+} x^b |y(x)| < +\infty \}. \]
Find the set \( A \). (Don’t panic! This problem is easier than it looks.)

**Solution:** Note that \( x = 0 \) is a regular singular point of the differential equation. The first thing to do is compute the indicial equation
\[ 2r(r - 1) + 9r + 6 = 0. \]
This has roots \( r_1 = -3/2 \) and \( r_2 = -2 \). Since \( r_1 - r_2 = 1/2 \) is not an integer can find linearly independent solutions
\[ y_1 = x^{r_1} p_1(x), \quad y_2 = x^{r_2} p_2(x), \]
where \( p_1(x) \), and \( p_2(x) \) are power series, convergent in a neighbourhood of \( x = 0 \), with \( p_1(0) \neq 0 \neq p_2(0) \). From this, we see
\[ \lim_{x \to 0^+} x^b y(x) = \lim_{x \to 0^+} c_1 x^{b + r_1} p_1(x) + c_2 x^{b + r_2} p_2(x), \]
where \( c_1, c_2 \) are arbitrary coefficients. Since \( p_1, p_2 \) are continuous, and not zero at \( x = 0 \), this limit is finite whenever \( b + r_1 \geq 0 \) and \( b + r_2 \geq 0 \). Plugging in the values of \( r_1 \) and \( r_2 \) we see that
\[ A = [2, \infty) \].