Introduction to the MMP (w/ a View to KRF)

Question: How do we find Canonical Representatives in a birational equivalence class?

Recall: A rational map \( \Phi: X \to Y \) is a map defined on \( X - V \) for \( V \) a variety of codim \( \geq 2 \). \( \Phi \) is birational if it has a rational inverse.

Let's try \( \dim_e X = 2 \).

Name: **Fact 1:** if \( \Phi: X \to Y \) birational, \( \dim_0 X = \dim_0 Y = 2 \),
then \( \Phi \) is a composition of blow ups and blow down.

**Fact 2:** If \( E \subset X \) is a \((-1)\)-curve, and \( E \cong \mathbb{P}^1 \), then

\[ \exists \text{ a map } \Phi: X \to Y \text{ s.t. } \Phi: X \setminus E \cong Y - \Phi(E) \]

and \( X = \text{Bl}_C Y \). (Castelnuovo).

So: we have a Name MMP:

\[ X \]

\[ \begin{cases} X : = Y \text{ if } \exists \text{ a } (-1) \text{-curve} \text{ s.t. } X \setminus E \cong Y - \Phi(E) \text{ and } X = \text{Bl}_C Y \text{ (Castelnuovo)} \end{cases} \]

\[ \begin{cases} \text{Yes} \quad \text{No} \end{cases} \]

\[ \Phi: X \to Y \]

\[ \text{blow down} \]
Since blow downs decrease the Betti number by 1, this process terminates, producing a minimal model.

**NB:** Minimal models need not be unique.

\[ \mathbb{P}^2 \cong \mathbb{P}^1 \times \mathbb{P}^1. \]

**Problem:** How do we study the geometry of Min. Mod.? 

"Not having a (-1)-curve" is not a very strong geometric condition.

**Ans:** (Mori) Ask instead: "is \( k_x \) NEF?"

**Recall:** \( E \) is NEF if \( E.C > 0 \) for all irreducible curves \( C \).

\[ \inf \left\{ C_i(E) \geq 0 \right\}. \]

**Or:** \( E \) is NEF if for every ample \( H \), \( E + cH \) is ample \( \forall c > 0. \)

**Observe:** if \( E \) is a divisor, \( E \) NEF, then \( rE \) is NEF \( \forall r \geq 0. \) Thus, we can define the NEF cone of \( X \) by \( \text{NEF}(X) = \{ \text{Divisors } E/\text{num of } \}, \)

**Note:** \( E, F \) are numerically eq. if \( E.C = F.C \) \( \forall \text{irr. curves } C. \)
We can also define the "Dual Cone" to \( \overline{\text{NE}}(X) \)

\[
\overline{\text{NE}}(X) = \{ [a_i] : [c_i] \text{ is the class of an irreducible curve and } a_i > 0 \}
\]

Notice that \( \overline{\text{NE}}(X) \) and \( \text{NEF}(X) \) are \underline{D\text{ual}} in the sense that \( \forall E \in \text{NEF}(X), \: \forall Y \in \overline{\text{NE}}(X) \quad E \cdot Y \geq 0. \) (irreducible + reduced)

Lemma: \( E \) is a \( ^g \text{Rational} \) \( (\equiv \mathbb{P}^1) \) \( (-1) \)-curve iff \( K_x \cdot E < 0. \)

\[ \text{Proof of Riemann-Roch} \]

\[ \underline{\text{So: If we want to write our N\text{ame} MMP in this new setting, then we want to look for } E \in \overline{\text{NE}}(X) \text{ s.t. } K_x \cdot E < 0. \text{ The key result is:} } \]

\[ \text{Thm (Cone Theorem)} (\text{in dim } 2). \]

Let \( X \) be a nonsingular projective surface. Then the closure \( \overline{\text{NE}}(X) = \overline{\text{NE}}(X) + \sum_{k \geq 0} R_k. \)

where

\[
\overline{\text{NE}}(X) = \{ z \in \overline{\text{NE}}(X) : K_x \cdot z \geq 0 \}
\]

and the \( R_k \) are half-lines such that \( R_{-103} \) are in \( \overline{\text{NE}}(X) \), and such that the \( R_k \) are of the form

\[ R_k = \overline{\text{NE}}(X) \cap L^\perp \text{ for some NEF line bundles } L. \text{ Moreover, the } R_k \text{ are discrete.} \]
Note: \( L = \{ z \in \mathbb{N} \mid \mathbf{x} : L \cdot z = 0 \} \).

**Thm.** (Contractum Theorem) (mod. \( n \)).

For each extreme Ray \( \mathcal{R}_e \), there exists an extreme contraction \( \phi : \mathcal{R}_e \to \mathbb{N} \), s.t.

\[ \phi(c) = p^t \iff [c] \in \mathcal{R}_e \text{ for any curve } c \in S. \]

**Examples:** (and Comparison w/ KRF).

1) Let \( X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(1)) \to \mathbb{P}^1 \). Then also:

\[ X = \mathbb{B} \mid \mathbb{P}^2 \to \mathbb{P}^2. \]

Consider the Divisors \( H = \mathcal{O}_{\mathbb{P}^2}(1) \), \( E = \text{Exc} \circ \mathcal{O}(1) \). The Intersection form on \( X \) is determined by:

\[ H \cdot H = 1, \quad H \cdot E = 0, \quad E \cdot E = -1, \]

by Adjunction. We have:

\[ K_X = -3H + E. \]

We also have the Fibre \( F = \mathcal{O}_{\mathbb{P}^1}(g) \).

Then \( F = H - E \) defines an effective curve class.

And \( F^2 = 0 \), \( F \cdot E = 1 \), \( F \cdot H = 1 \).

Clearly \( F, E \) generate the effective curve class.
So, the divisor \( aH + bE \) is NEF iff

\[
(aH + bE) \cdot H = b > 0 , \quad (aH + bE) \cdot E = -b > 0
\]

\[
(aH + bE) \cdot H - E = a + b > 0 .
\]

\[\Rightarrow \] \text{NEF}(x) \]

\[\Rightarrow \] \text{NE}(x)

\(1,2,3\) Evolution by KRF.

\[\text{Compute: } (i) -3H + E \cdot (H - E) = -3 + 1 = -2 < 0 . \]

\[\text{ (ii) } (-3H + E) \cdot E = -1 < 0 .\]

Applying the Cone Theorem: \( \text{NE}(x) = \text{NE}_{K_{\geq 0}}(x) + \text{Cone}(E) \).

But, \( \text{NE}_{K_{\geq 0}}(x) = \emptyset \), so \( \text{NE}(x) = \text{Cone}(E) \cap \text{NE}(x) \).

The rays are clearly \( H - E, E \), which generate \( \text{NE}(x) \).
The Contraction Theorem says we get maps

\[ \phi_E, \quad \phi_{H-E}. \]

Clearly

\[ \phi_E = \tilde{\pi} : \mathbb{P}^2 \to \mathbb{P}^2 \]

\[ \phi_{H-E} = \tilde{\pi} : \mathbb{P}^1 (\mathcal{O}(1)) \to \mathbb{P}^1 \]

What Does the KRF Say?

Recall that if \([c_wJ]\) is NEF, then we can solve the KRF to get \([w(4)]\), \([w(0)] = [c_wJ]\), and \([w] = -[c_1(x)] = K_x\).

Write: \( w_o = b_o H - a_o E \in \text{NEF}(X) \).

Case 1: \( b_o = 3a_o \), then \([w(4)] \to (0,0)\) and so \(X\) collapses to a pt.

Case 2: \( b_o > 3a_o \), then KRF exists until the path intersects the boundary of the NEF cone, at which point \( w_+ = \alpha H \). In particular \([w_+] \cdot E = 0\), so \(E\) has collapsed \(\Rightarrow X \to \mathbb{P}^2\). Thus, the KRF generates the map \(\phi_E\).

Case 3: \( b_o < 3a_o \): Then the KRF exists until the path intersects \(H-E\), so \([w] = \alpha (H-E)\) which has \(\alpha (H-E) \cdot (H-E) = 0\), so \(X \xrightarrow{\phi_{H-E}} \mathbb{P}^1\) is generated.
Note 1: For a rigorous pf see Song-Wenkave, KRF on Horzebruch Surfaces.

Note 2: The map generated is associated to the Extrema/ Ray $R_x$ where $R_x = \overline{NE}(x) \cap L^1$ for L the NEF line bundle defining the boundary of the cone.

Example 2: $X = P\left(\mathcal{O}(1) \oplus \mathcal{O}(-1)\right) \overset{\Pi}{\longrightarrow} \mathbb{P}^1$.

Two Divisors: $f = \Pi^{-1}(p)$ the class of a Florea.

$\exists = \left[\mathcal{O}(1)\right]$.

we have $f^2 = 0, \ \exists^2 = 0, \ \exists \cdot f = 1$. $\mathcal{K}_X = -2(f + \exists)$

Note that $0 \rightarrow \mathcal{O}(1) \hookrightarrow \mathcal{O}(1) \oplus \mathcal{O}(-1) \rightarrow \mathcal{O}(-1) \rightarrow 0$

and $\mu(\mathcal{O}(1)) = \frac{\deg(\mathcal{O}(1))}{\text{rk}(\mathcal{O}(1))} = 1 > \frac{\text{rk}(\mathcal{O}(1))}{\text{rk}(\mathcal{O}(1))} = \mu(\mathcal{O}(1) \oplus \mathcal{O}(-1))$

so $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ is unstable. The divisor $P(\mathcal{O}(1)) \subseteq X$ gives rise to an effective curve $(\exists - f)$

NB: This procedure is general for unstable Vector bundles see Lazarsfeld, Vol. 1 1.5.A.

$(\exists - f)^2 = -2$, so $\exists - f$ corresponds to the $(-2)$-curve
\[ X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \rightarrow X. \]

(See, e.g. Song-Wemkare: KRF or Horibechn Surfaces).

**NB:** The curve \( H \) is an example I also chose from the destabilizing sub-bundle \( \mathcal{O}(1)! \)

We compute:

\[ (-1,0) \quad f \quad \text{KRF} \quad (1,0). \]

\[ \text{Cone Theorem:} \quad \overline{\text{NE}}(X) = \left\{ a_1 \mathbb{Z} + a_2 \mathbb{Z} \in \text{NE}(X): -b > a_1 \mathbb{Z} + a_2 \mathbb{Z} \right\} \]

and \( \mathbb{Z} \in \text{NE}(X), k < 0. \)

So, there is a map \( \Phi, \) contracting \( \mathbb{Z}, \) and \( \Phi^1 = \pi: X \to \mathbb{P}^1. \)

The KRF exists until \( [\omega_1] = a_1 \mathbb{Z}, \) and \( \Phi^1 \text{KRF}. \)

By Song-Wemkare, the KRF contracts \( X \) to \( \mathbb{P}^1, \)

so produces the map given by Contracting the Ray dual to the intersection pt. of the Flow.
MMP in $Dm_c = 2$: (See Matsuki, p.41).

How do we study Minimal Models?

**Defn (Prop.)**
Let $X$ be a min. mod., $\dim X = 2$. Then $K_X \text{ NEF} \Rightarrow \Im K_X$ is base point free for $m \gg 0$ by the abundance Thm.

Thus, $\phi : \phi_{\Im K_X} : X \to X_{\text{can}}$ satisfies

(i) $\phi$ is a morphism with connected fibres onto a normal projective variety $X_{\text{can}}$.

(ii) For any curve $C \subset X$, $\phi(C) = \text{pt.} \iff K_X \cdot C = 0$

(iii) $\phi_{\Im K_X} \circ \phi^* X = \phi^* X$ for some ample $\alpha : X \to X_{\text{can}}$.

(iv) $\text{Kod}(X) = \dim X_{\text{can}}$.

**Abundance Thm**: $X$ has $\dim X = 2$, $K_X \text{ NEF}$, then $\Im K_X$ is base point free for $m \gg 0$.

**Higher Dimensions**:

Things change: we now have Singularities + Flips!
The Higher Div're analog of the Cone/Contraction Theorem produces maps of type

(a) $\phi : X \to Y$, $\dim X = \dim Y$, $\text{Cod}_{x} (\text{Ex}(\phi)) = 1$
   "Contracting a Divisor"

(b) "
   \underline{Flipping} type
   $\text{Cod}_{x} (\text{Ex}(\phi)) > 2$

(c) $\dim X > \dim Y$: Flopping type.

Example of a Flip:

Let $E = \mathcal{O}_{\mathbb{P}^{1}} \oplus \mathcal{O}_{\mathbb{P}^{1}} (-1) \oplus \mathcal{O}_{\mathbb{P}^{1}} \oplus \mathcal{O}_{\mathbb{P}^{1}} (-2) \to \mathbb{P}^{1}$, and let $X = \mathbb{P}(E) \to \mathbb{P}^{1}$

Take $P_{0} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{1}}) \subset X$, curve. Let $D_{00} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{1}} (-1))$
and $L_{00}$ be the line bundle associated to $D_{00}$.

Claim: $L_{00}$ is base pt. free, and $L_{00}$ is trivial.

Now take $L_{00} \to X \to Y$. Then $\phi (P_{0}) = \text{pt}$.
**Singularity:**

**Defn:** Let $X$ be a normal projective variety s.t. $K_X$ is a $Q$-Cartier divisor. Let $\hat{T}: \tilde{X} \to X$ be a resolution, and $\{E_i\}_{i \in I}$ the irreducible components of the exceptional locus of $\hat{T}$. There exists a unique collection $\alpha_i \in \mathbb{Q}$ s.t.

$$K_{\tilde{X}} = \hat{T}^* K_X + \sum_i \alpha_i E_i$$

Then $X$ is said to have:

(i) **Terminal singularities** if $\alpha_i > 0$, $\forall i$

(ii) **Canonical singularities** if $\alpha_i \geq 0$, $\forall i$

(iii) **log terminal singularities** if $\alpha_i > -1$, $\forall i$

(iv) **log canonical singularities** if $\alpha_i \geq -1$, $\forall i$.

**Example:** Consider the case $\{uv - w^2 = 0\}$ in $\mathbb{C}^3$, which is singular at $0 \in \mathbb{C}^3$. Let $C \subseteq \mathbb{P}^2$ be the curve defined by the case. We desingularize $X$ by taking

$$Y = \{\text{Total space of } \mathcal{O}(41)|_C \}.$$ The complement of the zero section is isomorphic to $X \setminus S_3$, and the zero section maps to $S_3$. Moreover $C \equiv \mathbb{P}^1$. By looking in local coordinates, it is easy to check that $K_Y|_C = \mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$. Thus

$$K_Y = \hat{T}^* K_X$$

and in this case, the singularity is canonical.
This result is born out by the

Theorem (Kollár et al.)

Let \( O \in X \) be a normal surface singularity over \( \mathbb{C} \). Then

\[ X \text{ is: } \quad \text{terminal } \iff \text{ smooth} \]

\[ \text{Canonical } \iff \mathbb{C}^2 / (\text{finite subgroup of } \text{SL}(2, \mathbb{C})) \]

\[ \text{Log Canonical } \iff \text{simple elliptic, cusp, smooth or a quotient of these by a finite group.} \]

Our example corresponded to \( \mathbb{C}^2 / (\mathbb{Z}/2\mathbb{Z}) \) where

\[ \mathbb{Z}/2\mathbb{Z} \cong \text{SL}(2, \mathbb{C}) \text{ is } \left\{ (-1, 0), (0, 1) \right\}. \]

Rmk: \( \mathbb{C}^3 / \left\{ \mathbb{Z}/2\mathbb{Z} \right\} \in \text{SL}(3, \mathbb{C}) \text{ is terminal!} \)

This is the case over the Veronese embedding of \( \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 \).

We can also formulate this analytically. If \( X \) is a cplx

mfd, and \( D_1, \ldots, D_n \) are divisors with local equations \( f_i \), then \( (X, \sum_i D_i) \) is log canonical if \( (\prod_i f_i)^{1-\varepsilon} \)

is locally \( L^2 \forall \varepsilon > 0 \). This is more related to singularities of \( \text{varieties in } \mathbb{P}^n \). (In Singularities in Codim 1).