Lecture 2: The Kähler-Ricci Flow

and the \( \bar{\partial} \) operator on Vectorfields.

**Intro**: This paper is, in some sense, a continuation of Lecture 1. In this paper we \textbf{DO NOT} assume the uniform Boundedness of the Rm tensor. The trade off is that we lose the Kähler-Gromov Cptness Theorem, and have to assume instead that the Smallest pos. eigenvalue \( \lambda_t \) is not degenerating along the flow. \textbf{Setup}: \( (X, \omega_0) \) compact, Fano.

Recall: NKRF

\[
\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= g_k j - R_{kij} = -\partial_j \partial_k \omega, \\
g_{ij} (0) &= (g_0), k_j \in \text{II} C_1 (x)
\end{aligned}
\]

Mabuchi: K-energy: Defined by it's value at a point, and it's variation \( \delta M (q) = -\frac{1}{V} \int_X \delta \varphi (R - h) w^n_x, \quad V = \int_X w^n_x = \text{II} C_1 (x) \)

Recall: In EPS. Stability + Convergence Lecture 1 J, it was shown that, if \( M > -C > -\infty \) on \( C_1 (x) \), then the function

\[
Y(t) = \int_X w^n_t \quad \text{has} \quad Y(t) \to 0 \quad \text{as} \quad t \to \infty.
\]

Moreover, we computed that
\[ \dot{y}(t) = (n+1) y(t) - \int x \nabla h_1^2 R \, d^n w - \int \nabla \Delta h_1^2 \, d^n w \]

In particular \[ \int_0^\infty \int x \nabla h_1^2 w \, dt = y(0) + (n+1) \int_0^\infty y(t) \, dt - \int_0^\infty \int x \Delta h_1^2 w \, dt \]

Now \( M_{ab} > -C \Rightarrow \left| \int y(t) \right| < \infty \), and Perelman's bound

for \( R \Rightarrow \) the RHS is finite, and so \[ \int_0^\infty \int x \nabla h_1^2 w \, dt < \infty \]

Thus \[ \| R - h \|_2 \to 0 \quad \text{as} \quad t \to \infty \]. In this paper,

Theorem 1: Assume that the Mabuchi K-energy is bounded

From below on \( \operatorname{MCG}(x) \). Let \( g_{\tilde{y}}(t) \) be any solution of the Kähler-Ricci flow, and let \( R(t) \) be the scalar curvature of \( g_{\tilde{y}}(t) \). Then

(i) \[ \| R(t) - h \|_2 \to 0 \quad \text{as} \quad t \to \infty \]

(ii) \[ \int_0^\infty \| R(t) - h \|_p^p \, dt < \infty \quad \forall \ p > 2 \].

Theorem 2 is an adaptation of the Result in Lecture 1
To the case when \( |R_{mm}| \) is NOT uniformly banded.

Theorem 2: Suppose we have a sol'n of the KRF, \( w(t) \) the Kähler Forms. Let \( \lambda \) be the smallest positive eigenvalue of \(-\bar{\partial} \partial \), acting on \( T^{1,0} X \).

(i) If \( \inf_{w \in \operatorname{MCG}(x)} M(w) > -\infty \)

(ii) \[ \| \]
Then the $g_{k,j}$ converge exponentially fast in $C^\infty$ to a KE.

(iii) If $g_{k,j}(t) \xrightarrow{C^\infty} g$ to a KE metric, then (A) and (S) are satisfied.

(iii) In particular, all convergence is exponential.

Before beginning the proof, we recall Perelman’s Results.

(i) if $u$ is normalized by $\frac{1}{V} \int_X e^{-u} w^n = 1$, then:

(i) $\exists C_0$ depending only on $g_{k,j}(0)$ s.t.

$$\|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R\|_{C^0} \leq C_0$$

(ii) Let $p > 0$ be given. $\exists c > 0$ depending only on $g_{k,j}(0)$ and $p$ s.t. $\forall x \in X$, and $t \geq 0$ and $r \in (0, p]$, we have:

$$\int_{B_r(x)} w^n > c r^{2n}$$

where $B_r(x)$ is the geodesic ball of radius $r$ centered at $x$, wrt $g(t)$.

The Smoothing Lemma:

Lemma 1: $\exists \delta, K > 0$ depending only on $n$ s.t. $\forall 2 \in (0, \delta]$ and any $t_0 \geq 0$, if $\|u(t)\|_{C^0} \leq \Sigma$, then

$$\left\| \nabla u(t_{t+2}) \right\|_{C^0} + \left\| R(t_{t+2}) - n \right\|_{C^0} \leq K \Sigma.$$
Key Point: if $\|u\| \to 0$, then $\|\nabla u\|, \|\Delta u\| \to 0$.

Proof: wlog, assume $t_0 = 0$. We computed the evolution equation

For $u$, $\partial_t \partial_k u = \partial_t \left( g^k_j - R^k_j \right) = \partial_j \partial_k g + \partial_j \partial_k g \partial_k l$

\[ \Rightarrow \partial_{\eta} \partial_k u + \partial_j \partial_k \Delta u \max_{\text{prc.}} \begin{cases} \hat{u} = \Delta u + u - b \end{cases} \]

where $b = b(t)$ is defined by $b = \frac{1}{\nu} \int_{\mathbb{R}} x e^{-u} \, dx$. (This is the necessary normalization for Perelman's results.)

In order to make this not depend on $b$, we define $c(t)$ by $c = c + b$, $c(0) = 0$. Now set $\hat{u}(t) = u(t) - c(t)$

we have $\|\hat{u}(t)\| \leq \varepsilon$, and $\partial_t \hat{u} = \dot{u} - \dot{c} = \Delta u + u + b - c - b$

\[ \Rightarrow \hat{u} = \dot{u} + \dot{c} = \Delta u + u + b - c - b \]

\[ \Rightarrow \Delta \hat{u} + \dot{u} = \Delta u + u + b - c - b \]

We compute:

\[ \begin{align*}
\frac{\partial u}{\partial t} &= 2 \hat{u} \dot{u} = 2 \hat{u} \left( \Delta u + \hat{u} \right) = (\Delta \hat{u})^2 - 2 |\nabla \hat{u}|^2 \\
\frac{\partial}{\partial t} |\nabla \hat{u}|^2 &= \partial_j \partial_k g \partial_j \partial_k u + \partial_j g \partial_j g \partial_k \partial_k u - \partial_j \partial_k g \partial_k l \\
&= \partial_j \partial_k g \partial_j \partial_k u + \nabla \hat{u}^2 + \partial_j g \partial_j \hat{u} \Delta u + 1 |\nabla \hat{u}|^2 \\
&\quad - \partial_j \partial_k g \partial_j \partial_k u - \partial_j g \partial_j \partial_k \hat{u} \Delta u + 1 |\nabla \hat{u}|^2
\end{align*} \]

\[ \Delta |\nabla \hat{u}|^2 \text{ Norm.}_{\text{coord.}} = \int_{\mathbb{R}^n} \partial_j \partial_k g \partial_j \partial_k u - \partial_j \partial_k g \partial_k l \]

\[ = \int_{\mathbb{R}^n} \partial_j \partial_k g \partial_j \partial_k u - \partial_j \partial_k g \partial_k l \]

\[ + \partial_j \partial_k g \partial_j \partial_k u + \partial_j \partial_k g \partial_j \partial_k u \]
\[ F_2 = (\Delta \hat{u}) g^{jk} \partial^k \hat{u} + \nabla \nabla \hat{u}^2 + \nabla \nabla \hat{u}^2 + \partial_j \hat{u} \partial_j g^{jk} \partial^k \hat{u} + \partial_j \hat{u} j^k \partial^k \hat{u} \]

Thus \( (\frac{\partial}{\partial t} - \Delta) |\nabla \hat{u}|^2 = 2|\nabla \hat{u}|^2 - |\nabla \nabla \hat{u}|^2 - \nabla \nabla \hat{u}^2 - F_1 - F_2 \)

\[ \partial_j \hat{u} j^k \partial^k \hat{u} = -2 (g^{jm} \partial_j \partial^m \hat{u} ^k + g^{jm} g^{rk} R_{mr}) \]

Thus \( -g^{jm} \partial^m \hat{u} ^k = g^{jm} g^{rk} R_{mr} \)

Thus \( F_2 = \partial_j \hat{u} \partial^j \hat{u} [ -g^{jm} g^{rk} \partial^m \hat{u} ^k ] + |\nabla \hat{u}|^2 = -F_1 + |\nabla \hat{u}|^2 \)

\( (\frac{\partial}{\partial t} - \Delta) |\nabla \hat{u}|^2 = |\nabla \hat{u}|^2 - |\nabla \nabla \hat{u}|^2 - |\nabla \nabla \hat{u}|^2 \)

\[ \frac{\partial}{\partial t} \hat{u} = \Delta \hat{u} + \Delta \hat{u} + |\nabla \nabla \hat{u}|^2 \]

(i) From (1) \( \frac{d(\hat{u})^2}{dt} \mid_{\text{max}} \leq 2(\hat{u})^2 \mid_{\text{max}} \implies \frac{\partial}{\partial t} (\hat{u})^2 \leq 0 \).

Thus \( (\hat{u})^2 \mid_{\text{max}} \leq \epsilon_2 \) on \([0,2]\) \( \Rightarrow ||\hat{u}(t)|| \leq \epsilon_2 \) for \( t \in [0,2] \).

(ii) \( \frac{\partial}{\partial t} (\epsilon^2 (\hat{u}^2 + t|\nabla \hat{u}|^2)) < \Delta (\epsilon^2 (\hat{u}^2 + t|\nabla \hat{u}|^2)) \)
Putting the minimum in,

\[ e^{-2t} \left( \| u \|^2 + t |\nabla u|^2 \right) \leq \varepsilon^2 \Rightarrow |\nabla u|^2 \leq \varepsilon^2 e^{-4t} \quad \forall t \in [1/2], \]

Using similar techniques we can show:

\[ |\Delta u|^2 (t) < 2ne^5 \varepsilon \quad \text{at} \ t = 2. \]

Recall, \( b = \int_X u e^{-u^*} \). In order to prove part (i) of Thm 1 it suffices to show that \( \Delta u \rightarrow 0 \). By Lemma 1, it is enough to show \( \| u \|_{C^0} \rightarrow 0 \). To do this, it suffices to show that \( b \), and \( \| u - b \|_{C^0} \rightarrow 0 \) as \( t \rightarrow \infty \).

Lemma: (Poincaré Inequality for the measure \( e^{-u^*} \))

\[
\int_X f^2 e^{-u^*} < \int_X |\nabla f|^2 e^{-u^*} + \left( \int_X f e^{-u^*} \right)^2
\]

\[ \forall f \in C^0(X). \]

pf:

Consider the elliptic operator \( L := -g^i_j \nabla_i \nabla_j u + g^i_j \nabla_i u \nabla_j u \).

Observe that \( L \) is elliptic, and self-adjoint with the inner product

\[ \langle \psi, \phi \rangle = \int_X \psi \phi e^{-u^*} \] for \( \psi, \phi \in C^0(X). \) Thus \( L \) has all real, non-negative eigenvalues. Note that if \( f \) is an eigenfunction with \( \lambda \) its eigenvalue, then

\[ \int_X |\nabla f|^2 e^{-u^*} = \langle Lf, f \rangle = \lambda \int_X f^2 e^{-u^*}. \]

So \( L \) has all real, non-negative eigenvalues, and it's eigenfunctions are orthogonal.
Show that if \( f \in \ker L \) (i.e., \( \int f e^{-\omega} = 0 \)), then \( \lambda > 1 \). To do this, we use Bernstein's trick. Differentiating the equation gives (wrt. \( \nabla_x \)):

\[
-g^j k \nabla_j \nabla_k f + j \nabla_j \nabla_k u \nabla_j f + g^j k \nabla_j u \nabla_k f.
\]

Now, multiply by \( g^m \nabla_m f \). We get:

\[
-g^j k \nabla_j \nabla_k f + j \nabla_j \nabla_k u \nabla_j f + g^j k \nabla_j u \nabla_k f \nabla_m f g^m + j \nabla_j \nabla_k f \nabla_m f g^m + j \nabla_j \nabla_k f \nabla_m f g^m.
\]

\[
= -g^j k \nabla_j \nabla_k f + R_{x^j k} \nabla_j f + ... \]

Now multiply by \( g^m \nabla_m f \) and integrate:

\[
\int g^j k \nabla_j \nabla_k f g^m \nabla_m f + g^m k \nabla_j \nabla_k f \nabla_j f + g^m k \nabla_j \nabla_k u \nabla_j f \nabla_m f g^m \nabla_k f
\]

\[
+ \int g^j k \nabla_j u \nabla_k f g^m \nabla_m f \nabla_j f e^{-\omega} = \lambda \int g^m \nabla_m f \nabla_j f e^{-\omega} \]

Integrate by parts in the 1st term. When \( \nabla_j \) lands on \( e^{-\omega} \), the result is killed by \( \nabla \). Also \( \nabla \nabla_j u = g^j \nabla_j f - R_{x^j k} \nabla_k f \), so
\[
\int_\mathbb{H} \nabla \tilde{f} \cdot e^{-u} w^n + \int_\mathbb{H} \nabla f \cdot e^{-u} w^n = \lambda \int_\mathbb{H} \nabla f \cdot e^{-u} w^n
\]

Thus \( \lambda > 1 \). Now in general, given \( f \in C^\infty(\mathbb{H}) \), write

\[
\tilde{f} = f - \frac{1}{V} \int_\mathbb{H} f e^{-u} w^n.
\]

Then \( \tilde{f} \perp \ker L \) wrt \( \langle \cdot, \cdot \rangle_u \).

Thus \( \tilde{f} \) is in the span of the eigenfunctions, and we have

\[
\frac{1}{V} \int_\mathbb{H} \nabla \tilde{f} \cdot e^{-u} w^n \leq \frac{1}{V} \int_\mathbb{H} \nabla f \cdot e^{-u} w^n = \frac{1}{V} \int_\mathbb{H} f e^{-u} w^n - \left( \frac{1}{V} \int_\mathbb{H} f e^{-u} w^n \right)^2
\]

Since \( \nabla \tilde{f} = \nabla f \), we're done \( \Box \).

Lemma 3: we have:

(i) \( 0 < -b \leq \|u - b\|_c \)

(ii) \( \|u - b\|^\infty \leq C \|u\|_{L^2} \|u\|_c \)

Proof: Note that the measure \( e^{-u} w^n \) has unit mass, and so we can apply Jensen's Formula. In particular

\[
b = \frac{1}{V} \int \nabla u \cdot e^{-u} w^n \leq \log \left( \frac{1}{V} \int e^u e^{-u} w^n \right) = 0.
\]

Moreover, \( \int e^{-u} w^n = 1 \) by assumption. In particular, \( \sup_x u > 0 \)

Thus

\[
0 < -b \leq \sup_x (u - b) \quad \text{proving (i)}.
\]
To prove (ii) set \( A = \sup_x |u-b| = |u-b|(x_0) \). Since \( U \) is \( C^\infty \) and \( r > 0 \) st. \( |u-b| > \frac{A}{2} \) on \( B_r(x_0) \). Let \( \rho \) be as in Perelman's non-collapsing result. Then, clearly we can take \( r = \frac{A}{2\nu \| U \|_{C^0}} \). If \( r < \rho \), then

\[
\int_{B_r(x_0)} (u-b)^2 \geq \left( \frac{A}{2} \right)^2 r^{2n} = \frac{A^2}{4} c \frac{A^n}{r^{4n}} \geq \frac{A^{2n+2}}{2^n \| U \|_{C^0}^{2n}} \geq \frac{A^{2n+2}}{\nu \| U \|_{C^0}^{2n}}.
\]

Note: Here we have used Perelman's non-collapsing result so that \( c \) is independent of time.

Thus,

\[
\| u-b \|_{L^\infty} \leq C \| u-b \|_{L^2} \| U \|_{C^0} \leq C \| U \|_{C^0} \| U \|_{L^2} \quad \text{(see below)}
\]

Now apply Lemma 2: we have:

\[
\int_{x(p)} (u-b)^2 \frac{w^n}{V} \leq C_2 \int (u-b)^2 \frac{w^n}{V} \leq C_3 \int \frac{\| U \|_{C^0}^2 w^n}{V} \leq C_4 \int \frac{\| U \|_{C^0}^2 w^n}{V}.
\]

If \( r > \rho \), then integrate over \( B_\rho \) to get an even stronger result.

Rule: (i) Perelman's non-collapsing estimate is crucial

(ii) we probably didn't need the full strength of Lemma 2

(iii) If \( Mab > -C \), then we know \( \| U \|_{L^2} \to 0 \).
the pf of part (i) of theorem (i).

Proof of Theorem 1 part (ii) and Theorem 2.
we begin by proving that Stability (A) and (S) yield exponential convergence, which is Part (i) of Thm 2. We do this by proving 2 Lemmas:

Lemma 5: if $\lambda > -c > -\infty$, and $\lambda_t > \lambda > 0$, then there exist $\mu, c > 0$, independent of $t$, such that

(i) $\|v(t)\|_{L^2(t)} \leq C e^{-\mu t}$

(ii) $\|u(t)\|_{C^0} + \|\nabla u(t)\|_{C^0} + \|R-N\|_{C^0} \leq C e^{-\frac{t}{2(1-n)}} \mu t$

(i) is obviously necessary. It turns out that (ii) is sufficient.

Lemma 6: Assume $R(t)$ has $\int_0^\infty |R-N| dt < \infty$. Then the KRF converges exponentially to a KE metric.

We first prove Lemma 6. The idea is that the integrability of $|R-N|_{C^0}$ implies a uniform bound for $\Phi$ (the Kähler Potential).
Let \( f_{j+1} = (f_j)_{j+1} + \Delta f_{j+1} \). Where we normalize \( \mathbf{q} \) by

\[
\mathbf{q} = \log \left[ \frac{\det (g_0 + \Delta \mathbf{q})}{\det (g_0)} \right] + \mathbf{q} + \mathbf{u}(\mathbf{t}) \quad \mathbf{q}(0) = \mathbf{c}_0
\]

where \( \mathbf{c}_0 = \int_0^\infty \left( -t \| \nabla \mathbf{q} \|_L^2 \right) dt + \frac{1}{V} \int \mathbf{u}(\mathbf{t}) \omega_0^n \)

(Why this is a good choice is taken up in Phong, Storm, Sisum which will be covered later C? ).

Then we have \( \mathbf{q} = \mathbf{q} - \log \left[ \frac{\det (g_0 + \Delta \mathbf{q})}{\det (g_0)} \right] - \mathbf{u}(\mathbf{t}) \)

Penelma’s Band: For \( \mathbf{u} \) implies that \( \mathbf{q} \) is bounded.

Thus, it suffices to bound the 2nd Term. But we have:

\[
\left| \left[ \log \frac{\omega^n}{\omega_0^n} \right] \right| \leq \left| \int_0^t \frac{d}{dt} \left( \log \frac{\omega^n}{\omega_0^n} \right) dt \right| = \left| \int_0^t \frac{\partial}{\partial t} \left( \log \frac{\omega^n}{\omega_0^n} \right) dt \right| = \left| \int_0^t \left( \frac{\partial}{\partial t} \right) \left( \frac{\omega^n}{\omega_0^n} \right) dt \right| = \left| \int_0^t \mathbf{R} \cdot \mathbf{n} dt \right| < C < \infty \quad \text{where } C \text{ is independent of time.}
\]

Thus \( \left| \mathbf{q} \right| \leq C \quad \text{independent of time. we now have:} \)

\[
\left| \mathbf{q} \right| \leq C_0 + \left| \mathbf{q} \right| + \left| \nabla \phi \right| + \left| \Delta \phi \right| < C \quad \text{independent of time.}
\]
Lemma ([PSSJ], 2.4)
with our choice of $c_0$, we have:

\[
\sup_{t \geq 0} \| \varphi \|_{c_0} \leq A_0 < \infty \quad \Longleftrightarrow \quad \sup_{t \geq 0} \| \varphi \|_{c_k} \leq A_k < \infty
\]

\[\forall k \in \mathbb{N}.\]

This is essentially the parabolic analogue of Yau-Aubin estimates.

Once we have uniform boundedness of the $\| \varphi \|_{c_k}$, we know that the metrics $g^{t}_{k_j}(t)$ are uniformly bounded. By Arzela-Ascoli, we can find times $t_m \to \infty$ s.t. $g(t_m)$ converge in $C^\infty$ and $g^{t}_{k_j}(t_m)$ converge in $C^\infty$. Observe that, the metrics $g^{t}_{k_j}(t_m)$ are uniformly equivalent.

\[
\exists C \text{ s.t. } 1 \leq g^{t}_{k_j}(t_m) \leq (g^{t}_{k_j})_{k_j} \leq C g^{t}_{k_j}(t_m)
\]

Now, uniform equivalence + convergence in $C^\infty$ for $g^{t}_{k_j}(t_m) \Rightarrow$ we have uniform curvature bounds for all derivatives.

Now, the assumption $\Rightarrow$ $|\Delta U|_{C^0} \to 0$ as $t_m \to \infty$. But uniform equivalence $\Rightarrow$ this is also true for $|\Delta_0 U|_{C^0}$. By the maximum principle $U(t) \to \text{const.}$

But $\int_{X} e^{-U} w = 1 \Rightarrow U \to 0$ as $t \to \infty$.

Thus $R_{k_j}^\infty - g_{k_j}^\infty = 0 \Rightarrow \varphi(\infty)$ is a potential for
Claim: The eigenvalues \( \lambda_t \geq \lambda > 0 \) as \( t \to \infty \).

Proof: Suppose not. Then \( \exists t_k \) s.t. \( \lambda_{t_k} \to 0 \). Extract a further subsequence (not relabeled) s.t. \( g(t_k) \to \) to KE metric. Note that the geometry is uniformly controlled along this sequence:

- Curvature control comes from \( g(t_k) \) converging in \( C^\infty \) and uniform equivalence.
- Inj. radius control follows from curvature control and Perelman's uniform diameter bound.

So: we can apply Kähler-Gromov compactness in the special case when \( J \) is fixed. (In particular, notice that in this case, the diffeomorphism group acting on \( J \) is trivial, so we have stability (B) trivially along the subsequence, and so we can apply the results of Lecture 1).

Thus \( 0 = \lim_{k \to \infty} \lambda_{t_k} = \lambda(g_{\infty}) > 0 \).

Now, since we know a KE metric exists, we know that \( Mab \geq -c > \infty \), and so Lemma 5 \( \Rightarrow ||Dull(4)|| \to 0 \) exponentially.
we now know that the eigenvalue $\lambda_t$ is not degenerating and that $|R_{\mu\nu}|$ is uniformly held along the flow. We can thus apply the same arguments as in Lecture 1 to show that $\|\nabla u\| \rightarrow 0$ and hence

$$H^{(5)}(w) \leq \frac{1}{c} \text{ we have uniform equivalence}$$

$$\|\nabla u\| \rightarrow 0 \quad \text{For any } k \text{ by Sobolev imbedding.}$$

But

$$\left| g_{ij} e_j \right| \leq \left| R_{\bar{k}j} - g_{\bar{k}j} \right| c_k \rightarrow 0$$

exponentially, and so the whole sequence converges.

we now prove Lemma 5:

recall from Lecture 1, we computed that

$$Y(\tau) \leq -2 \lambda_t Y - 2 \lambda_t \text{ Fut}(\Pi_t(\nabla u)) - \int_{\Omega} |\nabla u|^2 \langle R, n \rangle w^n$$

$$- \int_{\Omega} \nabla u \cdot \nabla^2 u (R_{\bar{k}j} - g_{\bar{k}j}) w^n.$$

For our case $1 = 0$ since $\lambda \phi > -c > \infty$, 2 we can control by Part (i) of Thm 1 (for $t$ large $\leq \frac{\lambda}{2} \int |\nabla u|^2 w^n$)

However, 3, we have no control. Previously (Lec. 1) we were able to control 3 by $c$ we assumed uniform curvature bounds. In the absence of this assumption we need to work harder.
Claim: \( \exists K_0 > 0 \text{ s.t.} \)

\[
\dot{Y}(t) \leq -\lambda Y(t) + \frac{\lambda}{2} Y^2(t) \cdot \prod_{j=1}^{N} \left[ Y(t-a_j) \right]^{\frac{\delta_j}{2}} \quad \forall t \geq K_0
\]

where \( N, a_j \in \mathbb{N} \), \( \delta_j \in \mathbb{R}^+ \), \( \sum_{j=1}^{N} \delta_j = 1 \).

Proof:

We want to estimate

\[
\left| \int \nabla u \nabla \tilde{u} \left( E_{k_j} - g_{k_j} \right) \right|
\]

\[
\leq \|u\|_{C^0} \left( \int |u|^2 \right)^{\frac{1}{2}} \left( \int |E_{k_j} - g_{k_j}|^2 \right)^{\frac{1}{2}}
\]

Now

\[
\int |E_{k_j} - g_{k_j}|^2 = \int |E_{k_j} - g_{k_j}|^2 \omega^n \overset{\text{IBP}}{=} \int |u|^2 \omega^n = \int |R-n|^2 \omega^n
\]

\[
\left( \int |R-n|^2 \omega^n \right)^{\frac{1}{2}} \leq \sup_x |R-n| \leq K \sup_x |u| (t-2)
\]

\[
\leq K \sup_x |u-b| (t-2) + |b| (t-2)
\]

Lemma 3

\[
\leq C \left\| \nabla u \right\|_{(t-2)} \left\| \nabla u \right\|_{(t-2)}^{\frac{1}{2}}
\]

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\]

So, we have shown that:

\[
\left| \int \nabla u \nabla \tilde{u} \left( E_{k_j} - g_{k_j} \right) \right| \leq C \left\| \nabla u \right\|_{(t-2)} \left\| \nabla u \right\|_{(t)}^{\frac{1}{2}} \left\| \nabla u \right\|_{(t-2)}^{\frac{1}{2}}
\]

Note moreover, the last 3 inequalities are general.
and show that \[ \| \nabla u \| (t) \leq C \| u - b \| (t-2) \]

\[ \leq C_2 \| \nabla u \|_{C^0} \| u \|_{L^2} \quad (\forall t) \]

By iterating this inequality, we can transfer weight from the \( C^0 \) norm to the \( L^2 \) norm.

\[ \text{Let } g(t) = \prod_j A(t - a_j j) \prod_k B(t - b_k k) \quad \text{where} \]

\[ A(t) = \| \nabla u \|_{L^2} \quad B(t) = \| \nabla u \|_{C^0} \quad \text{Note that, initially} \]

\[ \sum_j a_j = \sigma \quad \sum_k b_k = \Sigma \quad \text{have } \sigma + \Sigma = 2 \quad (1 + \frac{n}{n+1} + \frac{1}{n+1} = 2) \]

\[ a_j, b_k > 0 \quad \text{Then, iterating (X) shows} \]

\[ g(t) \leq C \tilde{g}(t) \quad \text{where } \tilde{g}(t) = \prod_j A(t - \tilde{a}_j j) \prod_k B(t - \tilde{b}_k k) \]

\[ \text{and } \tilde{a} = \sum_j \tilde{a}_j \quad \tilde{b} = \sum_k \tilde{b}_k \quad \text{have } \tilde{\sigma} + \tilde{\Sigma} = 2. \]

\[ \text{Ex:} \]

\[ \| \nabla u \|_{C^0} \| \nabla u \|_{L^2} \| \nabla u \|_{C^0} \| \nabla u \|_{L^2} \leq \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} \]

\[ \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} \]

\[ \text{Note: that } \tilde{\sigma} + \tilde{\Sigma} = 1 \text{ is clear by scaling.} \]

\[ \text{Note that every iteration increases } \tilde{\sigma}, \text{ and so eventually we will have } \tilde{\Sigma} < 1. \text{ Then, set } \tilde{a}_j = \tilde{\sigma} \tilde{\sigma}_{\tilde{a}_j}, \text{ and we get} \]
\[ \| v \|_{L^2(t_0)} + \| v \|_{L^2(t_2)} \leq H(t) \prod_{j} A(t - \tilde{a}_j) \delta_j \]

where \( H(t) = \sum_{k} B(t - b_k) \prod_{j} A(t - \tilde{a}_j) \delta_j \).

Now by construction \( \delta_j - \delta_j > 0 \). Thus, by Realman's bound \( B \) is banded along the flow, and by Lemma 4, we see \( H(t) \to 0 \) as \( t \to \infty \). Thus, we find for \( k_0 \) large such that

\[ \dot{Y}(t) \leq -\lambda Y(t) + \frac{\lambda}{2} \gamma_{\frac{1}{2}}(t) \prod_{j=1}^{N} \gamma_{\frac{1}{2}}(t - a_j) \delta_j \quad \forall t \geq k_0 \]

Let \( F(t) = Re^{-\mu t} \) for some \( R, \mu > 0 \) TBD.

Claim: \( Y(t) \leq F(t) \)

Proof: Suppose not, choose \( R \) large such that \( Y(t) < F(t) \) for \( t < t_0 \) and \( Y(t_0) = F(t_0) \). Then we can assume that

\[ \dot{Y}(t_0) \geq F(t_0) = -\mu F(t_0) \] By our previous inequality

\[ -\mu F(t_0) \leq -\lambda Y(t_0) + \frac{\lambda}{2} \gamma_{\frac{1}{2}}(t_0) \prod_{j} \gamma_{\frac{1}{2}}(t_0 - a_j) \delta_j \]

\[ -\mu F(t_0) \leq -\lambda F(t_0) + \frac{\lambda}{2} F(t_0) \prod_{j=1}^{N} F(t_0 - a_j) \]

\[ -\mu F(t_0) \leq -\lambda F(t_0) + \frac{\lambda}{2} F(t_0) \prod_{j} F(t_0 - a_j) \delta_j \]

\[ -\mu F(t_0) \leq -\lambda F(t_0) + \frac{\lambda}{2} F(t_0) \prod_{j=1}^{N} F(t_0 - a_j) \delta_j \]

\[ -\mu F(t_0) \leq -\lambda F(t_0) + \frac{\lambda}{2} F(t_0) \prod_{j} F(t_0 - a_j) \delta_j \]

\[ -\mu F(t_0) \leq -\lambda F(t_0) + \frac{\lambda}{2} F(t_0) \prod_{j=1}^{N} F(t_0 - a_j) \delta_j \]
\[-\lambda Re^{\mu t_0} + \frac{\lambda}{2} Re^{\frac{\mu t_0}{2}} e^{\frac{\mu}{2} (t_0 - a_j) \delta_j} = -\mu e^{\mu t_0} \Rightarrow -\lambda Re^{\mu t_0} + \frac{\lambda}{2} Re^{\frac{\mu t_0}{2}} + \mu e^{\frac{\mu}{2} a_j \delta_j} \Rightarrow -\mu e^{\mu t_0} \Rightarrow \]

By choosing \( \mu \) sufficiently small (\( a_j, \delta_j \) depend only on \( n \)) we can ensure this does not hold. Thus, \( y(t) \leq F(t) \) and so exponential decay follows.

Lemma 3 \( \Rightarrow \) \( \|u\|_{c^0} \) decays exponentially, and finally

Lemma 1 \( \Rightarrow \) \( \|R-n\|_{c^0}, \|u\|_{c^n} \) decay exponentially.

The proof of Theorem 1 part (ii) follows in much the same way. Theorem 2 parts (ii) and (iii) follow in a similar fashion as Lemma 6.