Introduction to the KRF, a.c.s

Perelman’s Functional.

Outline:
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(3) The Entropy Functional
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   (4.3) Uniform Sobolev Embedding for KRF.

(1) Intro. Without loss of generality, we assume $[\omega_0] = C_1(x)$ for $k \geq 1$.

The K-Ricci Flow is defined by
\[
\frac{d g_{ij}}{d t} = -Ric_{ij}
\] (1)

This is unnormalized. The normalized RF is
\[
\frac{d g_{ij}}{d t} = Kg_{ij} + (t) - Ric_{ij} \]
\[ g_{ij}(0) = (g_0)_{ij} \] (N1)

\[
9_{ij}(0) = (9_0)_{ij}
\]

\[ 9_{ij}(t) = \ldots \]
we can transform a solution to the RF to a solution of the RF on \([0, T]\) (where \(T\) is the max. existence time) by setting \(\tilde{g}(t) = (T-t)g\).
Let \((X,\omega)\) be a Kähler mfd. \(\dim_{\mathbb{C}} X = n\)

The Ricci flow is defined by
\[ \begin{aligned}
\frac{dg_{ij}}{dt} &= -R_{\bar{k}j}^i \\
g_{ij}^{(0)} &= (g_0)_{ij} \\
\end{aligned} \tag{1} \]

if \(\rho\) is the Ricci Form, then the flow is defined on forms by
\[ \begin{aligned}
\frac{dw}{dt} &= -\rho \quad \text{(1')} \quad \text{Suppose we knew that the flow existed on } [0,\varepsilon), \text{ and that } w(t) \text{ was a closed, positive, } \Omega \text{ Form } \forall t \in [0,\varepsilon) \text{. Since } C_1(X) = [\rho] \text{ is independent of the metric, we have: (taking cohomology classes of (1'))}
\end{aligned} \]

\[ \frac{d [w(t)]}{dt} = -[\rho(t)] = -[\rho(0)] \quad \text{Thus, } [w(t)] = [w(0)] - t[\rho(0)] \]

Thus \(\frac{d}{dt} w(t) - w(0) + t\rho(0)\) is exact for \(t \geq 0\). By the \(\delta\bar{\delta}\)-lemma we can find \(\phi(t)\) so that
\[ g_{ij}^{(t)} - (g_0)_{ij} = t \partial_{\bar{k}} \partial_k \log \det ((g_0)_{ij}) + \partial_{\bar{k}} \partial_k \phi(t). \]

So differentiating yields:
\[ \partial_{\bar{k}} \partial_k \phi = -R_{\bar{k}j}^{ij} + R_{\bar{k}j}^{ij} = \partial_{\bar{k}} \partial_k \log \det ((g_0)_{ij}) + t \partial_{\bar{k}} \log \det (g_{ij}^{(t)}) \]

\[ \partial_{\bar{k}} \partial_k \log \left[ \frac{\det (g_0 + t\partial_{\bar{k}} \log \det g_0 + t\partial_{\bar{k}} \phi(t))}{\det (g_0)} \right] \]
Applying the Maximum Principle yields the Scalar eq'n:

\[
\phi = \log \frac{\det (g_0 + \Delta \log \det g_0 + \Delta \phi)}{\det (g_0)}.
\]

For some constant \( c_1(t) \). By Parabolic Theory there is a unique solution. This shows that we have short time existence, and that the RF preserves the Kähler condition.

Consider now the case when \([w]_k = [\rho]_k = c_1(x)\).

This will be the primary case of interest, as we want to produce Einstein metrics. We can then consider the

Normalized KRF:

\[
\begin{align*}
\frac{dg}{dt} &= -Ric + Kg. \\
g(0) &= g_0.
\end{align*}
\]

(Note: \( K \) determined purely topologically.

In this case, the eq'n on Forms is:

\[
\begin{align*}
\frac{dw}{dt} &= -\rho + Kw \\
w(0) &= w_0.
\end{align*}
\]

(\( N^* \))

A solution to \((N^*)\) can be converted to a solution of the KRF by setting \( \tilde{g}(t) = (1-t)g(-\ln(1-t)) \) (when \( K = 1 \)).

Thus, we have short time existence, and preservation of the Kähler condition.

\[
\frac{\partial [w]}{\partial t} = 0
\]
using a similar argument we can write the PMA eq.

$$\frac{d\Phi}{dt} = \log \left[ \frac{\det (g_o + \delta \Phi)}{\det g_o} \right] + k \Phi - f_o + c_1(t)$$

For some function $c_1(t)$. $f_o$ is defined by $\rho_o - \frac{k \rho_o}{\Theta} = 2\Theta f$

Thm (Ca0).

For $k \geq 0$, the solution to the NKRF (NK*) exists on $[0, \infty)$, and is unique. When $k < 0$, the NKRF converges exponentially fast in every $C^k$ norm to the unique KE metric $g_{00}$ in the Kähler class $[\omega_J]$

Remark

This provides a RF proof of the Calabi Conjecture when $c_1 < 0$. Thus, we only care about the $c_1 > 0$ case (Fano). We will (who?) take $k=1$.

NKRF:

Pros: Volume is preserved, existence on $[0, \infty)$, Simpler Scalar equation (?).

Cons: Singularities occur at $\infty$, and so are difficult to analyze.

The Energy Functional:

We define $\Psi : \text{Met}(x) \times C^\infty(x) \to \mathbb{R}$ by

$$\Psi(g, f) = \int_x (R + 4\pi f^2) e^{-f} d\text{vol}$$
Properties: (a) $\mathcal{F}$ is diffeomorphism invariant.

If $\phi: X \rightarrow X$ is a diffeomorphism

then $\mathcal{F}(\phi^*g, f \circ \phi) = \mathcal{F}(g, f)$.

(b) if $c > 0$, be $\mathbb{R}$, then

$\mathcal{F}(cg, c^2 f + b) = e^{c^2 n - 2} \mathcal{F}(g, f)$.

We now compute the Variation of $\mathcal{F}$.

If we $C^\infty (X, S^2 T^* X)$, he $C^\infty (X)$ (here we regard $X$ as a 2n-dim' Real mfd. we Forget the cplx structure).

Then, set $V = g^{ij} V_{ij}$

$$
\delta \mathcal{F} (V,h) = - \int_X V_{ij} \left[ (Ric)_{ij} + \nabla_i \nabla_j f \right] e^{-f} \, d\mu

+ \int_X (V - h) \left( 2 \Delta f - \nabla^2 f + R \right) e^{-f} \, d\mu.
$$

Exercise: Do this Computation.

Hint: It's enough to do the computation at a point.

Then use normal coordinates.

and $\delta \mathcal{F} (V,h) = \frac{d}{ds} |_{s=0} \mathcal{F}(g + sv, f + sh)$.

Note: if $V_{ij} = -(Ric)_{ij} + \nabla_i \nabla_j f$,

$V = g^{ij} V_{ij} = -(R + \Delta f) = h$, then $\delta \mathcal{F} (V, h) \geq 0$. 
with \( \delta F(v_1h) = 0 \) iff \( (Ric)_ij = -\nabla_i \nabla_j f. \)

Thus, if \( f(t), g(t) \) solve

\[
\begin{align*}
\frac{dg}{dt}_ij &= -(Ric)_ij - \nabla_i \nabla_j f \quad (**)
\frac{df}{dt} &= -\Delta f - R \quad (\Delta \text{ is Laplace wrt } g(t)!) 
\end{align*}
\]

Then \( F(g(t), f(t)) \) is increasing!

\( \text{Note: (i) The equation } \frac{df}{dt} = -\Delta f - R \text{ is equivalent to the fact that } \frac{d}{dt} \left( \int e^{-f} dvol \right) = 0 \text{ along the flow } (**). \)

\( \text{for } g. \text{ That is, we are preserving the measure } e^{-f} dvol! \)

\( \text{c.ii) The evolution } \text{ for } f \text{ is a Backwards heat equation.} \)

\( \text{So in general, we can't solve an initial value problem.} \)

**KEY RESULT** (**) is Related to the RF by diffreos.

**Lemma**: Let \( g(t) \) solve RF, on \([0,T]\), and let \( f_T \) be a function on \( X \).

\( \text{(i) we can solve } \frac{\partial f}{\partial t} = -\Delta f + 4\nabla f^2 - R \quad \text{for } t \in [0,T] \)

\( f(T) = f_T. \)

\( \text{(ii) Given a solution } \phi(t), \text{ define a family of diffreos' } \phi(t): X \to X \text{ by } \)

\[
\begin{align*}
\frac{d\phi}{dt} &= -\frac{1}{2} \nabla^{g(t)} f(t) \\
\phi(0) &= id_X.
\end{align*}
\]
(This is a system of ODE, hence is solvable). Then
\[ \bar{g}(t) = \Phi^*(t) g(t), \quad \text{and} \quad \bar{f}(t) = \Phi^*(t) f(t) = f_{\bar{g}} \Phi(t) \] Solve
\[ (**). \]

**Sketch:**
(i) Set \( u = e^{-f} \), \( T = T - t \). Then
\[ \frac{du}{dT} = - \frac{\partial u}{\partial T} = u \frac{df}{dT} = u \left( -\Delta f + |\nabla f|^2 - R \right) = \Delta u - R u. \]
This is parabolic, and hence solvable.

(ii) It's mostly difficult to compute \((\Phi^*(t) g(t))'\),

It is easier to compute \( \frac{d}{ds} \left| \Phi^*(s) g(t) \right| \) and \( \frac{d}{dt} \left| \Phi^*(s) g(t) \right| \) at \( s = t \).

For the first we can identify this as a Lie Derivative along the \( \nabla f, \frac{1}{2} \nabla f \). We leave the details as an exercise.

We now obtain the Monotonicity of \( \bar{T} \):

**Proposition:** If \((g(t), f(t))\) is a solution to
\[ \frac{\partial g_{ij}}{\partial t} = -(\text{Ric})_{ij} \]
\[ \frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R \]
Then \( \bar{T}(g(t), f(t)) \) is monotonically, and \( \frac{dT}{dt} = 0 \) iff
\[ \frac{\partial g_{ij}}{\partial t} = -(\text{Ric})_{ij} - \nabla_i \nabla_j f \]
If $g_{ij}$ solves $R$F, $g(0)$ Kähler, then:

$$\frac{\partial g_{ij}}{\partial t} = -R_{ij}$$

$$\frac{\partial f}{\partial t} = -\Delta f + \left| \nabla_i \nabla_j f \right|^2 - R$$

( Go back To Variational Formula and set $V_{ij} = V_{i,j} = 0$ ).

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**The Entropy Functional.**

Define $W : \text{Met}(X) \times C^\infty(X) \times R^+ \rightarrow R$ by

$$W(f, g, \tau) = \left( 4\pi \tau \right)^n \int_X \left[ \tau \left( R + 16f^2 \right) + (f - 2n) \right] e^{-f} \text{dVol}.$$ 

**Exercise:** Compute The Variation of $W$ by using the Formula For $\delta W$.

**The Evolution Eqn's are:**

$$\left\{ \begin{array}{l}
\frac{\partial g_{ij}}{\partial t} = - \left( R_{ij} + \nabla_i \nabla_j f \right) \\
\frac{\partial f}{\partial t} = - \Delta f - R + \frac{n}{2\tau} \\
\frac{\partial \tau}{\partial t} = -1
\end{array} \right.$$ 

In a similar manner to the Gradient Flow eqn's for the functional $T$, these are coupled to the Ricci Flow

Follows: Via diffeomorphisms. The System For The Ricci Flow is
\[
\begin{align*}
\frac{\partial g_{ij}}{\partial t} &= -R_{ij} \\
\frac{\partial f}{\partial t} &= -\Delta f + |\nabla f|^2 - R + \frac{n}{4} \\
\frac{d\tau}{dt} &= -1
\end{align*}
\]

By a similar argument as before, we can solve these equations, and we obtain: If \((g, f, \tau)\) solve (II),

\[W(g(t), f(t), \tau(t)) \to \]

Observe: \(W\) is diffeomorphism independent, and

\[W(g_t, f_t, \tau_t) = W(g, f, \tau)\]

**The \(\mu\) Functional**

**Define:**

\[\mathcal{X}_{(g, \tau)} = \left\{ f \in C^\infty(X) \mid \int (4\pi \tau)^{-n/2} e^{-f} d\text{vol} = 1 \right\} \]

\[\mu(g, \tau) = \max_{f \in \mathcal{X}} W(g, f, \tau).\]

**Remarks:**

(i) \(\mu\) is diffeo. independent

(ii) \(\mu(g_t, \tau_t) = \mu(g, \tau)\)

(iii) \(\mu\) is increasing along the Ricci flow.

If choose \(f \in \mathcal{X}(g_{(t)}, \tau_{(t)})\). Solve, the system (II) (just the backward Heat eq'n).

\[\mu(g(0), \tau(0)) \leq W(g(0), f(0), \tau(0)) \leq W(g(t), f(t), \tau(t))\]
Use the fact that \( (\Box) \) preserves \( X \) in

\[
\left( \frac{\Gamma(n)}{2\pi} \right)^{-n/2} e^{-f(t)} \text{dVol}(t) = 1.
\]
Then take Inf on RHS.

**Lemma 6.23:** (Finiteness of \( \mu \)).

For any given \( g \) and \( \varepsilon > 0 \) on a closed manifold \( X \),

\[
\mu(g, \varepsilon) > -\infty.
\]

**Proof:** Since \( \mu(g, 1) = \mu(\varepsilon g, \varepsilon) \) and \( \varepsilon > 0 \), it suffices to prove that \( \mu(g, 1) > -\infty \). Equivalently, \( \exists C = C(g) \) such that \( W(g, f, t) > C \) for \( f \in X(g, \varepsilon) \).

Let \( V = (4\pi)^{-n/2} e^{-f/2} \). Then

\[
W(g, f, t) = \int_X \left( \frac{1}{2} |W|^2 + (R - 2\log w - \frac{n}{2} \log (4\pi) - n) w^2 \right) d\mu
\]

for \( w > 0 \) s.t. \( \int w^2 = 1 \). Clearly, it suffices to show

\[
\int w^2 \log w^4 d\mu \leq 2 \int \|W\|^2 d\mu + C.
\]

Which is a consequence of the following log Sobolev Inequality.
**Lemma (Log Sob.)**

Let \((M^n, g)\) be a closed Riemannian mfd. For any \(a > 0\), \(\exists C(a, g) \text{ s.t. if } \varphi > 0 \text{ has } \int \varphi^2 d\mu = 1, \text{ then}

\[
\int_{M} \varphi^2 \log \varphi \leq a \int \nabla \varphi^2 + C(a, g).
\]

**Proof** Sobolev \(\Rightarrow \left( \int \varphi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq C_S(M, g) \int \nabla \varphi^2 + \text{Vol}(M)^{-\frac{2}{n}} \)

It is easy to see that for \(x > 0\), \(x \log x \leq x^{\frac{3}{n}}\) (since \(x \leq e^x\) for every \(x\)). Thus

\[
\frac{2}{n} \int \varphi^2 \log \varphi \leq \int \varphi^{\frac{2}{n} + \frac{2}{n}} d\mu \leq 3 \int \varphi^{\frac{2}{n} + \frac{4}{n}} + \frac{1}{3} \int \varphi^2 d\mu
\]

\[
\leq 3 \left( \int (\varphi^2)^{\frac{n}{n-2}} \right)^{\frac{n-2}{2n}} \left( \int \varphi^{\frac{4}{n} \cdot \frac{2}{n}} \right)^{\frac{2}{n}} \leq \frac{1}{3} \int \varphi^2 d\mu
\]

\[
= \frac{1}{3} C_S(g)^{-1} \int \nabla \varphi^2 d\mu + \frac{1}{3} + \varepsilon \text{Vol}(M)^{-\frac{2}{n}} C_S(g)^{-1}
\]

Now set \(\varepsilon = \frac{2a C_S(g)}{n}\) and the result follows.

**Lemma:** For each \(g, \mathcal{C} \text{ \(\exists f \in \mathcal{C}(g, \mathcal{T})\), \(f\) smooth s.t.}

\[
\mathcal{M}(g, f, \mathcal{T}) = \mu(g, \mathcal{T}).
\]

**Proof** Omitted. But, the proof implies we can allow \(W^{1,2}\) Functions in the mfd.
No Local Collapsing.

Def'n: \((K\text{-non-collapsed})\)

Given \(p \in (0, \infty)\), and \(K > 0\), we say that the metric \(\hat{g}\) is \(K\text{-non-collapsed}\) below the scale \(\rho\) if for any metric ball \(B(x, r)\) with \(r < \rho\) satisfying \(1\text{Rm}(y) \leq r^{-2}\) \(\forall y \in B(x, r)\) we have

\[
\frac{\text{Vol}(B(x, r))}{r^n} \geq K.
\]

The following lemma relates this to non-collapse and shows non-collapse is equivalent to a injectivity radius bound.

Lemma: Let \((\hat{M}^n, \hat{g})\) be a complete Riemannian manifold and fix \(p \in (0, \infty)\).

(i) If \(\hat{g}\) is \(K\text{-non-collapsed}\) below the scale \(\rho\) for some \(K > 0\), then \(\exists \delta = \delta(p, K)\) s.t. if \(x \in \hat{M}\), \(r < \rho\), and \(1\text{Rm} \leq r^{-2}\) in \(B(x, r)\) then \(\text{inj}(x) \geq \delta r\).

(ii) \(\text{inj}(x) > \delta \Rightarrow K\text{-non-collapsed}\) ... See B. Chow Lemma 6.54.

Using the \(W\) functional, we can prove Non-collapse under a weaker curvature assumption.
Proposition: Let $g(t)$ be a solution of the NKRF. Then there exists $C > 0$, depending only on $g(0)$, such that for any $g(t)$ with $|t| < 1$ on $B_{g(t)}(x, 1)$.

Proposition': Let $\tilde{g}(t)$ be a solution of the KRF. Then there exists $K = K(\tilde{g}(0)) > 0$ such that if $|t| < \frac{1}{r^2}$, in a ball $B_{\tilde{g}}(p, r)$, we have $\text{Vol}(B_{\tilde{g}}(p, r)) \geq Kr^{2n}$. 