Let $M$ be a complex manifold.

**Def** A function $g : M \to \mathbb{R}$ is **quasi-plurisubharmonic** if it can be locally written as a sum of a smooth function and a psh fn.

**Def** For a continuous $C^{1,1}$-form $\gamma$ on $M$, $\text{PSH}(M, \gamma)$ will denote the class of quasi-psh $\psi$ on $M$ s.t. $dd^c \psi + \gamma \geq 0$ (\(\gamma\)-psh fn).

**Note** If $\gamma \geq 0$, constants $c \in \text{PSH}(\mathbb{M}, \gamma)$.

- $\text{PSH}(M, \gamma)$ is closed under operation of maximum and regularized maximum.

See Demailly, Lecture I, 5.17-9.18

**Def** The Le/long number of $g$ at $a \in M$ is

$$V_g(a) = \lim_{z \to a} \frac{g(z)}{\log |z-a|}$$

Or

$$V_g(a) = \lim_{r \to 0} \left( \frac{\sup_{|z-a|=r} g(z)}{\log r} \right)$$
Theorem 1

Let \( \omega \) be a positive continuous \((1,1)\)-form on a compact \(\mathbf{C}^n\)-manifold \(M\). For every \( \gamma \in \text{PSH}(M, \omega) \), there exists a sequence \( \gamma_j \in \text{PSH}(M, \omega) \) such that \( \gamma_j \) decreases to \( \gamma \).

Theorem 2

Let \( M \) be a complex manifold with a positive hermitian form \( \omega \). Assume that \( \gamma \) is a continuous \((1,1)\)-form on \( M \), and let \( \gamma_j \in \text{PSH}(M, \gamma) \) be such that the Lelong number \( \nu_{\gamma_j}(z) = 0 \) for every \( z \in M \). Then for every open \( M' \subset M \) we can find \( \varepsilon_j \searrow 0 \) and \( \gamma_j' \in \text{PSH}(M', \gamma + \varepsilon_j \omega) \cap C^0(M') \) decreases to \( \gamma \) in \( M' \).

Theorem 3

Let \( (M, \omega) \) be a compact Kähler manifold.

For \( \gamma, \psi \in \text{PSH}(M, \omega) \cap L^p(M) \),

\[
\frac{\psi \wedge (dd^c \gamma)^n}{\psi \wedge (dd^c \psi)^n} \leq \frac{\gamma \wedge (dd^c \psi)^n}{\gamma \wedge (dd^c \gamma)^n}.
\]
Proof:

WLOG, we may assume $g \leq -1$.

Since $w$ is positive, constants $\in \text{PSH}(M, w)$ so we have a sequence

$$\mathcal{E} = \max \left\{ g, -\lambda \right\} \in \text{PSH}(M, w) \cap L^\infty(M)$$

which decreases to $g$.

Then we may assume $g$ is bounded.

Then $V_g(z) = 0$ if $z \in M$ $\Rightarrow$ Apply Thm 2 with $y = w$, $M' = M$ (allowed since $M$ compact).

Then $\exists \epsilon > 0$ and $\psi_j \in \text{PSH}(M, \lambda_j; w) \cap C^0(M)$, with $\lambda_j = 1 + \epsilon$

such that $\psi_j \wedge g$ in $M$ (can assume $\psi_j \leq 0$)

Let $g_j = \frac{\psi_j}{\lambda_j}$. Since $\psi_j \leq 0$, $\psi_j \wedge g$, and $\lambda_j > 1$,

we have $\psi_j \wedge g$.

Since $\psi_j \in \text{PSH}(M, \lambda_j; w) \cap C^0(M)$, $\ddc \psi_j + \lambda_j w \geq 0$

$\Rightarrow \ddc \frac{\psi_j}{\lambda_j} + w \geq 0$

$\Rightarrow \ddc \frac{g_j}{\lambda_j} + w \geq 0$

$\Rightarrow g_j \in \text{PSH}(M, w) \cap C^0(M)$

and $g_j \wedge g$. \square
Pf of Thm.2

In the flat case, use the standard smooth regularization by convolution.

If \( p(\tau) = \delta^{(1,1)}(\tau) \in C^\infty(D^n) \) is s.t.

\begin{enumerate}
\item \( \hat{p} = 0 \)
\item \( p(\tau) = 0 \) for \( \tau \in D^1 \)
\item \( \int_{D^n} p d\lambda = 1 \)
\item \( p_\delta(x) = \delta^{-n} p(\delta x) \) for \( \delta > 0 \)
\end{enumerate}

Then set

\[ U_\delta(z) = (u \times p_\delta)(z) = \int_{D^n} u(z - \delta \omega) p(\omega) d\lambda(\omega). \]

When \( u \) is psh \( U_\delta \downarrow u \) as \( \delta \downarrow 0 \).

When \( u \) is continuous \( U_\delta \to u \) locally uniformly.

Lemma

Assume \( U, V \subset \subset D^n \), let \( F: U \to V \) be biholo.

Let \( u \in \text{PSH}(U) \) s.t. \( v_u(z) = 0 \) \( \forall \ z \in U \)

Define \( U_\delta \circ F := (u \circ F^{-1})_\delta \circ F \)

Then \( U_\delta \circ F \to 0 \) locally uniformly as \( \delta \to 0 \).
let $u^F_{x,\delta}$ denote the regularization of $u_x$ in $U_\beta$

By Lemma 4,

\[ u^F_{x,\delta} - u_{x,\delta} = u^F_{x,\delta} - u_{x,\delta} + (u_x - u_\beta)^F \rightarrow f_x - f_\beta \]

locally and in $U_\alpha \cap U_\beta$ as $\delta \rightarrow 0$.

Let $n_\alpha$ be smooth in $U_\alpha$ s.t.

\[ n_\alpha = 0 \quad \text{in } V_\alpha \]
\[ n_\alpha = -1 \quad \text{away from a compact subset of } U_\alpha \]

Then $dd^cn_\alpha > -Cw$ for some constant $C$ depending on $\varepsilon$.

Let

\[ y_{\delta} := \max_{\alpha} \left( u_{x,\delta} - f_x + \frac{\varepsilon n_\alpha}{\varepsilon} \right) \]

by * if $\delta$ is sufficiently small, the values on $\varepsilon n_\alpha = -1$ do not affect the maximum, then $y_{\delta}$ is continuous.

(If consider regularized maximum in stead of maximum in above)

defn we get $y_{\delta}$ smooth

Using * we get $y_{\delta} \in \text{PSH} (M^1, \omega, \text{fem})$

Also, $y_{\delta} \rightarrow y_0$ as $\delta \rightarrow 0$
Let $\hat{u}_s(z) := \frac{\max U}{B(z, s)}$.

If $u$ is psh $\Rightarrow \hat{u}_s(u)$ is log convex in $S$.

$\Rightarrow \hat{u}_s$ is continuous, psh in $U_s := \{ z \in U; \overline{B(z,s)} \subset U \}$ and decreases to $u$ as $s \to 0$.

By log convexity, for $a \geq 1, r > 0$ fixed and $s < 1$,

$$0 \leq \hat{u}_{as} - \hat{u}_s \leq \frac{\log a}{\log(\frac{r}{s})} (\hat{u}_r - \hat{u}_s)$$

Then if $\hat{u}_s$ Lebesgue $s$ vanish, if $a > 0$,

$$(\hat{\ast}) \quad \hat{u}_{as} - \hat{u}_s \to 0 \quad \text{locally uniformly as } s \to 0.$$

Also, let $\hat{u}_s^F := (\hat{u} \circ F^{-1})_s \circ F = \max_{F^{-1}(\overline{B(F(z), s)})} U$

For a fixed $K \subset U$, we can find $A > 1, s_t$ for $z \in K$ and $s < 1$,

$$\overline{B(F(z), s)} \subset F(\overline{B(z, As)}) \quad F(\overline{B(z, s)}) \subset \overline{B(F(z), As)}$$

Then on $K$, $\hat{u}_s^F \leq \hat{u}_{As}$, $\hat{u}_s \leq \hat{u}_{As}$.

$$(\ast) \quad \hat{u}_s^F - \hat{u}_s \to 0 \quad \text{locally uniformly in } U \text{ as } s \to 0$$

Then Lemma 5. For $w \in \text{PSH}(U), w(z) = 0 \quad \forall z \in U$.  

$\hat{u}_s - \hat{w}_s \to 0 \quad \text{locally uniformly in } U \text{ as } s \to 0$.