1 (20 pts). Find the limit
\[
\lim_{x \to \infty} \cos(e^{-x}).
\]

Solution Let \( t = e^{-x} \) then \( t \) has limit 0 as \( x \) goes to \( \infty \). It follows that
\[
\lim_{x \to \infty} \cos(e^{-x}) = \lim_{t \to 0} \cos t = \cos 0 = 1.
\]

2 (20 pts). Find the limit
\[
\lim_{x \to 2} \frac{\sqrt{x} + 2 - \sqrt{2x}}{x^2 - 2x}.
\]

Solution The limit is of type \( \frac{0}{0} \). We want to rationalize the numerate by multiply it by its conjugate and then cancel a term like \( x - 2 \):
\[
\lim_{x \to 2} \frac{\sqrt{x} + 2 - \sqrt{2x}}{x^2 - 2x} \cdot \frac{x + 1}{x + 1} = \lim_{x \to 2} \frac{(x + 2) - (2x)}{(x^2 - 2x)(\sqrt{x} + 2 + \sqrt{2x})} = \lim_{x \to 2} \frac{-1}{x(\sqrt{x} + 2 + \sqrt{2x})} = -\frac{1}{8}.
\]

3 (20 pts). Find values of \( a \) and \( b \) such that \( f \) is differentiable everywhere:
\[
f(x) = \begin{cases} 
x + 1 & \text{if } x \leq 1 \\
x^2 + ax + b & \text{if } x > 1
\end{cases}
\]

Solution The function is defined by two piece of differentiable function. Thus for any \( a \) and \( b \), the function differential at every point except at \( x = 1 \). The function of derivatives other than \( x = 1 \) is given by
\[
f'(x) = \begin{cases} 
1 & \text{if } x < 1 \\
2x + a & \text{if } x > 1.
\end{cases}
\]
For \( f \) to be differentiable at \( x = 1 \), it is necessary and sufficient that the function is continuous at \( x = 1 \) with two half derivatives equal:

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)
\]

\[
\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^-} f'(x).
\]

By our formulae for \( f(x) \) and \( f'(x) \), we see that these two conditions are equivalent to

\[
1 + 1 = 1 + a + b, \quad 1 = 2 + a.
\]

We have unique solution

\[
a = -1, \quad b = 2.
\]

4 (20 pts). Use intermediate value theorem for function

\[
f(x) = x - \cos x
\]

to prove that the equation

\[
\cos x = x
\]

has one solution in the interval \((0, 1)\).

**Proof** The solution of \( x = \cos x \) is equivalent to the solution \( f(x) = 0 \) in the interval \( x \in (0, 1) \). As \( f \) is continuous, by the intermediate value theorem, any value \( N \) between \( f(0) \) and \( f(1) \) will equal to \( f(c) \) for some \( c \in (0, 1) \). For our purpose, we take \( N = 0 \). Thus we need to show that two values \( f(0) \) and \( f(1) \) have different signs. Let us compute them separately as follows:

\[
f(0) = 0 - \cos 0 = -1 < 0, \quad f(1) = 1 - \cos 1 > 0.
\]

Here in the second equality we use the fact that \( \cos x < 1 \) for all \( x \) except multiples of \( 2\pi \).

5 (20 pts). Find an equation of the tangent line to the curve

\[
y = e^x - x^2
\]

at the point \((0, 1)\).

**Solution** The slope of the tangent line at \((0, 1)\) is given by the derivative:

\[
\frac{dy}{dx}|_{x=0} = (e^x - 2x)|_{x=0} = e^0 - 2 \cdot 0 = 1.
\]

The tangent line at \((0, 1)\) is given by

\[
y - 1 = 1 \cdot (x - 0), \quad \text{or} \quad y = x + 1.
\]