# CALCULUS I – MIDTERM EXAM #1
## OCTOBER 4, 2007
## SOLUTION

1 (20 pts). Find the limit

$$\lim_{x\to\infty} \cos(e^{-x}).$$

*Solution* Let $t = e^{-x}$ then $t$ has limit 0 as $x$ goes to $\infty$. It follows that

$$\lim_{x\to\infty} \cos(e^{-x}) = \lim_{t\to 0} \cos t = \cos 0 = 1.$$

2 (20 pts). Find the limit

$$\lim_{x\to 2} \frac{\sqrt{x+2} - \sqrt{2x}}{x^2 - 2x}.$$

*Solution* The limit is of type $\frac{0}{0}$. We want to rationalize the numerate by multiply it by its conjugate and then cancel a term like $x - 2$:

$$\lim_{x\to 2} \frac{\sqrt{x+1} - \sqrt{2x}}{x^2 - 2x} \cdot \frac{\sqrt{x+2} + \sqrt{2x}}{\sqrt{x+2} + \sqrt{2x}} = \lim_{x\to 2} \frac{(x+2) - (2x)}{(x^2 - 2x)(\sqrt{x+2} + \sqrt{2x})}$$

$$= \lim_{x\to 2} \frac{2 - x}{x(x-2)(\sqrt{x+2} + \sqrt{2x})} = \lim_{x\to 2} \frac{-1}{x(\sqrt{x+2} + \sqrt{2x})}$$

$$= \frac{-1}{2 \cdot (\sqrt{2+2} + \sqrt{2\cdot})} = -\frac{1}{8}$$

3 (20 pts). Find values of $a$ and $b$ such that $f$ is differentiable everywhere:

$$f(x) = \begin{cases} x + 1 & \text{if } x \le 1 \\ x^2 + ax + b & \text{if } x > 1 \end{cases}$$

*Solution* The function is defined by two piece of differentiable function. Thus for any $a$ and $b$, the function differential at every point except at $x = 1$. The function of derivatives other than $x = 1$ is given by

$$f'(x) = \begin{cases} 1 & \text{if } x < 1 \\ 2x + a & \text{if } x > 1 \end{cases}.$$

For $f$ to be differentiable at $x = 1$, it is necessary and sufficient that the function is continuous at $x = 1$ with two half derivatives equal:

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1-} f(x) = f(1)$$

$$\lim_{x\to 1+} f'(x) = \lim_{x\to 1-} f'(x).$$

By our formulae for $f(x)$ and $f'(x)$, we see that these two conditions are equivalent to

$$1 + 1 = 1 + a + b, \qquad 1 = 2 + a.$$

We have unique solution

$$a = -1, \qquad b = 2.$$

4 (20 pts). Use intermediate value theorem for function

$$f(x) = x - \cos x$$

to prove that the equation

$$\cos x = x$$

has one solution in the interval $(0, 1)$.

*Proof* The solution of $x = \cos x$ is equivalent to the solution $f(x) = 0$ in the interval $x \in (0, 1)$. As $f$ is continuous, by the intermediate value theorem, any value $N$ between $f(0)$ and $f(1)$ will equal to $f(c)$ for some $c \in (0, 1)$. For our purpose, we take $N = 0$. Thus we need to show that two values $f(0)$ and $f(1)$ have different signs. Let us compute them separately as follows:

$$f(0) = 0 - \cos 0 = -1 < 0, \qquad f(1) = 1 - \cos 1 > 0.$$

Here in the second equality we use the fact that $\cos x < 1$ for all $x$ except multiples of $2\pi$.

5 (20 pts). Find an equation of the tangent line to the curve

$$y = e^x - x^2$$

at the point $(0, 1)$.

*Solution* The slop of the tangent line at $(0, 1)$ is given by the derivative:

$$\frac{dy}{dx}|_{x=0} = (e^x - 2x)|_{x=0} = e^0 - 2 \cdot 0 = 1.$$

The tangent line at $(0, 1)$ is given by

$$y - 1 = 1 \cdot (x - 0), \quad \text{or} \quad y = x + 1.$$