Period integrals in triple product

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1 Introduction

Let $F$ be a number field and $E$ a cubic extension of $F$. Let $\mathbb{A}$ and $\mathbb{A}_E = \mathbb{A} \otimes E$ denote the algebra of adeles of $F$ and $E$ respectively. Let $B$ be a quaternion algebra over $F$. Let $\pi$ be a cuspidal automorphic representation
representation of $B_E^\times$. We also use $\pi$ denote to the space of automorphic forms corresponding to $\pi$. For $f \in \pi$, we define the period integral
\[
\ell(f) := \int_{[B^\times]} f(h) dh
\]
where $[B^\times]$ denote the group $\A^\times B^\times \backslash B_\A^\times$ and $dh$ is the invariant measure on $[B^\times]$ with volume 1. Notice that the period vanishes if the central character of $\pi$ has nontrivial restriction on $\A^\times$. For simplicity, we assume that $\pi$ has unitary central character which has trivial restriction on $\A^\times$.

2 Theta lifting

In this section we use $V$ to denote a vector space of even dimension $m$ over a number field $F$. Let $n$ be a positive integer and let $\text{Sp}(n)$ denote the symplectic group for the standard sympectic form $J := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ on $F^{2n}$.

Let $S(V_n^m)$ denote the space of Bruhat-Schwartz functions on $V^m_n$. Fix a non-trivial character $F \backslash \A$. Then we have a Weil representation $\omega$ of $\text{Sp}(n)_\A \times O(V)_\A$. The action is the product of local actions of $O(V)_v \times \text{Sp}_n(F_v)$ on $S(V(F_v)^n)$. The action of $O(V)_v$ is simply given by translation:
\[
\omega(h) \phi_v(x) = \phi_v(h^{-1} x), \quad \phi_v \in S(V^n_v), \quad h \in O(V)_v.
\]

For the action of $\text{Sp}_n(F_v)$ we notice that $\text{Sp}_n$ is generated by the following elements:
\[
M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a \in \text{GL}_n(F) \right\}
\]
\[
N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in \text{Sym}_n(F) \right\}
\]
\[
J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.
\]

The following are some formulae
\[
\omega(m(a)) \phi_v(x) = \chi_V(\det a)|\det a|^{m/2} \phi_v(x a),
\]
\[
\omega(n(b)) \phi_v(x) = \psi_v(\text{tr}(bQ(x))) \phi_v(x),
\]
where $\chi_V$ is the character defined by Hilbert symbol $\chi_V(a) = (a, \det(V))_F$ and $Q(x) = \frac{1}{2}(x_i, x_j)$ be the moment matrix for $x = (x_1, \cdots, x_n)$,

$$\omega(J)\phi_v(x) = \gamma\hat{\phi}_v(x)$$

where $\gamma$ is an 8th root of unity, and $\hat{\phi}$ is the Fourier transform of $\phi$:

$$\hat{\phi}_v(x) = \int_{V(F_v)^n} \phi(y)\psi_v(\sum_i x_i y_i) dy$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots y_n)$.

**Extend to similitude**

We may extend this action to $\text{GO}(V)$ by

$$\omega(h)\phi(x) = \nu(h)^{-mn/4}\phi(h^{-1}x).$$

The action of $\text{GO}(V)$ and the action of $\text{Sp}_n$ do not commutate. The commutator is given by

$$[\omega(h), \omega(g)] = \omega[d(h), g]$$

where $d(h) = \begin{pmatrix} 1_n & 0 \\ 0 & \nu(h)1_n \end{pmatrix}$, and $\nu$ is the similitude factor morphism from $\text{GO}(V)$ to $\mathbb{G}_m$. It follows that the Weil representation can be extented to the group $\text{R}$ generated by $\text{Sp}_n$ and $\text{GO}(V)$ with the above commutator relation:

$$hgh^{-1} = d(h)gd(h)^{-1}.$$  

We may also write

$$\text{R} = \{(h, g) \in \text{GO}(V) \times \text{GSp}_n : \nu(g) = \nu(h)\}$$

where $\nu$ is also the similitude factor morphism from $\text{GSp}_n$ to $\mathbb{G}_m$ with $\text{GO}(V)$ consists of elements $(d(h), h)$ and $\text{Sp}_n$ consists of $(1, g)$. We have canonical isomorphisms:

$$\text{R}/\text{Sp}_n \simeq O(V), \quad \text{R}/O(V) \simeq \text{GSp}_n^+$$

where $\text{GSp}_n^+$ is the subgroup of $\text{GSp}_n$ with similitude in $\nu(\text{GO}(V))$. The symbol $(h, g)$ means the product $h \cdot d(z)^{-1}g$. We may also write $(g, h)$ as product $gd(h)^{-1}$. The connection is

$$(h, g) = (h, d(h)gd(h)^{-1}).$$
In either expression, the center of $R$ consists of scales elements $(z, z)$.

For $\phi \in \mathcal{S}(V^n_k)$ a Bruhat-Schwartz function, we can define the theta function on $R(\mathbb{A})$ by

$$\theta_\phi(h, g) = \sum_{x \in V^n} \omega(h, g)\phi(x), \quad h \in \text{GO}(V)_k.$$ 

This clear an automorphic function on $R$ with trivial central character. For a cusp form $\varphi$ on $\text{GSp}_n(\mathbb{A})$ we can define its theta-lifting on $\text{GO}(V)$ as by

$$\theta^\varphi_\phi(h) = \int_{[\text{Sp}_n]} \theta_\phi(g_1 h, h)\varphi(g_1)dg_1, \quad \nu(g) = \nu(h)$$

For an automorphic representation $\sigma$ on $\text{GSp}_n$, we denote the span of $\theta^\varphi_\phi$ by $\Theta(\sigma)$. This is an automorphic representation of $\text{GO}(V)$ with the same central character as $\sigma$. Notice that for any fixed $\phi$, the space $\theta^\varphi_\phi$ spanned by all $\varphi$ is already stable under action by $\text{GO}(V)_k$.

Similarly, for a form $f$ on $\text{GO}(V)$ we can define its theta lifting $\text{GSp}_n^+(\mathbb{A})$ on by

$$\theta^f_\phi(g) = \int_{[\mathcal{O}(V)]} \theta_\phi(h_1 h, g)f(h_1 h)dh_1 \quad \nu(g) = \nu(h).$$

We can define the space $\Theta(\pi)$ spanned by $\theta^f_\phi$ for $f \in \pi$ a cuspidal representation of $f \in \pi$.

**Siegel–Weil formula**

In this subsection we want to identify the integral

$$I_\phi(g) := \int_{[\mathcal{O}(V)]} \theta_\phi(h_1 h, g)dh_1$$

where $\nu(h) = \nu(g)$. With respect to the Siegel parabolic subgroup, we have the Iwasawa decomposition

$$\text{GSp}_n(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K$$

where $K$ is the product of the maximal compact subgroup $K_v$ as follows:

At archimedean place $v$ if we identify $F_{v}^{2n} \simeq \mathbb{C}^n$ as real vector space so that the symplectic form on $F_{v}^{2n}$ corresponds to the imaginary part of the
standard hermitian form on $\mathbb{C}^n$, then we may take $K_v = GU(n)\mathcal{O}_v^{2n}$, the group of the unitary similitude with similitudes in $\{\pm 1\}$.

At a finite place $v$, we take $K_v$ to be the subgroup of $\text{GSp}_n$ which fixes the lattice $\mathcal{O}_v^{2n}$ and with similitudes in $\mathcal{O}_v^\times$.

For $\phi \in S(V^n_\mathbb{A})$ and $s \in \mathbb{C}$, define a function on $\text{GSp}_n(\mathbb{A})$ by
\[
\Phi(s, g) = \nu(g)^{-n} \omega(g_1) \phi(0) \det a(g_1)^{s-s_0}
\]
where $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & \nu(g)^{-1} \end{pmatrix}$, $g \in \text{Sp}_n(\mathbb{A})$, and $a(g_1)$ is defined by the Iwasawa decomposition in $\text{Sp}_n(\mathbb{A}) = N_1 M_1 K_1$ analogous as above for $\text{GSp}_n(\mathbb{A})$, and $s_0 = (m-1-n)/2$. Then $\Phi(g)$ is in the normalized induction $\text{Ind}_{\text{P}(\mathbb{A})}^{\text{GSp}_n(\mathbb{F})}(\nu^{-n} \chi_{V} \cdot |\cdot|^s)$. We may define the Eisenstein series by
\[
E(g, a, \phi) = \sum_{\gamma \in \text{P}(\mathbb{F}) \setminus \text{GSp}_n(\mathbb{F})} \Phi(s, \gamma g).
\]

**Theorem 2.1** (Siegel-Weil formula). The Eisenstein series $E(g, s, \phi)$ is holomorphic at $s = s_0$ and
\[
E(g, s_0, \phi) = \kappa I_\phi(g), \quad \kappa = \begin{cases} 1 & \text{if } m > n + 1 \\ 2 & \text{if } m \leq n + 1 \end{cases}
\]

**Shimuzu lifting**

Let $B$ be a quaternion algebra over $\mathbb{F}$ and let $V$ be the space $B$ over $\mathbb{F}$ with quadratic form $q(x) = x\bar{x}$. Then we have an action $B^\times \times B^\times$ on $V$ by
\[(b_1, b_2)x = b_1 x b_2^{-1}.
\]
This defines a morphism $B^\times \times B^\times \longrightarrow \text{GO}(V)$. This morphism determines an extension
\[
1 \longrightarrow \mathbb{G}_m \longrightarrow (B^\times \times B^\times) \times < \iota > \longrightarrow \text{GO}(V) \longrightarrow 1
\]
where $< \iota >$ is the subgroup of $V$ generated by the canonical involution $\iota(x) = \bar{x}$. In this way, $(B^\times \times B^\times)/\Delta$ can be consider as a subgroup of $\text{GO}(V)$ of index 2, where $\Delta$ is the $F^\times$ diagonally embedded into $B^\times \times B^\times$. Now we consider case $n = 2$ with $\text{GSp}(1) = \text{GL}_2$. Let $\sigma$ be a cusp form on $\text{GL}_2$. We consider the space $\Theta(\sigma)$ of forms on $\text{GO}(V)$.
Theorem 2.2 (Shimizu). The space $\Theta(\sigma)$ is either 0 or a cuspidal form of $GO(V)$ and is nonzero if and only if $\sigma$ has a nonzero Jacquet Langlands lifting $\pi$ on $B^\times$. Moreover, as space of functions on $B^\times \times B^\times$, it is given by

$$\Theta(\sigma)|_{B^\times \times B^\times} = \pi \times \tilde{\pi}.$$ 

3 Application to periods integrals

Now we want to apply the results in the above section to the periods integrals in §1. It is somehow easy to evaluate the square of this integral to periods of on some orthogonal group:

$$|\ell_B(f)|^2 = \int_{[B^\times \times B^\times]} f(h_1)\tilde{f}(h_2)dh_1dh_2. $$

Notice that $\tilde{f}$ belongs to the space $\tilde{\pi} = \pi$. Thus we have a period of form $\pi \times \tilde{\pi}$. By Shimizu’s lifting on $GO(V)_E$ and $GL(2)_E$, the representation $\pi \otimes \tilde{\pi}$ is the theta lifting of the Jacquet-Langlands lifting $\sigma$ of $\pi$ on $GL(2)_E$. Thus there is a $\phi \in S(V_{E,A})$ and $\varphi \in \sigma$ such that

$$f \otimes \tilde{f} = \theta^\varphi_\phi. $$

Thus the periods integral has the form

$$|\ell(f)|^2 = \int_{[B^\times \times B^\times]} \theta^\varphi_\phi(h)dh = \int_{[GO(V)]} \theta^\varphi_\phi(h)dh. $$

Here the second equality follows from Prasad’s thesis. Interchange the order of the integrations, the period can be written as

$$\int_{[SL(2)_E]} dg \int_{[GO(V)]} \theta_\phi(gd(h), h) \varphi(gd(h))dh$$

$$= \int_{SL(2)_E} dg \int_{[GO(V)/O(V)]} dc \int_{[O(V)]} \theta_\phi(gd(h), h_1d(h)) \varphi(gd(h))dh.$$

where $\nu(h_c) = \nu(h)$. The collection $d(h)g$ forms the subgroup $GL(2)^0_E$ of $GL(2)_E$ of elements with determinant in $F$. Thus the above integral can be written as

$$|\ell(f)|^2 = \int_{[GL(2)_E^0]} \varphi(g)I_\phi(g)dg$$
where \( I_\phi(g) \) is a function on \( \text{GL}(2)^0_E \):

\[
I_\phi(g) = \int_{[O(V)]} \theta_\phi(g, h_1 h e) dh_1.
\]

If we consider \( V_E = V \otimes E \) as a vector space over \( F \), then we have a symplectic form on \( E^2 \) by

\[
E^2 \otimes E^2 \to E^{\text{trdf}} \to F.
\]

In this way, we have a Weil representation of \( O(V) \times \text{Sp}(E^2) \) on \( S(V_E, A) \) and this can be extended to the subgroup of \( \text{GO}(V) \times \text{GSp}(E^2) \) of elements \((g, h)\) with the same norm. This action is compatible with previous action by the subgroup \( R \) in \( \text{GO}(V)_E \times \text{GL}(2)_E \) for the inclusion \( \text{GO}(V)_E \subset \text{GO}(V)_E \), and \( \text{GL}(2)^0_E \subset \text{GSp}(E^2) \). Thus we may extend \( I_\phi \) to a function on \( \text{GSp}(E^2) \).

Apply the Siegel-Weil formula, \( I_\phi \) may be identified with an Eisenstein series \( E(g, 0, g) \) on \( \text{GSp}(E^2) \). In summary we have proved the following formula

**Theorem 3.1.** Assume that \( f \otimes \bar{f} = \theta_\phi \varphi \), then

\[
|\ell(f)|^2 = 2 \int_{\text{GL}(2)_E^0} \varphi(g) E(g, 0, \phi) dg
\]

where \( E(g, s, \phi) \) is the Eisenstein series on \( \text{GSp}(E^2) \simeq \text{GSp}(3) \):

\[
E(g, s, \phi) = \sum_{\gamma \in P(F) \setminus \text{GSp}_3(F)} \nu(g)^{-3-3s} \omega(g_1) \phi(0) |a(g_1)|^{2s}
\]

where \( g_1 = \begin{pmatrix} 1 & 0 \\ 0 & \nu(g)^{-1} \end{pmatrix} \in \text{Sp}_3 \).

**4 Rankin-Selberg integrals**

In this section, we want to study the integral defined in Theorem 3.1. Recall that we have defined

\[
\Phi(g) := \nu(g)^{-3-3s} \omega(g_1) \phi(0) |a(g_1)|^{2s}.
\]
Notice that $\chi_V = 1$ as $\det(V) \in (F^\times)^2$, the function satisfied the equation

$$\Phi(pg) = |a(p)|^{2+2s}|\nu(p)|^{-3-3s}\Phi(g).$$

In other words, $\Phi \in \text{Ind}_P^{\text{GSp}_6}(\lambda_s)$ where $\lambda_s$ is the character

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto |\det a/\det d|^{s/2}.$$

We define the Eisenstein series

$$E(g, \Phi) = \sum_{\gamma \in P \backslash \text{GSp}_6} \Phi(\gamma g).$$

We would like to consider the integral

$$(4.1) \quad \int_{[G]} \varphi(g)E(g, \Phi)dg,$$

where $G := \text{GL}(2)^0_F$ and $[G] = \mathbb{A}^\times G(F)\backslash G(\mathbb{A})$. Notice that at $s = 0$, we still get the period in Theorem 3.1.

Let $X$ be the variety over $F$ of the maximal isotropic subspaces of the space $E \oplus E$. Then $X$ has a transitive action by $\text{GSp}_6$ and the stabilizer of $x_0 := 0 \oplus E$ is $P$. Thus we may consider $P \backslash \text{GSp}_6$ as $X$ with right action by $\text{GSp}_6$. Let define the function

$$\Phi_x(g) = \Phi(\gamma_x g)$$

for some $\gamma_x \in \text{GSp}_6$ such that $x = x_0\gamma_x$. Let $P^x$ denote the stabilizer of $x$ in $\text{GSp}_6$, then $\Phi^x$ has the same transformation property for $P^x$ as $\Phi$ for $P$. Thus we have

$$E(g, \Phi) = \sum_{x \in X} \Phi_x(g)$$

Now we decompose this sum into orbits under the right action of $\text{GSp}_6$. For each $x \in X$, let $G_x = G \cap P_x$ denote the stabilizer of $x$ in $G$. Then we have

$$E(g, \Phi) = \sum_{x \in X/G} \sum_{h \in G_x(F)\backslash G(F)} \Phi_x(hg).$$
It follows that
\[
\int \varphi(g)E(g, \Phi)dg = \sum_{x \in X/G} \int_{[G]} \varphi(g) \sum_{h \in G_x \backslash G(F)} \Phi(\gamma_x hg)dg
\]
\[
= \sum_{x \in X/G} \int_{\mathbb{A}^\times G_x(F) \backslash G(\mathbb{A})} \varphi(g)\Phi_x(g)dg
\]
\[
= \sum_{x \in X/G} \int_{G_x(\mathbb{A}) \backslash G(\mathbb{A})} \varphi(hg)\Phi_x(hg)dh
\]

Using the fact that $G_x$ is the stabilizer $P_x$ of $x$ in $\text{GSp}_6$, we see that $\Phi_x(hg) = \lambda_x(h)\Phi_x(g)$ for a character of $P_x$. Thus we have that
\[
\int_{[G]} \varphi(g)E(g, \Phi)dg = \sum_{x \in X/G} \int_{G_x(\mathbb{A}) \backslash G(\mathbb{A})} \Phi_x(g)dg \int_{[G_x]} \varphi(hg)\lambda_x(h)dh
\]

Notice that $\lambda_x(n) = 1$ for any unipotent element $n$ in $P_x$ (even when it is not in the radical of $P_x$). Thus if $G_x$ contains a normal subgroup as a unipotent radical $N_x$ of a parabolic subgroup of $G$. Then the last integral over $[G_x]$ can be written as
\[
\int_{\mathbb{A}^\times G_x(F)N_x(\mathbb{A}) \backslash G_x(\mathbb{A})} \lambda(h)dh \int_{N_x(F) \backslash N_x(\mathbb{A})} \varphi(nhg)dn.
\]
This last integral vanishes as $\varphi$ is cuspidal. Such an orbit $xG$ is called negligible.

By computation of Piatetski-Shapiro and Rallis, all $x$ except one $x_1$ on $X$ are negligible. More precisely, $x_1$ corresponding to the subspace $F \oplus E^0$ where $E^0$ is the subspace of $E$ of elements of trace $0$. The corresponding stabilizer is
\[
G_1 = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\}, \quad a, b \in F^\times, x \in E_0
\]
The corresponding character $\lambda_x = 1$. Thus in the above integrals, we may replace $\varphi$ by
\[
\int_{[E^0]} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx.
\]
Let $\psi$ be a nontrivial character of $F \backslash \mathbb{A}$ and let $W$ be the Whittaker functional $\psi(\text{tr}_{E/F}x)$ on $E \backslash \mathbb{A}_E$. Then we have Fourier expansion of $\varphi$ by
\[
\varphi(g) = \sum_{\xi \in E^\times} W \left( \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right).
\]
Bring this to the above integral over \([E_0]\) to obtain
\[
\int_{[E_0]} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = \int_{[E_0]} \psi(\text{tr}(\xi x)) W \left( \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right) dx
= \sum_{\xi \in F^\times} W \left( \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right).
\]

Bring this this to our original integral to obtain:
\[
\int_{[G]} \varphi(g) E(g, \Phi) dg = \int_{A^\times \cdot G_1(F) \cdot N_1(A) \cdot G(A)} \Phi(\gamma g) \sum_{\xi \in F^\times} W \left( \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right) dg
= \int_{A^\times \cdot N_1(A) \cdot G(A)} \Phi(\gamma g) W(g) dg.
\]

Here \(\gamma \in \text{Sp}_3(F)\) such that
\[(0 \oplus E)\gamma = F \oplus E^0\]

Notice that we may integrate \(N_1 \setminus N(E) \simeq N(F)\) in the above integral to replace \(\Phi_1\) by
\[
\widetilde{\Phi}(g) = \int_A \Phi \left( \gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(-x) dx.
\]

In summary, we have proved the following:

**Theorem 4.1.** Consider the integral (4.1). Let \(N_1\) denote the subgroup of \(G\) of elements of the form \(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\) with \(x \in E_0\), the trace zero elements of \(E\). Let \(\Phi_1\) be a function on \(\text{GSp}_6\) defined by \(\Phi_1(g) = \Phi(\gamma_1 g)\) with \(\gamma_1\) an element in \(\text{GSp}_6\) such that
\[(0 \oplus E)\gamma_1 = F \oplus E_0.\]

Let \(\psi\) be a character of \(F\) which induced a character \(\psi_E\) by composing with trace \(\text{tr}_{E/F}\). Let \(W(g)\) be the Whittaker function of \(\varphi\) with respect to a character \(\psi_E\). Then
\[
\int_{[G]} \varphi(g) E(g, \Phi) dg = \int_{A^\times \cdot N(A) \cdot G(A)} \widetilde{\Phi}(g) W(g) dg.
\]
5 Local integrals

In this section we want to compute local factor in Theorem 4.1. For simplicity, we let $F$ and $E$ be the local fields, and $\Phi_s$ be an element in $\text{Ind}_{P}^{\text{GSp}_3}(\lambda_s)$ for each $s \in \mathbb{C}$ such that the restriction of $\Phi_s$ on a fixed maximal compact subgroup $K$ of $\text{GSp}_3$ is independent of $s$, and that $W(g)$ be the whittaker function for a representation $\pi$ of $\text{GL}_2(E)$ whose central character has trivial restriction on $F \times$ for a character of the form $\psi_E = \psi_F \circ \text{tr}_{E/F}$. We want to computer the integral

$$Z(\Phi_s, W) := \zeta(2s + 2)\zeta(4s + 2) \int_{F^x N_1 \backslash \text{GL}(2)^0} \Phi_s(\gamma g)W(g)dg,$$

where $N_1$ is the element $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x \in E_0$, and $\gamma$ is an element in $\text{GSp}_3$ such that

$$F \oplus E_0 = (0 \oplus E)\gamma.$$

It can be shown that when for $\text{Re}(s)$ sufficiently large, the above integral is absolutely convergent. When $F$ is non-archimedean with residue field $\mathbb{F}_q$, then the above integral defines a rational function of $q^s$ and admits a common denominator.

In the following, we give formula for unramified case.

Theorem 5.1 (Piateski-Shapiro – Rallis). Assume the following

1. $F$ is nonarchimedean of residue field $\mathbb{F}_q$ of characteristic $p \neq 3$, and $E/F$ is unramified, and $\psi$ has order 1;

2. $\Phi_s$ is the the unique element in $\text{Ind}_{P}^{\text{GSp}_3}(\lambda)$ fixed by $\text{GSp}_3(\mathcal{O}_F)$ and normalized by $\Phi_s(e) = 1$;

3. the representation $\pi$ of $\text{GL}_2(E)$ is unramified and $W(g)$ and $W(g)$ is the unique Whittaker function on $\text{GL}_2(E)$ fixed by $\text{GL}(\mathcal{O}_E)$ and normalized by $W(e) = 1$.

Then we have the following formula for $Z(\Phi_s, W)$:

1. For $E = F \oplus F \oplus F$ and $\pi$ on $\text{GL}_2(E) = \text{GL}_2(F)^3$ is given by

$$\sigma = \pi(a, b) \otimes \pi(a', b') \otimes \pi(a'', b''),$$
we have
\[ Z(\Phi_s, W) = \det(I_8 - A \otimes B \otimes C q^{-s-1/2})^{-1} \]
with
\[ A = \begin{pmatrix} q^{-a} & 0 \\ 0 & q^{-b} \end{pmatrix}, \quad B = \begin{pmatrix} q^{-a'} & 0 \\ 0 & q^{-b'} \end{pmatrix}, \quad C = \begin{pmatrix} q^{-a''} & 0 \\ 0 & q^{-b''} \end{pmatrix}. \]

2. For \( E = K \oplus F \) with \( K \) an unramified quadratic extension of \( F \), and \( \sigma \) on \( GL_2(K) \times GL_2(F) \) given by
\[ \pi = \pi(a, b) \otimes \pi(a', b') \]
then we have
\[ Z(\Phi_s, W) = \det(I_8 - (I_2 \otimes A \otimes B) \circ t' \cdot q^{-s-1/2})^{-1} \]
with
\[ A = \begin{pmatrix} q^{-2a} & 0 \\ 0 & q^{-2b} \end{pmatrix}, \quad B = \begin{pmatrix} q^{-a} & 0 \\ 0 & q^{-b} \end{pmatrix}, \]
where \( t'(v_1 \otimes v_2 \otimes v_3) = v_2 \otimes v_1 \otimes v_3. \)

3. For \( E \) a field, \( \pi = \pi(a, b) \), we have
\[ Z(\Phi, W) = \det(I_8 - (I_2 \otimes I_2 \otimes I_2 \otimes A) q^{-s-1/2} \circ t)^{-1} \]
where
\[ A = \begin{pmatrix} q^{-3a} & 0 \\ 0 & q^{-3b} \end{pmatrix}, \quad t(v_1 \otimes v_2 \otimes v_3) = v_2 \otimes v_3 \otimes v_1. \]

Then we have the following complete results about the local integrals when \( E \) is split.

**Theorem 5.2.** Assume that \( E \) is split \( E = F^3 \) and \( \pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \) be the corresponding decomposition.

1. There exists a local Euler factor \( L(s, \pi_1 \otimes \pi_2 \otimes \pi_3) \) such that, for any local data \( (\Phi, W) \) the quotient
\[ \tilde{Z}(\Phi_s, W) := \frac{Z(\Phi_s, W)}{L(s + 1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)} \]
is entire as a function of \( s \).
2. Let $\sigma_i, \ i = 1, 2, 3$ be the representation of the Weil-Deligne group of $F$ associate to $\pi_i$ by the local Langlands correspondence. Then
\[
L(s, \pi_1 \otimes \pi_2 \otimes \pi_3) = L(s, \sigma_1 \otimes \sigma_2 \otimes \sigma_3).
\]

3. If $F$ is non-archimedean, then there is a pair $(\Phi_s, W)$ such that
\[
\tilde{Z}(\Phi_s, W) \equiv 1.
\]

4. If $F$ is archimedean, there exists a finite collection of Whittaker function $W_j$ and section $\Phi_{j,s}$ such that
\[
\sum_j \tilde{Z}(\Phi_{0,j}, W) = 1.
\]

6 Tate’s Survey

In this and next section, we want to introduce the notion of Weil-Deligne group and Langlands correspondence used in Theorem 5.2.

Weil group

Let $F$ be a local field. If $F$ is nonarchimedean with residue field $\mathbb{F}_q$, then we define the Weil group $W_F$ to be the subgroup of $\text{Gal}(\overline{F}/F)$ whose image in $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \simeq \hat{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}$ and is generated by the Frobenius $x \rightarrow x^q$. The norm $\| \cdot \|_F$ on $W_F$ is define by $\sigma(x) \simeq x^{\|\sigma\|}$. The class field theory gives an isomorphism
\[
F^\times \longrightarrow W^{ab}.
\]

Notice that $W_F$ is solvable as for any finite Galois extension $K$ of $F$, the Galois group $\text{Gal}(K/F)$ is filtered by decomposition subgroups $\text{Gal}(K/F)^i$ of elements $\sigma$ such that $\text{ord}_K(\sigma(\pi_K) - \pi_K) \geq i + 1$. The quotient of these groups subgroups of $G_0/G_1 \subset \text{Gal}(k(K)/k(F))$, and $G_i/G_{i+1} \subset \overline{\mathbb{F}_q}$.

If $F \simeq \mathbb{C}$, then the Weil group is defined to be $W_F = F^\times$.

If $F = \mathbb{R}$ then $W_F$ is defined to be the union $F^\times \cup jF^\times$ with $j^2 = -1$ and $jx = \bar{x}j$. 

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Representations of $W_F$

If $F$ is archimedean, the only characters of $F$ are of form $z^N\omega_s$ where $z$ is an embedding $F \subset \mathbb{C}$ and $\omega_s$ is $s$-power of the norm map. This form is unique if we require that $N = 0$ or $1$ when $F$ is real. If $F$ is complex this are the only irreducible representations. If $F$ is real, then $F$ has a subgroup $F^\times$ of index $2$. All other irreducible representations are of the form $\text{Ind}_{F/F}(z^{-N}\omega^s)$ with $N > 0$.

If $F$ is non-archimedean of characteristic $p$, then any irreducible representation of $W_F$ of degree prime $p$ is induced from a character. This shows that the two dimensional representations of $W_F$ when $p \neq 2$ are induced from a character on $W_K$ for a quadratic extension $K$ of $F$. If $p = 2$, then we can show that non primitive representations are quotient of two representations induced by characters. More generally, by Brauer’s theorem, the K-group of representations of $W_F$ is generated by representations induced from characters.

$L$-series and $\epsilon$-factors

If $F = \mathbb{R}$, then we define

$$L(s, x^{-N}) = \Gamma_R(s) = \pi^{-s/2}\Gamma(s/2), \quad N = 0, 1$$

If $F = \mathbb{C}$, then

$$L(s, z^{-N}) = \Gamma_C = 2(2\pi)^{-s}\Gamma(s), \quad N \geq 0.$$  

If $F$ is non-archimedean with a uniformizer $\varpi$,

$$L(s, \chi) = \begin{cases} 
(1 - \chi(\varpi))^{-1} & \text{if } \chi \text{ is unramified} \\
1 & \text{if } \chi \text{ is ramified}
\end{cases}$$

For a non-trivial additive character $\psi$ of $F$, we may define the $\epsilon$-factor so that the functional equation is true:

$$\int \hat{\phi}(x)\chi^{-1}|x|^{1-s}d^\times x \quad \frac{L(1-s, \chi^{-1})}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi)\frac{\int \phi(x)\chi(x)|x|^sd^\times x}{L(s, \chi)}.$$  

Here $\hat{\phi}$ is the Fourier transform with respect to the self-dual Haar measure $dx$ on $F$:

$$\hat{\phi}(x) = \int \phi(y)\psi(xy)dx$$
Using Brauer’s theorem, we can define L-function and $\epsilon$ for any representations of $W_F$.

**Weil-Deligne group**

The Weil-Deligne group of a non-archimedean local field $F$ is the semiproduct $W'_{F} := \mathbb{G}_a \rtimes W_F$ as a group scheme where $W_F$ acts as $waw^{-1} = \|w\|a$. Thus a representation of $W'_F$ on a vector space $V$ will have the form

$$\rho'(a, w) = e^{aN} \rho(w)$$

where $\rho$ is a representation of $W_F$ and $N$ is a nilpotent element in $\text{End}(V)$ such that

$$\rho(w)N \rho(w)^{-1} = \|w\|N.$$ 

We say that $\rho'$ is admissible if $\rho$ is semisimple.

It is easy to check that a representation $\rho'$ is irreducible if and only if $N = 0$ and $\rho$ is irreducible. An admissible representation $\rho'$ is indecomposable if and only if $\rho' = \rho' \otimes \text{sp}(n)$ where $\text{sp}(n)$ is a representation of $W'_F$ on

$$V = \mathbb{Q}^n = \sum \mathbb{Q}e_i$$

$$\rho(w)e_i = \|w\|^i e_i, \quad Ne_i = e_{i+1}.$$ 

We have the following definition of $L$-functions and $\epsilon$-factors:

$$L(s, \rho') = \det(1 - q^{-s} \Phi|_{V'_N})^{-1}.$$ 

$$\epsilon(s, \rho', \psi) = \epsilon(s, \rho, \psi) \det(-\Phi|_{V'/V'_N})$$

where $\Phi \in W_F$ is a geometric Frobenius.

**$\ell$-adic representations**

Let $F$ be a local field. Let $\ell$ be a prime not equal to the characteristic $p$ of $F$. Let

$$t_{\ell} : I_F \rightarrow \mathbb{Z}_{\ell}(1) = \lim_{\rightarrow} \mu_{\ell^n}$$

be the nonzero homomorphism defined by

$$t_{\ell}(\sigma) = \frac{\sigma(p^{1/\ell^n})}{p^{1/\ell^n}}.$$ 

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Then
\[ t_\ell(w\sigma w^{-1}) = \|w\| t_\ell(\sigma), \quad \sigma \in I, \ w \in W_F. \]

Fix a geometric Frobenius element in \( W_F \). Then for any representation \( \rho' \) of \( W'_F \) on a finite dimensional space \( V \) over \( \bar{\mathbb{Q}}_\ell \), we can associate an \( \ell \)-adic representation of \( G_F \) by
\[
\rho_\ell(\Phi^n\sigma) = \rho(\Phi^n\sigma) \exp(\ell(t_\ell(\sigma)N)), \quad \sigma \in I, \ n \in \mathbb{Z}.
\]

In this way, we can define \( L \)-functions and \( \epsilon \)-factors for \( \ell \)-adic representations.

### 7 Local Langlands correspondences

Let \( \mathcal{G}_F(n) \) be the set of isomorphism classes of admissible representations of \( W'_F \) of degree \( n \). Let \( \mathcal{A}_F(n) \) be the set of isomorphism classes of admissible representations of \( \text{GL}_n(F) \). Then by work of Harris and Taylor et al., there is a canonical projection
\[
\pi_F : \mathcal{G}_F(n) \rightarrow \mathcal{A}_F(n), \quad \rho' \mapsto \pi(\rho')
\]
such that

1. \( \pi_F(\rho''(\chi)) = \pi(\rho'')(\chi) \) for any character \( \chi \) of \( F^\times \).

2. \( \det(\rho') \) corresponds to \( \omega_{\pi_F(\rho')} \), the central character of \( \pi_F(\rho') \) via the local class field theory.

3. \( \pi_F(\rho'') = \pi_F((\rho'')^\vee) \).

4. \( L(s, \rho_1' \otimes \rho_2') = L(s, \pi_F(\rho_1') \times \pi_F(\rho_2')) \).

5. \( \epsilon(s, \rho_1' \otimes \rho_2', \psi) = \epsilon(s, \pi_F(\rho') \times \pi_2'(\rho_2), \psi) \).

6. \( \pi_F \) preserve conductors.

7. If \( F \) is a finite Galois extension of a field \( F_0 \), then \( \pi_F \) is compatible with the natural action of \( \text{Gal}(F/F_0) \) on \( \mathcal{G}_F \) and \( \mathcal{A}_F \).

In nonarchimedean case, in this correspondence, the unramified representation corresponds to unramified representations; irreducible representations corresponding to supercuspidal representations; indecomposable representations \( \rho \otimes \text{sp}(r) \) corresponding to essentially square integrable representations.
$Q(\rho, \rho(1), \ldots, \rho(r-1)$ which is defined as the unique irreducible quotient of the induced from a parabolic subgroup of $\text{GL}(n)$ with Levi group $\text{GL}(m)^r$ where $\rho \otimes \rho(1) \otimes \cdots \rho(r-1)$ is defined.

Of course the most difficult part if the correspondence between irreducible one in $\mathcal{G}_F$ and supercuspidal one in $\mathcal{A}_F$. In case of $n = 2$ when the characteristic $p \neq 2$, the supercuspidal representations corresponding to the induced representation from characters on a quadratic fields are constructed from Weil representations.

When $F = \mathbb{C}$, the irreducible representations of $\text{GL}_n(\mathbb{C})$ are parabolically induced from characters on $\text{GL}_1(\mathbb{C})$. The Langlands correspondence is thus given by class field theory.

When $F = \mathbb{R}$, the irreducible representations of $\text{GL}_n(\mathbb{R})$ are parabolically induced from representations following representation characters on $\text{GL}_1(\mathbb{R})$ or the representation $D_\ell \otimes |\cdot|^s$ on $\text{GL}_2(\mathbb{R})$ where $D_\ell$ ($\ell \in \mathbb{N}$) is the weight $\ell + 1$ representations on the holomorphic functions on $H^\pm = \mathbb{C} \setminus \mathbb{R}$ such that

$$\|f\|^2 = \int_{H^\pm} |f(z)|^2 |y|^{N+1} \frac{|dx\,dy|}{y^2} < \infty.$$ 

The action is given as usual:

$$D_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = \frac{1}{(cz+d)^{\ell+1}} f \left( \frac{az+b}{cz+d} \right).$$

The one dimensional representations are classified by characters of $W^{\text{ab}}_\mathbb{R} = \mathbb{R}$ via class field theory. The two dimensional representations are induced from $z^N |\cdot|^s$ which corresponds to the representations $D_N \otimes |\cdot|^{(s+N)/2}$.

### 8 Explicit formula

In this section we want to give a precise formula for the period $|\ell(f)|^2$ under the condition that $\pi$ has trivial restriction on $\mathbb{A}^\times$. We assume that $f$ is decomposable with respect to a decomposition $\pi = \otimes \pi_v$: $f = \otimes f_v$. The functional $f \mapsto \ell(f)$ gives a $B_\mathbb{R}^\times$-invariant linear form on $\pi$. Thus non-vanishing of one $\ell(f)$ will implies that we have nonzero $B_\mathbb{R}^\times$-invariant functional on each $\pi_v$. The following is an criterion for existence of such a criterion in terms of epsilon factors $\epsilon_v(\pi, \frac{1}{2}) = \pm 1$:
Theorem 8.1 (Prasad). Consider the space $\text{Hom}_{B_v^\times}(\pi_v, \mathbb{C})$ of $B_v^\times$-invariant linear form on $\pi_v$. Then

$$\dim \text{Hom}_{B_v^\times}(\pi_v, \mathbb{C}) \leq 1.$$ 

Moreover, $\dim \text{Hom}_{B_v^\times}(\pi_v, \mathbb{C}) \neq 0$ if and only if one of the following two conditions holds:

1. $B_v$ is split if $\epsilon(\pi_v, \frac{1}{2}) = 1$;
2. $B_v$ is nonsplit if $\epsilon(\pi_v, \frac{1}{2}) = -1$.

About the global function, we have the following:

Theorem 8.2 (Jacquet’s conjecture; Harris-Kudla-Prasad). The space $\text{Hom}_{B_E^\times}(\pi, \mathbb{C})$ is at most one dimensional. It is nonzero only if the following conditions hold:

1. $B_v$ is nonsplit exactly at the place $v$ where $\epsilon(\pi, \frac{1}{2}) = -1$;
2. $L(\pi, \frac{1}{2}) \neq 0$.

We would like to prove the following:

Theorem 8.3.

$$\frac{|\ell(f)|^2}{(f, f)} = \frac{\zeta_E(2)L(1/2, \pi, r_8)}{\zeta_F(2)L(1, \pi, \text{ad}_9)} \prod_v \beta_v(f).$$

Here $r_8$ and $\text{ad}_9$ are the standard 8-dimensional representation of $L$-group $\text{GL}_2(\mathbb{C})^3$ of $B_E^\times$, and

$$\beta_v(f) = \frac{\zeta_{F_v}(2)L(1/2, \pi_v, \text{ad}_9)}{\zeta_{E_v}(2)L(1/2, \pi_v, r_8)} \int_{F_v^\times \backslash B_v^\times} (\pi(g) f_v, f_v) dg.$$

9 Inner product formula

In this section we want to compute the inner product of form on an inner form of $\text{GL}_2$ in terms if Shimuzu lifting. More precisely, let $B/F$ be a quaternion algebra over a number field and let $f_1, f_2 \in \pi$ be an element in a cuspidal representation of $B_{E\cap F}^\times$, and $\phi \in S(V_{\mathbb{A}})$ and $\varphi \in J\text{L}(\pi)$ so that

$$f_1 \otimes \bar{f}_2 = \theta_{\phi}^\varphi.$$
Assume everything is decomposable, we want to compute the inner product \((f_1, f_2)\) in terms of local terms in
\[
\phi \otimes \varphi = \otimes \phi_v \otimes \varphi_v \in \mathcal{S}(V_\mathbb{A}) \otimes \mathcal{JL}(\pi) = \otimes (\mathcal{S}(V_v) \otimes \mathcal{JL}(\pi_v)).
\]

By definition,
\[
(f_1, f_2) = \int_{|B^\times|} f(h) \bar{f}(h) dh = \int_{|B^\times|} \theta^\pi_{\phi}(h) dh.
\]

Notice that the diagonal embedding
\[
B^\times \longrightarrow B^\times \times B^\times / \Delta(F^\times) \longrightarrow \text{GO}(V)
\]
is given by the conjugation of \(B^\times\) on \(V = B\). Let \(V = V_1 \oplus V_2\) with \(V_1\) consisting of scale elements, and \(V_2\) consisting of trace free elements. Then \(B^\times / F^\times\) can be identified with \(O(V_2)\). Assume that \(\phi = \phi_1 \otimes \phi_2\) with \(\phi_i \in \mathcal{S}(V_i(\mathbb{A}))\). Then
\[
\theta_\phi(g, h) = \theta_{\phi_1}(g, 1) \theta_{\phi_2}(\bar{g}, h), \quad (g, h) \in \text{SL}_2(\mathbb{A}) \times O(V_2)(\mathbb{A}).
\]
where \(\bar{g} \in \tilde{\text{SL}}_2(\mathbb{A})\) lifts \(g\). It follows that
\[
(f_1, f_2) = \int_{[O(V_2)]} \varphi(h) \theta_{\phi_1}(\bar{g}, 1) \theta_{\phi_2}(\bar{g}, h) dg = \int_{[\text{SL}_2]} \varphi(g) \theta_{\phi_1} (\bar{g}, 1) \left( \int_{[O(V_2)]} \theta_{\phi_2}(\bar{g}, h) dh \right) dg.
\]

By Siegel-Weil the last integration defines an Eisenstein series:
\[
\int_{[O(V_2)]} \theta_{\phi_2}(\bar{g}, h) dh = \sum_{\gamma \in P(F) \setminus \text{SL}_2(F)} \omega(\gamma \bar{g}) \phi_2(0).
\]

Notice that the sum on the right hand may not be convergent. To regularize this integral, we may insert \(\delta(\gamma g)^s\) for \(\text{Re}(s) > 0\) then take limit \(s \to 0\). We ignore this process and take formal computation. Bring this to our inner product to obtain:
\[
(f_1, f_2) = \int_{[\text{SL}_2]} \varphi(g) \theta_{\phi_1} (\bar{g}, 1) \sum_{\gamma \in P(F) \setminus \text{SL}_2(F)} \omega(\gamma \bar{g}) \phi_2(0) dg
\]
\[
= \int_{P(F) \setminus \text{SL}_2(\mathbb{A})} \varphi(g) \theta_{\phi_1} (\bar{g}) \omega(\bar{g}) \phi_2(0) dg
\]
\[
= \int_{P(F) N(\mathbb{A}) \setminus \text{SL}_2(\mathbb{A})} \left( \int_{N(F) \setminus N(\mathbb{A})} \varphi(n g) \theta_{\phi_1} (n \bar{g}) dn \right) \omega(\bar{g}) \phi_2(0) dg.
\]
Compute the inside integral, we need to compute $\theta_{\phi_1}(ng)$ for $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$:

$$\theta_{\phi_1}(ng) = \sum_{\xi \in F} \omega(n\tilde{g})\phi_1(\xi) = \sum_{\xi} \psi(\frac{1}{2}\text{tr}(x\xi))\omega(\tilde{g})\phi_1(\xi).$$

Thus the inner integral can be written as

$$\sum_{\xi \in F} \omega(\tilde{g})\phi_1(\xi) \int_{F\backslash A} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(\text{tr}(x\xi^2)) dx.$$

Since $\varphi$ is a cusp form, the integral here vanishes if $x = 0$. If $x \neq 0$, it is given by $W(\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix})$ for $W$ the whittaker function for $\varphi$ for the character $\psi$. On the other hand for $\xi \neq 0$, $\omega(\tilde{g})\phi(\xi) = \omega(\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \tilde{g})\phi_1(1)$: It follows that

$$(f_1, f_2) = \int_{P(F) \backslash N(\mathbb{A}) \backslash SL_2(\mathbb{A})} \sum_{\xi \in F^x} \omega(\tilde{g})\phi_1(\xi)W(\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} g) \omega(\tilde{g})\phi_2(0) dg$$

$$= \int_{N(\mathbb{A}) \backslash SL_2(\mathbb{A})} W(g)\omega(\tilde{g})\phi_1(1)\omega(\tilde{g})\phi_2(0) dg.$$

This formula suggests that we may define local inner product by

$$(f_{1v}, f_{2v}) = B_v(\phi_v, W_v)$$

where

$$B(\phi_v, W_v) = \int_{N(F_v) \backslash SL_2(F_v)} W_v(g)\omega(g)\phi(1) dg$$

where $\phi_v \in S(V)$ with $V = B$ and $W_v \in JL(\pi_v)$.

### 10 Integration of matrix coefficients

We now apply the above section to compute the integration of matrix coefficients. So let us go back to the setting in §8. We want to compare the integral of matrix coefficients. Start with

$$f_1 \otimes \tilde{f}_2 = \theta^*_{\phi}$$
between forms a pair for forms in \( \pi \in A_0(B_{E,A}^\times) \) and form on \( JL(\pi) \in A_0(\text{GL}_2(E \otimes A)) \), we have a formula

\[
(f_1, f_2) = \int_{N(A_E) \backslash \text{SL}_2(A_E)} W(g) \omega(g) \phi(1) dg.
\]

Here \( W \) is the Whittaker function corresponding to \( \varphi \). For \( h \in B_{A}^\times \), the matrix coefficient is given by

\[
(\pi(h)f_1, f_2) = \int_{N(A_E) \backslash \text{SL}_2(A_E)} W(gd(h)) \omega(gd(h), (h, 1)) \phi(1) dg \\
= |\nu(h)|_{A}^{-3} \int_{N(A_E) \backslash \text{SL}_2(A_E)} W(gd(h)) \omega(d(h)^{-1}gd(h)) \phi(h^{-1}) dg
\]

The integration of the matrix coefficients is given by

\[
\int_{F_{A}^\times \backslash B_{A}^\times} (\pi(h)f_1, f_2) dh \\
= \int_{F_{A}^\times \backslash B_{A}^\times} |\nu(h)|_{A}^{-3} dh \int_{N(E_A) \backslash \text{SL}_2(E_A)} W(gd(h)) \omega(d(h)^{-1}gd(h)) \phi(h^{-1}) dg
\]

We may first integrate over \( B^1 \) of elements \( b \) in \( B \) with norm \( \nu(b) = 1 \) to obtain

\[
\int_{C} |\nu(c)|_{A}^{-3} dc \int_{N(E_A) \backslash \text{SL}_2(E_A)} W(gd(c))dg \int_{B_{1}^A / \pm 1} \omega(d(c)^{-1}gd(c))\phi(c^{-1}h^{-1}) dh
\]

where \( C = B_{A}^\times / B_{A}^1 \cdot F_{A}^\times \simeq \{ \pm 1 \} \). The first two integrals can be combined to a single integral over \( F_{A}^\times N(E_A) \backslash \text{GL}_2(F_A)^0 \). Thus we obtain

\[
\int_{F_{A}^\times N(E_A) \backslash \text{GL}_2(F_A)^0} |\nu(g)|_{A}^{-3} W(g) dg \int_{B_{1}^A / \pm 1} \omega(d(g)^{-1}g)\phi(h g^{-1}h^{-1}) dh
\]

In summary, we have shown that the local matrix coefficients is given by

\[
\int_{F_{A}^\times B_{A}^\times} (\pi(h)f_1, f_2) dh = \int_{F_{A}^\times N(E_A) \backslash \text{GL}_2(E_A)^0} W(g) F_\phi(g) dg
\]

where

\[
F_\phi(g) = |\nu(g)|_{A}^{-3} \int_{B^1} \omega(d(g)^{-1}g)\phi(h g^{-1}h^{-1}) dh.
\]
Let us check if \( F(g) \) has right translation under \( F^\times \) and \( N(E_v) \). For \( z \in F^\times_v \),
\[
d(zg)^{-1}zg = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}
\]
and \( h_{zg} = zh_g \). Thus integrant changes
\[
|z^2|_h^{-3}|z|^2_E = 1.
\]
Similarly, for \( n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \),
\[
d(n(g)^{-1}ng = n(x\nu(g))d(g)g, \quad h_{ng}^{-1} = h_g
\]
and the integrant changes \( \psi_E(x\nu(g)\nu(g)^{-1}) = \psi_E(x) \).

To prove the Theorem 8.3, it suffices to compare \( F(g) \) with \( \tilde{\Phi}(g) \) defined in Theorem 4.1:
\[
\tilde{\Phi}(g) = \int_H \Phi \left( \gamma \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) g \right) \psi(-x)dx.
\]
Here
\[
\Phi(g) = |\nu(g)|^{-3}\omega(d(g)^{-1}g)\phi(0) = \omega(gd(g)^{-1})\omega(h_g)\phi(0)
\]
and \( \gamma \in \text{GSp}_3 \) such that
\[
(0 \oplus E)\gamma = F \oplus E_0.
\]
Here \( h_g \in \text{GO}(V_h) \) with norm \( \nu(g) \).

Take the decomposition \( E^2 = F^2 \oplus E_0^2 \) which induce an embedding \( H := \text{SL}_2 \) into \( \text{GSp}(E^2) \) by acting on the first factor. Then we may take \( \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in H \). The element \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) acts diagonally on the above decomposition in the expression of \( \tilde{\Phi} \). By writing \( \phi \) as linear combination of according to \( S(\mathbb{A}_E^2) = S(\mathbb{A}_F^2)S(\mathbb{A} \otimes E_0^2) \), we see that we see that the action of \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) on the second factor does not effect the result. Thus we may replace this action is given in \( H \). Then it is easy to see that \( \tilde{\Phi} \) is the Fourier coefficient of the following Eisenstein series:
\[
\tilde{\Phi}(g) = \int_{F \setminus \mathbb{A}_F^2} E_H \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \psi(-x)dx.
\]
\[ E_H(g) = \sum_{\gamma \in P_{H(F)} \setminus H(F)} \omega(\gamma g(dg)^{-1}) \omega(h_g) \phi(0). \]

Now we apply Siegel-Weil formula: we obtain

\[ E_H(g) = \int_{B^1(F) \setminus B^1(A)} \omega(gd(g)^{-1}) \omega(h_g) \phi(h^{-1}\xi) dh. \]

It follows that

\[ E_H \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \int_{B^1(F) \setminus B^1(A)} \omega(gd(g)^{-1}) \omega(h_g) \phi(h^{-1}\xi) \psi(x\nu(\xi)) dh \]

\[ \Phi(g) = \int_{B^1(F) \setminus B^1(A)} \omega(gd(g)^{-1}) \omega(h_g) \phi(h^{-1}\xi) \psi((\nu(\xi) - 1))x) dh dx \]

\[ = \int_{B^1(F) \setminus B^1(A)} \omega(gd(g)^{-1}) \omega(h_g) \phi(h^{-1}\xi) dh \]

\[ = |\nu(g)|^{-3} \int_{B^1(h)} \omega(d(g)^{-1}g) \omega(h_g) \phi(h^{-1}) dh \]

\[ = F_{\phi}(g) \]

This completes the proof of the theorem.