Small points and Berkovich metrics

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September 6, 2010

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Introduction

The aim of this paper is to develop an intersection theory for metrized line bundles on varieties over finitely generated fields in terms of Berkovich spaces and apply it to equidistribution problems of small points.

In §1, we review the basic definition of Berkovich space introduced by [Be]. We consider more general situation than Berkovich. We also introduce continuous and norm-equivariant metrics for line bundles, and show that at so called enclosed points such metric exists uniquely.

In §2, we define our integrable line bundles as limits of line bundles on integral models over $\mathbb{Z}$ with certain topology. We define intersections of these line bundles with values in §3 as limits of either Gillet–Soulé intersections or Deligne pairings [GS, GS2, De].

In §4, we study the notion of small points, propose an equidistribution conjecture, and prove an equidistribution result (Theorem 4.7) about the
specialization of small points. Finally, in §5 we apply our theory to polarized
dynamical systems over finitely generated fields, and show that the equidistri-
bution conjecture implies the equidistribution of Galois orbits of preperiodic
points.

The equidistrbution conjecture is known to be true either in number field
case by Yuan [Yu] or the function field case by Faber [Fab] and Gubler [Gu2].
The equidistribution result in Theorem 4.7 is crucial in the consideration of
preperiodic points in another recent paper [YZ] of the authors.

While our work is a continuation of previous works [Zh2, SUZ, Yu] on
varieties over number field but has been inspired by Gubler [Gu1], Moriwaki
[Mo2], and Chambert-Loir [Ch]. The work of Gubler and Chambert-Loir
convince us to work on Berkovich spaces, while Moriwaki’s work suggests a
way to take limits and a Northcott property. The difference between our
treatment and Moriwaki’s is that he uses polarization on the base while we
don’t. Our theory is compatible with base changes, but his theory is not.

1 Berkovich space and metrics

Berkovich spaces

Let $X$ be a scheme and denote by $X^{an}$ the associated Berkovich analytic
space. The definition in [Be] works in this general setting. If $X$ is covered by
affine schemes Spec$A$ then $X^{an}$ as a set is covered by the affinoid (Spec$A)^{an}$ of
multiplicative seminorms on $A$. For each $x \in (\text{Spec}A)^{an}$, the corresponding
norm is a composition

$$A \rightarrow \kappa_x \xrightarrow{|\cdot|} \mathbb{R}$$

where $\kappa_x$ is a residue field and $|\cdot|$ is a valuation (multiplicative norm) on
$\kappa_x \rightarrow \mathbb{R}$. We write the first map as $f \mapsto f(x)$. Then the image of $f$ under the
composition is written as $|f(x)|$, and we also write it as $|f|_x$ for convenience.
Then the multiplicative seminorm (on an affinoid) corresponding to $x \in X^{an}$
is just $|\cdot|_x$. The topology on $X^{an}$ is the weakest one such that all $(\text{Spec}A)^{an}$
are open, and such that the function $|f(x)|$ is continuous on $X^{an}$ for all $f$. If
$X$ is separated then $X^{an}$ is Hausdorff.

We have a continuous map $X^{an} \rightarrow X$ by sending $x$ to the kernel of the
norm in Spec$A$. The fiber is simply the the set of norms on the residue fields.
We also have a natural section $X \hookrightarrow X^{an}$ by sending a point $x$ to the trivial
norm the residue field $\kappa_x$. 

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Two elements \( x \) and \( y \) of \( X^{\text{an}} \) are called \textit{norm-equivalent} if they have the same image in \( X \) and are induced by equivalent norms on the residue field. Thus there are numbers \( a, b > 0 \) such that \( |\cdot|_x^a = |\cdot|_y^b \) locally. The set of norm-equivalence classes of in \( X^{\text{an}} \) is called the set of \textit{places} of \( X \) and is denoted as \( M(X) \). Most of our theory depends only on \( M(X) \) despite it does not have a good topology as \( X^{\text{an}} \) does.

One important example is \((\text{Spec}\, \mathbb{Z})^{\text{an}}\). It is a tree with vertices \( V := \text{Spec}\, \mathbb{Z} \cup \{\infty\} \) and the edges \( E_p \) for each \( p \in V \setminus \{0\} \) corresponding to norms \( |\cdot|_p^t \) with \( t \geq 0 \) if \( p \neq \infty \) and \( 0 \leq t \leq 1 \) if \( p = \infty \). The space \((\text{Spec}\, \mathbb{Q})^{\text{an}}\) is identified with the complement of \( V \setminus \{0\} \). The set of places of \( \text{Spec}\, \mathbb{Q} \) will be the union of \( \{0\} \) and open parts of \( E_p \). We write \((\text{Spec}\, \mathbb{Z})_p^{\text{an}} = E_p \).

If \( f : X \rightarrow Y \) is a morphism of schemes, then we get a morphism \( f^{\text{an}} : X^{\text{an}} \rightarrow Y^{\text{an}} \). If \( f \) is proper, so is \( f^{\text{an}} \); if \( f \) has connected fibers, then \( f^{\text{an}} \) has arwise connected fibers. If \( T \) is a subset of \( Y^{\text{an}} \) then we can define \( T\)-adic analytic space by base change

\[
 f_T^{\text{an}} : X_T^{\text{an}} \longrightarrow T. 
\]

When \( Y = \text{Spec} \, k \) and \( T = \{v\} \) corresponds to a valuation of \( k \), we write \( X_T^{\text{an}} \) as \( X_v^{\text{an}} \). If \( v \) is the trivial norm, then we simply write it as \( X_k^{\text{an}} \).

If \( X \) is integral with function field \( K \), then \((\text{Spec}\, K)^{\text{an}}\) is identified with the set of \textit{generic points} on \( X^{\text{an}} \), i.e., the preimage of the generic point in \( X \). If \( X \) is a projective variety over a field \( k \), then for any valuation \( v \) of \( k \), \( X_v^{\text{an}} \) gives an \textit{algebraic compactification} for \((\text{Spec}\, K)_v^{\text{an}}\).

Denote \( X^{\text{an}}_{\infty} := X_{(\text{Spec}\, \mathbb{Z})^{\text{an}}}^{\text{an}} \), which is the archimedean part of \( X^{\text{an}} \). Since every archimedean norm of a field \( k \) is given by a complex conjugacy class of embeddings \( k \rightarrow \mathbb{C} \) and a norm \( |\cdot|^t \) \((t \in (0, 1])\) on \( \mathbb{C} \), we see that

\[
 X^{\text{an}}_{\infty} = (0, 1] \times X(\mathbb{C})/\text{complex conjugation}. 
\]

Here the second factor can be identified with the set of closed points in the scheme \( X_{\mathbb{R}} \).

**Enclosed points**

For any non-archimedean point \( x \in X^{\text{an}} \), let \( C(x) \) denote the set of the points norm-equivalent to \( x \). If \( x \notin X \), then we have a natural parameterization

\[
 (0, \infty) \rightarrow C(x), \quad t \mapsto x_t
\]
given by $| \cdot |_x = | \cdot |_{\kappa}$. It is clear that $C(x)$ has a limit point $x_0 \in X$ given by the trivial norm on $\kappa(x)$, and thus it is just the image of $x$ in $X$.

We say that $x$ is *enclosed* if the limit $x_\infty \in X$ exists as $t \to \infty$, i.e., the morphism $\tilde{x} : \text{Spec} \kappa \to X$ can be extended into a morphism $\text{Spec} R_x \to X$ where $R_x$ is the subring of elements in $\kappa_x$ with norm $\leq 1$. It is clear that either $x_\infty = x$ if $x \in X$, or $x_\infty$ is given a divisor in the Zariski closure $\overline{x}_0$ of $x_0$. In this way, $x_\infty$ represents the trivial norm on $R_x/m_x$. Conversely, for any point $x_0 \in X$, any divisor $x_\infty$ in the Zariski closure $\overline{x}_0$ defines an equivalent class of enclosed points.

Let $\tilde{X}^{\text{an}}$ denote the subset of $X^{\text{an}}$ consisting of enclosed points. Then $\tilde{X}^{\text{an}}$ contains $X$ and is called the *enclosed locus* of $X^{\text{an}}$. For an affine space, $(\text{Spec} A)^{\text{an}}$ is the set of bounded seminorms on $A$. Here are some examples of enclosed locus:

- It is easy to have $(\text{Spec} \mathbb{Z})^{\text{an}} = (\text{Spec} \mathbb{Z})^{\text{an}}$.
- If $K$ is a field, then $\tilde{\text{Spec}} K$ is just the point given by the trivial norm.
- For a discrete valuation ring $R$, the enclosed locus $(\tilde{\text{Spec}} R)^{\text{an}}$ consists of the line segment described as above.

**Proposition 1.1.** Let $X \to Y$ be a morphism of schemes. Then the induced map $X^{\text{an}} \to Y^{\text{an}}$ sends enclosed elements to enclosed elements:

$$\tilde{X}^{\text{an}} \longrightarrow \tilde{Y}^{\text{an}}.$$ 

Moreover, if this morphism is proper, then a point in $X^{\text{an}}$ is enclosed if and only if it has an enclosed image, i.e.,

$$\tilde{X}^{\text{an}} = X_{\tilde{Y}^{\text{an}}}^{\text{an}}.$$ 

**Proof.** The first assertion is clear. For the second assertion, use the properness criterion. \qed

If $X$ is an integral scheme with function field $K$, then $\text{Spec} (K)_X^{\text{an}}$ is the set of norms induced from divisors of $X$. 

4
Berkovitch metrics

Let $X$ be a scheme and $L$ be a line bundle on $X$. It induces a line bundle $L^{an}$ on $X^{an}$. At each point $x \in X^{an}$, the residue $L^{an}(x)$ is the same as the residue of $L$ on the image of $x$ in $X$. By a Berkovich metric $\| \cdot \|$ on $L$ we mean a metric on $L^{an}$ compatible with norms on $\mathcal{O}_X$. More precisely, to each point $x$ in $X^{an}$ with residue field $\kappa$ we assign a norm $\| \cdot \|_x$ on the $\kappa$-line $L^{an}(x)$ which is compatible with the norm of $\kappa$ in the sense that

$$\| \ell \|_x = |f| \cdot \|\ell\|_x, \quad f \in \kappa, \quad \ell \in L^{an}(x).$$

We say that the metric $\| \cdot \|$ on $L$ is continuous if for any section $\ell$ of $L^{an}$ on an open subset $U$ of $X^{an}$, the function $\|\ell(x)\| = \|\ell(x)\|_x$ is continuous on $x \in U$. We say that the metric is norm-equivariant if for any two norm-equivalent points $x$ and $y$ with $|\cdot|_x = |\cdot|_y$, one has $\| \cdot \|_x = \| \cdot \|_y$.

We denote by $\text{Pic}(X)$ (resp. $\mathcal{P}ic(X)$) the Picard group of isomorphism classes (resp. the category ) of line bundles on $X$. Let $\widehat{\text{Pic}}(X^{an})$ (resp. $\widehat{\mathcal{P}ic}(X^{an})$) denote the group of isometry classes (resp. the category) of line bundles on $X$ endowed with continuous and norm-equivariant metrics. Thus we have a morphism

$$\widehat{\text{Pic}}(X^{an}) \to \text{Pic}(X), \quad \widehat{\mathcal{P}ic}(X^{an}) \to \mathcal{P}ic(X).$$

The fibers are homogeneous spaces of the group of metrics on $\mathcal{O}_X$.

If $s$ is a rational section of a metrized line bundle with divisor $D$, then $- \log \|s\|$ define a Green’s function for the divisor $D^{an}$ in the sense that this function is continuous and norm-equivariant and has logarithmic singularity near $D$. Conversely, given any divisor $D$ and a Green’s function $g$ on $X^{an}$ with logarithmic singularity along $D^{an}$ in $X^{an}$, then $e^{-g}$ defines a metric on $\mathcal{O}(D^{an})$. The pair $(D,g)$ is called an arithmetic divisor. Let $\widehat{\text{Div}}(X^{an})$ denote the group of arithmetic divisors. Then we have just described a homomorphism

$$\widehat{\text{Div}}(X^{an}) \to \widehat{\text{Pic}}(X^{an}).$$

Let $\phi : X \to Y$ be a morphism of schemes and $\mathcal{M}$ be a line bundle on $Y$ with a metric on $Y^{an}$. Then one has a metric on $\phi^*\mathcal{M}$ defined by the pull-back map.

Analogously, for a morphism $X \to Y$ and a subset $T$ of $Y^{an}$, we can define $T$-adic metrics on $L$ by only assigning the metrics on the fibers of $L^{an}$ above $X^{an}_T$. In the case $T = \widehat{X}$ is the enclosed locus of $X^{an}$, we have the following result.

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Proposition 1.2. Let \( L \) be a line bundle on \( X \). Then \( L \) has a unique continuous and norm-equivariant Berkovich metric on \( \tilde{X}^{\mathrm{an}} \).

Proof. First let us prove the existence. Let \( x \in \tilde{X}^{\mathrm{an}} \) be a enclosed point with ring of integers \( R_x \) and maximal ideal \( m_x \). Then we have a morphism \( \text{Spec} R_x \to X \) extending the image of \( x \) in \( X \). The pull-back of \( L \) on \( \text{Spec} R_x \) is a locally free module \( M \) of rank one on \( R_x \), so it is free. We define the metric on \( M \) so that the generator gets value 1. This is well-defined as different generator differs by an element in \( R_x^\times \) which has norm 1. We left the reader to check the continuity and the norm-equivariance.

For the uniqueness, assume that we have two continuous and norm-equivariant metrics \( \| \cdot \| \) and \( \| \cdot \|' \) of \( L \) on \( \tilde{X}^{\mathrm{an}} \). The we have a continuous function
\[
\phi : \tilde{X}^{\mathrm{an}} \to \mathbb{R}, \quad x \mapsto \log \frac{\| \cdot \|_x}{\| \cdot \|'_x}.
\]
It is norm-equivariant in the sense that for any two norm-equivalent points \( x \) and \( y \) with \( \| \cdot \|_x = \| \cdot \|_y \), one has
\[
a \phi(x) = b \phi(y).
\]
We need to prove \( \phi = 0 \) identically.

For any \( x \in \tilde{X}^{\mathrm{an}} \) and \( t \in (0, \infty) \), the point \( x_t \in \tilde{X}^{\mathrm{an}} \) given by \( \| \cdot \|_{x_t} = \| \cdot \|_x^t \) is norm-equivalent to \( x \) by definition. Thus we have
\[
\phi(x_t) = t \phi(x).
\]
The continuity of \( \phi \) at \( x_\infty \) as \( t \to \infty \) implies that \( \phi(x) = 0 \).

Since
\[
X_\infty^{\mathrm{an}} = (0, 1] \times X(\mathbb{C})/\text{complex conjugation},
\]
an archimedean metric of a line bundle \( L \) on \( X_\infty^{\mathrm{an}} \) is exactly given by a continuous hermitian metric on \( X(\mathbb{C}) \) invariant under the complex conjugation.

2 Integrable metrics

Conventions on arithmetic models

By a projective arithmetic variety (resp. open arithmetic variety) we mean an integral scheme, projective (resp. quasi-projective) and flat over \( \mathbb{Z} \).
By a projective model of an open arithmetic variety $\mathcal{U}$ we mean an open embedding $\mathcal{U} \hookrightarrow \mathcal{X}$ into a projective arithmetic variety $\mathcal{X}$ such that the complement $\mathcal{X} \setminus \mathcal{U}$ is the support of an effective Cartier divisor.

Let $K$ be a finitely generated field over $\mathbb{Q}$. By a projective arithmetic model (resp. open arithmetic model) of $K$ we mean a projective arithmetic variety (resp. open arithmetic variety) with function field $K$.

Let $K$ be a finitely generated field over $\mathbb{Q}$ and $X$ be a projective variety over $K$. By a projective arithmetic model (resp. open arithmetic model) of $X/K$ we mean a projective and flat morphism $X \to \mathcal{B}$ where:

- $\mathcal{B}$ is a projective arithmetic model (resp. open arithmetic model) of $K$;
- The generic fiber of $\mathcal{X} \to \mathcal{B}$ is $X \to \text{Spec}K$.

Model metrics

Let $K$ be a finitely generated field over $\mathbb{Q}$, $X$ a projective variety over $K$, and $L$ a line bundle on $X$.

Let $\mathcal{X} \to \mathcal{B}$ be a projective model of $X/K$, and $\mathcal{L}$ be a Hermitian line bundle on $\mathcal{X}$ with generic fiber $\mathcal{L}_K$ isomorphic to $L$. This model induces a metric $L^\text{an}$ on $\mathcal{X}^\text{an}$, and thus a metric of $L^\text{an}$ on $X^\text{an}$. We call the metric of $L^\text{an}$ the model metric induced by $\mathcal{L}$ and denote it by $\| \cdot \|_{\mathcal{L}}$. Thus we have defined morphisms:

$$\hat{\text{Div}}(\mathcal{X}) \longrightarrow \hat{\text{Div}}(X^\text{an}), \quad \hat{\text{Pic}}(\mathcal{X}) \longrightarrow \hat{\text{Pic}}(X^\text{an}), \quad \hat{\mathcal{P}ic}(\mathcal{X}) \longrightarrow \hat{\mathcal{P}ic}(X^\text{an}).$$

It is clear that these are injective since they are injective if we replace $X^\text{an}$ by $\mathcal{X}^\text{an}$ and since $X^\text{an}$ is dense in $\mathcal{X}^\text{an}$.

The unions of the images of these objects over all models $(\mathcal{X}/\mathcal{B}, \mathcal{L})$ are denoted by

$$\hat{\text{Div}}(X^\text{an})_\text{mod}, \quad \hat{\text{Pic}}(X^\text{an})_\text{mod}, \quad \hat{\mathcal{P}ic}(X^\text{an})_\text{mod}.$$

$\mathcal{D}$-topology

Let $\mathcal{U}$ be an open arithmetic variety. Projective models $\mathcal{X}$ of $\mathcal{U}$ form a projective system. Using pull-back morphisms, we define

$$\hat{\text{Div}}(\mathcal{U}) := \lim_{\mathcal{X}' \to \mathcal{X}} \hat{\text{Div}}(\mathcal{X}), \quad \hat{\mathcal{P}ic}(\mathcal{U}) := \lim_{\mathcal{X}' \to \mathcal{X}} \hat{\mathcal{P}ic}(\mathcal{X}), \quad \hat{\text{Pic}}(\mathcal{U}) := \lim_{\mathcal{X}' \to \mathcal{X}} \hat{\text{Pic}}(\mathcal{X}).$$
As usual, for each divisor $\overline{E} \in \widehat{\text{Div}}(\mathcal{U})$ we can construct an arithmetic line bundle $\mathcal{O}(\overline{E})$ in $\widehat{\text{Pic}}(\mathcal{U})$.

On a projective model $\mathcal{X}$, an arithmetic divisor $\overline{D} = (\mathcal{D}, g) \in \widehat{\text{Div}}(\mathcal{X})$ is effective (resp. strictly effective) if $\mathcal{D}$ is an effective (resp. strictly effective) divisor on $\mathcal{X}$ and the Green’s function $g \geq 0$ (resp. $g > 0$) on $\mathcal{X}(\mathbb{C}) - |\mathcal{D}(\mathbb{C})|$. Consequently, a $\mathbb{Q}$-divisor $\overline{D} \in \widehat{\text{Div}}(\mathcal{X})_{\mathbb{Q}}$ is called effective (resp. strictly effective) if for some positive integer $n$, the multiple $n\overline{D}$ is an effective (resp. strictly effective) divisor in $\widehat{\text{Div}}(\mathcal{X})_{\mathbb{Q}}$. For any $\overline{D}, \overline{E} \in \widehat{\text{Div}}(\mathcal{X})_{\mathbb{Q}}$, we write $\overline{D} > \overline{E}$ or $\overline{E} < \overline{D}$ if $\overline{D} - \overline{E}$ is effective. It is a partial order on $\widehat{\text{Div}}(\mathcal{X})_{\mathbb{Q}}$, and compatible with pull-back morphisms. Thus we can talk about effectivity and the induced partial ordering on $\widehat{\text{Div}}(\mathcal{U})$.

Fix a projective model $\mathcal{X}$ and an effective Cartier divisor $\mathcal{D}$ with support $\mathcal{X} \setminus \mathcal{U}$. It induces a topology on $\widehat{\text{Div}}(\mathcal{U})_{\mathbb{Q}}$ as follows. Let $g$ be any Green’s function of $\mathcal{D}$ such that the arithmetic divisor $\overline{D} = (\mathcal{D}, g) \in \widehat{\text{Div}}(\mathcal{X})$ is strictly effective. Then a neighborhood basis at 0 is given by

$$B(\epsilon, \widehat{\text{Div}}(\mathcal{U})_{\mathbb{Q}}) := \{ \overline{E} \in \widehat{\text{Div}}(\mathcal{U})_{\mathbb{Q}} : -\epsilon \overline{D} < \overline{E} < \epsilon \overline{D} \}, \quad \epsilon \in \mathbb{Q}_{>0}.$$ 

By translation, it gives a neighborhood basis at any point. The topology does not depend on the choice of $\mathcal{X}$ and $\mathcal{D}$. In fact, if $\overline{D'} \in \widehat{\text{Div}}(\mathcal{X}')$ is another effective divisor with support $\mathcal{X}' \setminus \mathcal{U}$, then we can find a third model dominating both $\mathcal{X}$ and $\mathcal{X}'$. Then we can find $r > 1$ such that $r^{-1} \overline{D} < \overline{D'} < r \overline{D}$.

Let $\widehat{\text{Div}}(\mathcal{U})_{\text{adlc}}$ be the completion of $\widehat{\text{Div}}(\mathcal{U})_{\mathbb{Q}}$ and $\widehat{\text{Pic}}(\mathcal{U})_{\text{adlc}}$ be the completion of the principal divisors. The group of adelic line bundles on $\mathcal{U}$ is defined to be

$$\widehat{\text{Pic}}(\mathcal{U})_{\text{adlc}} = \widehat{\text{Div}}(\mathcal{U})_{\text{adlc}}/\widehat{\text{Pr}}(\mathcal{U})_{\text{adlc}}.$$ 

Alternatively, we can define $\widehat{\text{Pic}}(\mathcal{U})_{\text{adlc}}$ to be the group of isomorphism classes of objects in the completion $\widehat{\text{Pic}}(\mathcal{U})_{\text{adlc}}$ of $\widehat{\text{Pic}}(\mathcal{U})_{\mathbb{Q}}$ with respect to the $\mathcal{D}$-topology defined by the following neighborhood of the trivial line bundle:

$$B(\epsilon, \widehat{\text{Pic}}(\mathcal{U})_{\mathbb{Q}}) := \{ \mathcal{L} \in \widehat{\text{Pic}}(\mathcal{U})_{\mathbb{Q}} : \mathcal{L} \simeq \mathcal{O}(Z) \text{ for some } Z \in B(\epsilon, \widehat{\text{Div}}(\mathcal{U})_{\mathbb{Q}}) \}.$$ 

The functor $\widehat{\text{Pic}}(\cdot)_{\text{adlc}}$ is contravariant for projective morphisms.

**Proposition 2.1.** The natural morphisms

$$\widehat{\text{Div}}(\mathcal{U}) \rightarrow \widehat{\text{Div}}(\mathcal{U}^\text{an}), \quad \widehat{\text{Pic}}(\mathcal{U}) \rightarrow \widehat{\text{Pic}}(\mathcal{U}^\text{an}), \quad \mathcal{P}\text{ic}(\mathcal{U}) \rightarrow \widehat{\text{Pic}}(\mathcal{U}^\text{an})$$

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can be extended continuously into morphisms
\[ \hat{\text{Div}}(U)_{\text{adlc}} \to \hat{\text{Div}}(U^{\text{an}})_{\mathbb{Q}}, \quad \hat{\text{Pic}}(U)_{\text{adlc}} \to \hat{\text{Pic}}(U^{\text{an}})_{\mathbb{Q}}, \quad \hat{\text{Pic}}(U)_{\text{adlc}} \to \hat{\text{Pic}}(U^{\text{an}})_{\mathbb{Q}}. \]

**Proof.** The image of \( B(\epsilon, \hat{\text{Div}}(U)_{\mathbb{Q}}) \) in \( \hat{\text{Div}}(U^{\text{an}}) \) is the space of continuous functions bounded by \( \epsilon g_{D^{\text{an}}} \). Thus \( \hat{\text{Div}}(U^{\text{an}}) \) is certainly complete with respect to this topology. This gives the extension of the first map. The other two follows from this one. \( \square \)

**Remark 2.2.** The map \( \hat{\text{Pic}}(U)_{\mathbb{Q}} \to \hat{\text{Pic}}(U)_{\text{ms}} \) is injective. Equivalently, the quotient \( D \)-topology on \( \hat{\text{Pic}}(U)_{\mathbb{Q}} \) is separable. Namely, there is no non-zero element \( \overline{E} \in \hat{\text{Pic}}(U) \) such that \(-\epsilon \overline{D} < \overline{E} < \epsilon \overline{D} \) for any \( \epsilon > 0 \). In fact, assume that it is true. We can assume that \( \overline{D} \) and \( \overline{E} \) are realized on the same projective model \( X \). Denote \( n = \dim U - 1 \). Then for any ample line bundles \( \overline{L}_1, \ldots, \overline{L}_n \) in \( \hat{\text{Pic}}(X) \), we have
\[ (\epsilon \overline{D} \pm \overline{E}) \cdot \overline{L}_1 \cdots \overline{L}_n > 0, \quad \forall \epsilon > 0. \]
It follows that
\[ \overline{E} \cdot \overline{L}_1 \cdots \overline{L}_n = 0. \]
Then \( \overline{E} = 0 \) by the Hodge index theorem for arithmetical divisors in [Mo1].

**Remark 2.3.** In [Fal], the definition of the Faltings height of an abelian variety uses the Petersson metric on the Hodge bundle of the (open) Siegel modular variety \( \mathcal{A}_g \) over \( \text{Spec} \mathbb{Z} \). The metric has logarithmic singularity along the boundary \( D \). One can check that the corresponding metrized line bundle actually lies in \( \hat{\text{Pic}}(\mathcal{A}_g)_{\text{ms}} \). Since \( \mathcal{A}_g \) is only a stack, to make the statement rigorous one needs to put a level structure on it.

**Relative case**

Let \( K \) be a finitely generated field over \( \mathbb{Q} \), and \( X \) be a projective variety over \( K \). The set of open arithmetic models \( U \to V \) of \( X/K \) form a projective system. Define
\[
\begin{align*}
\hat{\text{Div}}(X)_{\text{adlc}} & : = \lim_{U \to V} \hat{\text{Div}}(U)_{\text{adlc}}, \\
\hat{\text{Pic}}(X)_{\text{adlc}} & : = \lim_{U \to V} \hat{\text{Pic}}(U)_{\text{adlc}}, \\
\hat{\text{Pic}}(X)_{\text{adlc}} & : = \lim_{U \to V} \hat{\text{Pic}}(U)_{\text{adlc}}.
\end{align*}
\]
Then all these can be embedded to the corresponding objects on $X^{an}$. Notice that for any element $\mathcal{L}$ in $\widehat{\text{Pic}}(X^{an})$, there are two models $(\mathcal{X}_1/\mathcal{B}_1, \overline{\mathcal{L}}_1)$ and $(\mathcal{X}_2/\mathcal{B}_2, \overline{\mathcal{L}}_2)$ with the same underlying bundle $L$ such that

$$\| \cdot \|_{\mathcal{L}_1} \leq \| \cdot \|_{\mathcal{L}} \leq \| \cdot \|_{\mathcal{L}_2}.$$ 

If $X = \text{Spec}(K)$, we also denote them by

$$\widehat{\text{Div}}(K)_{\text{adlc}}, \quad \widehat{\text{Pic}}(K)_{\text{adlc}}, \quad \widehat{\text{Pic}}(K)_{\text{adlc}}.$$ 

**Integrable metrics**

We say that an adelic line bundle $\mathcal{L} \in \widehat{\text{Pic}}(U)_{\text{adlc}}$ is *semipositive* if it can be given by a Cauchy sequence $\{(\mathcal{X}_m, \overline{\mathcal{L}}_m)\}_m$ where each $\overline{\mathcal{L}}_m$ is ample on $\mathcal{X}_m$ in the sense of [Zh1]. We say that a line bundle in $\widehat{\text{Pic}}(U)_{\text{adlc}}$ is *integrable* if it is equal to the difference of two semipositive ones. Denote the subgroup of integrable elements by $\widehat{\text{Pic}}(U)_{\text{int}}$. Analogously, we introduce $\widehat{\text{Pic}}(U)_{\text{int}}$ as a subcategory of $\widehat{\text{Pic}}(U)_{\text{adlc}}$, and also $\widehat{\text{Pic}}(X)_{\text{int}}$ and $\widehat{\text{Pic}}(X)_{\text{int}}$ for any projective variety $X$ over a finitely generated field.

**Remark 2.4.** We may extend all the above discussions to the function field situation. Fix a field $k$ and a finitely generated extension $K$ over $k$. For a quasi-projective variety $U$ over $k$, and a projective variety $X$ over $K$, we can define notion

$$\widehat{\text{Div}}(U/k)_{\text{int}}, \quad \widehat{\text{Pic}}(U/k)_{\text{int}}, \quad \widehat{\text{Pic}}(U/k)_{\text{int}},$$

$$\widehat{\text{Div}}(X/k)_{\text{int}}, \quad \widehat{\text{Pic}}(X/k)_{\text{int}}, \quad \widehat{\text{Pic}}(X/k)_{\text{int}}.$$ 

If $k$ itself is finitely generated over $\mathbb{Q}$, then the above objects are base changes of the “arithmetic objects” defined above. More precisely, if the $k$-variety $U$ is the base change of an open arithmetic variety $U_\mathbb{Z}$ over $\mathbb{Z}$, the objects in the first row are the base changes of corresponding objects for $U_\mathbb{Z}$. For the second row, view $K$ as a finitely generated field over $\mathbb{Q}$ without the constant field. Then we also have base change from objects for $X$ to $X/k$ in the second row.

### 3 Arithmetic intersections and heights

Let $K$ be a finitely generated field over $\mathbb{Q}$ of transcendental degree $d$, and $X$ be a projective variety over $K$ of dimension $n$. In this section, we will
introduce two intersection maps:
\[ \widehat{\text{Pic}}(X)^{n+1}_{\text{int}} \rightarrow \widehat{\text{Pic}}(K)_{\text{int}}, \quad \widehat{\text{Pic}}(K)^{d+1}_{\text{int}} \rightarrow \mathbb{R}. \]

They give definition of heights of subvarieties of \( X \).

**Deligne Pairing**

Let \( \pi : \mathcal{U} \rightarrow \mathcal{V} \) be a flat and projective morphism of open arithmetic varieties over \( \text{Spec}\mathbb{Z} \) of relative dimension \( n \) whose generic fiber is \( X \rightarrow \text{Spec}K \). Then we have an inductive family given by the Deligne pairing
\[ \text{Pic}(\mathcal{U})^{n+1}_{\text{int}} \rightarrow \text{Pic}(\mathcal{V}), \quad (\mathcal{L}_1, \cdots, \mathcal{L}_{n+1}) \mapsto \langle \mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_{n+1} \rangle. \]

Our goal is to extend this intersection to line bundles in \( \widehat{\text{Pic}}(X)_{\text{int}} \). As the case of [Zh2] and [Mo3], we can not expect the intersection for all adelic line bundles, but only for the integrable ones.

**Proposition 3.1.** The above multilinear map extends to a multilinear maps
\[ \widehat{\text{Pic}}(\mathcal{U})^{n+1}_{\text{int}} \rightarrow \widehat{\text{Pic}}(\mathcal{V})_{\text{int}}, \]
\[ \widehat{\text{Pic}}(X)^{n+1}_{\text{int}} \rightarrow \widehat{\text{Pic}}(K)_{\text{int}}. \]

**Proof.** By linearity, we only need to extend the image for semipositive line bundles \( \mathcal{L}_1, \cdots, \mathcal{L}_{n+1} \) in \( \widehat{\text{Pic}}(\mathcal{U})_{\text{int}} \) for any open model \( \pi : \mathcal{U} \rightarrow \mathcal{V} \) of \( X \rightarrow \text{Spec}K \).

For each \( i = 1, \cdots, n+1 \), suppose that \( \mathcal{L}_i \) is given by the Cauchy sequence \( \{ (\mathcal{X}_m, \mathcal{Z}_{i,m}) \}_m \) with each \( \mathcal{Z}_{i,m} \) ample on models \( \mathcal{X}_m \) over a projective model \( \mathcal{B} \) of \( \mathcal{V} \). Here we assume that the integral model \( \mathcal{X}_m \) is independent of \( i \). It is always possible. Apply Raynaud’s flattening theorem ([Ra], Theorem 1, Chapter 4). After blowing up \( \mathcal{B} \) and replacing \( \mathcal{X}_m \) by its pure-transform, we get a flat family \( \pi_m : \mathcal{X}_m \rightarrow \mathcal{B}_m \). The goal is to show the convergence of
\[ \langle \mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_{n+1} \rangle := \lim_{m \rightarrow \infty} \langle \mathcal{L}_{1,m}, \mathcal{L}_{2,m}, \cdots, \mathcal{L}_{n,m} \rangle. \]

We can further assume that for each pair \( m < m' \), the map \( \pi_{m'} \) dominates \( \pi_m \) and
\[ \mathcal{L}_{i,m'} - \mathcal{L}_{i,m} \simeq \mathcal{O}(\mathcal{Z}_{i,m,m'}), \quad \mathcal{Z}_{i,m,m'} \in B(\epsilon_m, \text{Div}(\mathcal{U})_Q). \]
Here \( \{ \epsilon_m \}_{m \geq 1} \) is a sequence decreasing to zero.

We claim that for any \( m < m' \),

\[
\langle L_{1,m}, L_{2,m}, \ldots, L_{n+1,m} \rangle - \langle L_{1,m'}, L_{2,m'}, \ldots, L_{n+1,m'} \rangle \in B(\epsilon_m \deg X, \widehat{\text{Pic}}(U)_\mathbb{Q}).
\]

where

\[
\deg(X) = \sum_{i=1}^{n} \deg(L_{1,K} \cdot L_{2,K} \cdots L_{i-1,K} \cdot L_{i+1,K} \cdots L_{n+1,K}).
\]

Then the sequence \( \langle L_{1,m}, L_{2,m}, \ldots, L_{n+1,m} \rangle \) is Cauchy in \( \text{Pic}(U)_\mathbb{Q} \) under the \( D \)-topology, and thus the convergence follows.

Note that

\[
\langle L_{1,m}, \ldots, L_{n+1,m} \rangle - \langle L_{1,m'}, \ldots, L_{n+1,m'} \rangle = \sum_{i=1}^{n} \langle L_{1,m}, \ldots, L_{i-1,m}, L_{i,m} - L_{i,m'}, L_{i+1,m'}, \ldots, L_{n+1,m'} \rangle.
\]

Fix an \( i \) and let \( \ell_i \) be a section of \( L_{i,m} - L_{i,m'} \) with divisor \( Z_{i,m,m'} \). Since all \( L_{j,m} (j < i) \) and \( L_{j,m'} (j > i) \) are ample, they have sections \( \ell_j (j \neq i) \) so that \( \text{div} (\ell_j) \) for \( 1 \leq j \leq n + 1 \) intersects properly on \( X_{m'} \). In this way, the bundle

\[
\langle L_{1,m}, \ldots, L_{i-1,m}, L_{i,m} - L_{i,m'}, L_{i+1,m'}, \ldots, L_{n+1,m'} \rangle
\]

has a section \( \langle \ell_1, \ldots, \ell_{n+1} \rangle \) with divisor \( \pi_* (\text{div} \ell_1 \cdot \text{div} \ell_2 \cdot \cdots \cdot \text{div} \ell_{n+1}) \). Since \( \text{div} (\ell_i) = Z_{i,m,m'} \) is bounded by \( \epsilon_m \pi^* D \), where \( D \) is a fixed boundary divisor of \( V \), and since all other \( \text{div} (\ell_j) (j \neq i) \) are ample, we see that \( \pi_* (\text{div} \ell_1 \cdot \text{div} \ell_2 \cdot \cdots \cdot \text{div} \ell_{n+1}) \) is bounded by

\[
\epsilon_m \pi_* \langle L_{1,m} \cdots L_{i-1,m} \cdot \pi^* D \cdot L_{i+1,m'} \cdots L_{n+1,m'} \rangle
\]

\[
= \epsilon_m \deg(L_{1,K} \cdots L_{i-1,K} \cdot L_{i+1,K} \cdots L_{n+1,K}) \cdot D.
\]

It finishes the proof. \( \square \)

**Intersection numbers**

Let \( K \) be a field finitely generated over \( \mathbb{Q} \) with transcendental degree \( d \). For each projective model \( X \) of \( K \), we have an intersection pairing \( \widehat{\text{Pic}}(X)^{d+1} \to \mathbb{R} \). It easily extends to a pairing \( \widehat{\text{Pic}}(U)^{d+1} \to \mathbb{R} \) for any open arithmetic model \( U \) of \( K \).
Proposition 3.2. The intersection pairing above extends uniquely to a multilinear and continuous homomorphism

$$\hat{\text{Pic}}(K)^{d+1}_{\text{int}} \longrightarrow \mathbb{R}.$$ 

Proof. It suffices to prove the similar result for $\hat{\text{Pic}}(U)_{\text{int}}$ for any open arithmetic variety $U$ with function field $K$. We need to define $(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1})$ for any $\mathcal{L}_1, \ldots, \mathcal{L}_{d+1} \in \hat{\text{Pic}}(U)_{\text{int}}$.

Let $\mathcal{X}$ be a projective model of $U$. Replacing $U$ by an open subset if necessary, we may assume that $U$ is the complement of an ample divisor $D$ in $\mathcal{X}$. Complete $D$ to an ample divisor $\mathcal{D}$, and use it to define the $\mathcal{D}$-topology.

As in the proof in the previous proposition, we may assume that $\mathcal{L}_i$ is given by a Cauchy sequence $\{(\mathcal{X}_m, \mathcal{L}_{i,m})\}_m$ with each $\mathcal{L}_{i,m}$ ample on a projective model $\mathcal{X}_m$ dominating $\mathcal{X}$. Assume for any $m' > m$,

$$\mathcal{L}_{i,m'} - \mathcal{L}_{i,m} \in B(\epsilon_m, \hat{\text{Pic}}(U))$$

with $\epsilon_m \to 0$. For any subset $I \subset \{1, \ldots, d+1\}$, consider the sequence

$$\alpha_{I,m} := \mathcal{D}^{d+1-|I|} \prod_{i \in I} \mathcal{L}_{i,m}.$$ 

We want to prove by induction that $\{\alpha_{I,m}\}_m$ is a Cauchy sequence. When $I$ is the full set, we have the proposition.

There is nothing to prove if $I$ is an empty set. Assume the claim is true for any $|I| < k$ for some $k > 0$. Then for any $I$ with $|I| = k$, we have

$$\mathcal{D}^{d+1-k} \prod_{i \in I} \mathcal{L}_{i,m} - \mathcal{D}^{d+1-k} \prod_{i \in I} \mathcal{L}_{i,m'} = \mathcal{D}^{d+1-k} \sum_{i \in I} \prod_{j \in I, j < i} \mathcal{L}_{j,m} (\mathcal{L}_{i,m} - \mathcal{L}_{i,m'}) \prod_{j \in I, j > i} \mathcal{L}_{j,m'}.$$ 

Its absolutely value is bounded by

$$\epsilon_m \mathcal{D}^{d+1-k+1} \sum_{i \in I} \prod_{j \in I, j < i} \mathcal{L}_{j,m} \prod_{j \in I, j > i} \mathcal{L}_{j,m'} \leq \epsilon_m \mathcal{D}^{d+1-k+1} \sum_{i \in I} \prod_{j \in I, j < i} \mathcal{L}_{j,m} \prod_{j \in I, j > i} (\mathcal{L}_{j,m} + \epsilon_m \mathcal{D}).$$
The last term is a linear combination of $\alpha_{I',m}$ with $|I'| < k$. The coefficients of the linear combination grow as $o(\epsilon_m)$. It follows that $\alpha_{I,m}$ is a Cauchy sequence.

\begin{remark}
The intersection pairing defined by the above two propositions can be extended to the function field situation described in Remark 2.4. If the constant field $k$ is finitely generated over $\mathbb{Q}$, then the Deligne pairing in Proposition 3.1 is compatible with the following “base changes”:

\[ \mathcal{P}ic(X) \longrightarrow \mathcal{P}ic(X/k), \quad \mathcal{P}ic(K) \longrightarrow \mathcal{P}ic(K/k). \]

\end{remark}

### Adelic heights

Let $K$ be a field finitely generated over $\mathbb{Q}$ with transcendental degree $d$ and let $X$ be a projective variety over $K$ of dimension $n$. Let $\mathcal{L}$ be a semipositive element in $\widehat{\mathcal{P}ic}(X)_{\text{int}}$ with ample generic fiber $L = \mathcal{L}_K$.

For any closed $\overline{K}$-subvariety $Z$ of $X$, define the **adelic height** of $Z$ as

\[ h_\mathcal{L}(Z) := \frac{\langle \mathcal{L}|_{Z_{\text{gal}}} \rangle^{\dim Z + 1}}{(\dim Z + 1) \deg_L(Z_{\text{gal}})} \in \widehat{\mathcal{P}ic}(K)_{\text{int}}. \]

Here $Z_{\text{gal}}$ denotes the minimal $K$-subvariety of $X$ containing $Z$, $\mathcal{L}|_{Z_{\text{gal}}}$ denotes the pull-back in $\widehat{\mathcal{P}ic}(Z_{\text{gal}})_{\text{int}}$, and the self-intersections are taken as the Deligne pairing in the sense of Proposition 3.1. It gives a map

\[ h_\mathcal{L} : |X_{\overline{K}}| \longrightarrow \widehat{\mathcal{P}ic}(K)_{\text{int}}. \]

Here $|X_{\overline{K}}|$ denotes the set of closed $\overline{K}$-subvarieties of $X$. In particular, we have a height map of algebraic points:

\[ h_\mathcal{L} : X(\overline{K}) \longrightarrow \widehat{\mathcal{P}ic}(K)_{\text{int}}. \]

The class of $h_\mathcal{L}$ modulo bounded functions depends only on the class of $L$ in $\mathcal{P}ic(X)$.

If $K$ is a number field, then $\widehat{\mathcal{P}ic}(K)_{\text{int}} = \widehat{\mathcal{P}ic}(K)_{\text{adlc}} = \widehat{\mathcal{P}ic}(O_K)$. We have the degree map $\deg : \widehat{\mathcal{P}ic}(K)_{\text{int}} \rightarrow \mathbb{R}$. In that case, $\deg h_\mathcal{L}$ is the same as the usual height.
The height function satisfies the Northcott property. For any $D \in \mathbb{R}$ and $\alpha \in \widehat{\text{Pic}}(K)_{\text{int}}$, the set

$$\{x \in X(K) : \deg(x) < D, \ h_{\mathcal{L}}(x) < \alpha\}$$

is finite. It follows from the Northcott property of Moriwaki’s height described below.

**Moriwaki heights**

Let $\mathcal{H}$ be any element in $\widehat{\text{Pic}}(K)$. For any closed $K$-subvariety $Z$ of $X$, define the *Moriwaki height of $Z$ with respect to $\mathcal{H}$* as

$$h_{\mathcal{L}}^{\mathcal{H}}(Z) := h_{\mathcal{L}}(Z) \cdot \mathcal{H}^{d} = \frac{\langle L|_{Z_{\text{gal}}} \rangle^{\dim Z + 1} \cdot \mathcal{H}^{d}}{(\dim Z + 1) \deg_{L}(Z_{\text{gal}})} \in \mathbb{R}.$$ 

If both $\mathcal{L}$ and $\mathcal{H}$ are realized on some model $\mathcal{X} \to \mathcal{B}$, then $h_{\mathcal{L}}^{\mathcal{H}}$ is just the height function introduced in [Mo2]. If $\mathcal{H}$ is nef and big on $\mathcal{B}$, Moriwaki prove the height satisfies the Northcott property. Namely, for any $D \in \mathbb{R}$ and $A \in \mathbb{R}$, the set

$$\{x \in X(K) : \deg(x) < D, \ h_{\mathcal{L}}^{\mathcal{H}}(x) < A\}$$

is finite.

**Remark 3.4.** As in Remark 2.4 and Remark 3.3, we may extend the adelic heights and Moriwaki heights to the function field situation. The Moriwaki height has been studied by Faber [Fab] in the case of transcendental dimension one and by Gubler [Gu2] in the general case.

## 4 Small points and Equidistribution

Let $X$ be a projective variety over a finitely generated field $K$ over $\mathbb{Q}$, and $\mathcal{L}$ be an integrable arithmetic line bundle over $X$. Then we have height function

$$h_{\mathcal{L}} : \ X(K) \to \text{Pic}(K)_{\text{int}}.$$
Small points

A sequence \( \{x_m\} \) of points in \( X(\overline{K}) \) is called a sequence of \textit{adelic small points} if \( h_{\overline{L}}(x_m) \) has limit 0 in \( \text{Pic}(K)_{\text{int}} \). Recall that the topology in \( \text{Pic}(K)_{\text{int}} \) is taken as the induced topology on \( \text{Pic}(\mathcal{V})_{\text{int}} \) for open arithmetic varieties \( \mathcal{V} \) over \( \mathbb{Z} \) with function field \( K \).

For a polarization \( \overline{H} \), we can also define \textit{Moriwaki small points} in terms of heights \( h_{\overline{H}} \). Sometimes we call it \( h_{\overline{H}} \)-small to emphasize the dependence on the polarization. It is clear that “adelic small” implies “Moriwaki small”.

If \( K \) is a number field, both \( \deg h_{\overline{L}} \) and \( h_{\overline{L}} \) are equal to the usual height function \( h_{\overline{L}} \). Then we can check that both smallness are equivalent to the usual one given by \( h_{\overline{L}}(x_m) \to 0 \).

Remark 4.1. We can also define the smallness of points in function field situation. If the constant field \( k \) is finitely generated over \( \mathbb{Q} \), then we have two notions of small points. The one using heights valued in \( \hat{\text{Pic}}(K)_{\text{int}} \) is called the arithmetic smallness, while the one using heights valued in \( \hat{\text{Pic}}(K/k)_{\text{int}} \) is called the geometric smallness. It is clear that the arithmetic smallness implies the geometric smallness.

Equidistribution

Let \( v \) be a valuation of \( K \), i.e., a multiplicative norm \( |\cdot|_v : K \to \mathbb{R}_{\geq 0} \). It could be either archimedean or non-archimedean. Then we have the Berkovich analytic space \( X_v^{\text{an}} \) and the Chamber-Loir measure

\[
d\mu_{\overline{L},v} = \frac{1}{\deg_L(X)} c_1(\overline{L}) \dim X \]

on \( X_v^{\text{an}} \) attached to \( (X, \overline{L}) \). Here \( L = \overline{L}_K \) is the generic fiber of \( \overline{L} \) on \( X \). Also for each point \( x \in X(\overline{K}) \), we have the measure

\[
\mu_{x,v} = \frac{1}{\deg(x)} \delta_{O(x)}
\]

on \( X_v^{\text{an}} \). Here \( \delta_{O(x)} \) is the Dirac measure for the Galois orbit \( O(x) = \text{Gal}(\overline{K}/K)x \) in \( X_v^{\text{an}} \).

A sequence \( \{x_m\}_{m \geq 1} \) of points is said to be \textit{generic} if for any proper closed subvariety \( Y \) of \( X \) contains only finitely many terms of the sequence.
A sequence \( \{x_m\}_{m \geq 1} \) of points is said to be \textit{Galois equidistributed on} \( X_v^{\text{an}} \) \textit{with respect to} \( d\mu_{\overline{L},v} \) if the weak convergence

\[
\mu_{x_m,v} \rightarrow d\mu_{\overline{L},v}
\]

holds on \( X_v^{\text{an}} \).

**Conjecture 4.2.** Assume that \( \overline{L} \) is semipositive with ample generic fiber \( \overline{L}_K \). Then \( h_{\overline{L}}(X) = 0 \) if and only if there is a generic sequence of adelic small points on \( X(\overline{K}) \).

Moreover, if \( x_m \) is a generic sequence of adelic small points in \( X(\overline{K}) \), then \( x_m \) is Galois equidistributed on \( X_v^{\text{an}} \) with respect to \( d\mu_{\overline{L},v} \) for any valuation \( v \) of \( K \).

**Remark 4.3.** If \( K \) is a number field, the conjecture is proved by Yuan [Yu] by extending the previous work of Szpiro–Ullmo–Zhang [SUZ] for archimedean place \( v \) and point-wise positive \( c_1(\mathcal{L}, \| \cdot \|_v) \).

**Remark 4.4.** If the valuation \( v \) is trivial, or more generally in the function field case where \( K \) is a finitely generated field over a base field \( k \), the conjecture is proved independently by Faber [Fab] and Gubler [Gu3]. In fact, we can find a subfield \( L \) of \( K \) containing \( k \) so that \( K/L \) has transcendental degree 1. Then the adelic smallness implies Faber’s smallness.

**Remark 4.5.** An analogue of the conjecture for Moriwaki small points holds for any place \( v \) of \( \mathbb{Q} \). More precisely, let \( \mathcal{U} \rightarrow \mathcal{V} \) be an open arithmetic model for \( X/K \) such that \( \overline{L} \) and \( \overline{H} \) can realized on this model. Assume that \( \overline{H} \) is nef with \( \overline{H}^{\dim \mathcal{U}} = 0 \). Moriwaki [Mo2] proved

\[
\frac{1}{\deg(x_m)} \delta_{x_m} \wedge c_1(\overline{H})^d_v \rightarrow \frac{1}{\deg(X)} c_1(\overline{L})^n_v \wedge c_1(\overline{H})^d_v.
\]

Here \( n = \dim X \) and \( d \) is the transcendental degree of \( K \) over \( \mathbb{Q} \). \( \overline{x}_m \) denotes the Zariski closure in \( \mathcal{U}_{\mathbb{Q}} \), and the weak convergence is on the space \( \mathcal{U}_{\mathbb{Q},v}^{\text{an}} \). Moriwaki proves this for archimedean places under a positivity assumption of curvatures of \( \overline{L} \) and \( \overline{H} \). The general case can be done using the method in [Yu].

**Equidistribution of small 0-cycles**

Here we extend Conjecture 4.2 to small 0-cycles. Recall that a 0-cycle is a finite linear combination of closed points with rational coefficients. So it is
of the form
\[ z = \sum_{k} a_k z_k, \quad a_k \in \mathbb{Q}, \ z_k \in X_0, \]
where \( X_0 \) denote the set of closed points on \( X \). We need to define some notions:

- **The degree of** \( z \) **is defined to be**
  \[ \deg z := \sum_{k} a_k \deg z_k. \]
  The cycle \( z \) is called **unitary** if \( a_k \geq 0 \) and \( \deg z = 1 \).

- If \( v \) is a valuation of \( K \), then the **Dirac measure of** \( z \) on \( X_v^{an} \) **is defined to be**
  \[ \delta_{z,v} := \sum_{k} a_k \delta_{z_k,v} \]
  where \( \delta_{z_k,v} \) is the Dirac measure of \( z_k \) on \( X_v^{an} \). Notice that the total mass of this measure is 1.

- For a subvariety \( Y \) of \( X \), the **\( Y \)-degree of** \( z \) **is defined to be**
  \[ \deg_Y(z) := \sum_{z_k \in Y} a_k \deg(z_k). \]

- **The height of** \( z \) **is defined to be**
  \[ h(z) := \sum_{k} a_k (\mathcal{L}|_{z_k}). \]

Let \( \{z_m\}_m \) be a sequence of unitary 0-cycles on \( X \). We further define the following notions:

- The sequence \( \{z_m\}_m \) is called a **sequence of small 0-cycles with respect to** \( \mathcal{L} \) if \( h(z_m) \to 0 \).

- The sequence \( \{z_m\}_m \) is called a **generic sequence** if for any subvariety \( Y \) of \( X \), the \( Y \)-degree \( \deg_Y(z_m) \to 0 \).
The sequence \( \{z_m\} \) is said to be \textit{Galois equidistributed on} \( X_v^{\text{an}} \) with respect to \( d\mu_{\mathcal{Z},v} \) if the weak convergence
\[
\delta_{z_m,v} \longrightarrow d\mu_{\mathcal{Z},v}
\]
holds on \( X_v^{\text{an}} \).

\textbf{Conjecture 4.6.} Assume that \( \mathcal{Z} \) is semipositive with ample generic fiber \( \mathcal{Z}_K \). Then \( h_{\mathcal{Z}}(X) = 0 \) if and only if there is a generic sequence of small unitary 0-cycles on \( X \).

Moreover, if \( z_m \) is a generic sequence of small 0-cycles on \( X \), then \( z_m \) is Galois equidistributed on \( X_v^{\text{an}} \) with respect to \( d\mu_{\mathcal{Z},v} \) for any valuation \( v \) of \( K \).

It is easy to see that Conjecture 4.6 implies Conjecture 4.2. In fact, for any small and generic sequence \( x_m \in X(K) \), take \( z_m = \overline{x}_m / \deg x_m \) where \( \overline{x}_m \) is the image of \( x_m \) considered as a morphism \( x_m : \text{Spec} K \to X \).

Next, we show that Conjecture 4.2 implies Conjecture 4.6 when \( K \) is a number field. Let \( z_m \) be a generic sequence of small 0-cycles on \( X \). A basic result asserts that the space of semipositive measures of total mass 1 on a compact Hausdorff space is sequentially compact under the weak topology. In our case, any subsequence of \( \{\delta_{z_m,v}\}_m \) has a convergent subsequence. Therefore, it suffices to prove that the equidistribution is true for some subsequence of \( \{z_m\}_m \).

Order the set of irreducible proper subvarieties of \( X \) as
\[ Y_1, Y_2, \ldots, Y_{\ell}, \ldots. \]
It is possible because the set is countable. Write \( Y'_\ell = \cup_{i \leq \ell} Y_i \). Since \( \deg_{Y'_\ell}(z_m) \to 0 \) as \( m \to \infty \), inductively replacing \( z_m \) by a subsequence, we may assume that
\[
\deg_{Y'_\ell}(z_m) \leq \frac{1}{m}, \quad \forall m.
\]
After replacing \( z_m \) by a subsequence, we may assume that
\[
h(z_m) \leq \frac{1}{m^2}.
\]
Here \( h = h_{\mathcal{Z}} = \deg h_{\mathcal{Z}} \) is the usual height function on \( X \) since \( K \) is a number field.
In terms of closed points, write
\[ z_m = \sum_i a_{m,i} z_{m,i}. \]

For each \( m \), consider the following two partial sums of \( z_m \):
\[ \alpha_m = \sum_{z_{m,i} \not\in Y'_m} a_{m,i} z_{m,i}, \quad \beta_m = \sum_{h(z_{m,i}) \geq 1/m} a_{m,i} z_{m,i}. \]

It follows that \( \deg(\alpha_m) \leq 1/m \). And
\[ h(\beta_m) \geq \sum_{h(z_{m,i}) \geq 1/m} a_i \deg z_{m,i} \cdot \frac{1}{m} = \frac{\deg \beta_m}{m}. \]

Since \( h(z_m) \leq 1/m^2 \), it follows that \( \deg(\beta_m) \leq 1/m \).

Now we have decomposition
\[ z_m = z'_m + \gamma_m, \]
where
\[ z'_m = \sum_{z_{m,i} \not\in Y'_m, \, h(z_{m,i}) < 1/m} a_{m,i} z_{m,i}, \]
\[ \gamma_m = \sum_{z_{m,i} \in Y'_m \text{ or } h(z_{m,i}) \geq 1/m} a_{m,i} z_{m,i}. \]

Then \( \gamma_m \leq \alpha_m + \beta_m \) and thus \( \deg(\gamma_m) \leq 2/m \). The Dirac measure of \( \gamma_m \) converges to 0 on \( X^\text{an}_v \). It suffices to prove the Dirac measure of \( z''_m = z'_m / \deg(z'_m) \) converges to \( d\mu_v \).

We still have \( h(z''_m) \to 0 \). It is equivalent to \( h_{\overline{\mathcal{O}}}(z''_m) \to 0 \) in \( \mathcal{Pic}(K)_{\text{int}} \) since \( K \) is assumed to be a number field. So the union \( \bigcup_m |z''_m| \) of supports of \( z''_m \) is generic and small. Apply Conjecture 4.2, any subsequence of \( \bigcup_m |z''_m| \) is equidistributed on \( X^\text{an}_v \). Then it is easy to see that \( z''_m \) is equidistributed.

**Reduction of small cycles**

Let \( \mathcal{U} \to \mathcal{V} \) be an open arithmetic model of \( X/K \) over \( \mathbb{Q} \). Let \( s \) be a point of \( V := \mathcal{V}_\mathbb{Q} \). Then for any closed cycle \( Z \) on \( X/K \), we can define the reduction \( Z_s := \overline{Z}|_s \) of \( Z \) as the restriction to \( X_s := \mathcal{U}_s \) of the Zariski closure \( \overline{Z} \) of \( Z \) in \( V \).
Theorem 4.7. Let \( v \) be a place of \( K \) and let \( z_m \) be a generic sequence of small unitary 0-cycle on \( X/K \). Then for any point \( s \) as above, the sequence of reduction \( z_{m,s} \) is also unitary, small, and generic. If either \( v \) is trivial on \( \mathbb{Q} \), or \( s \) is a close point, then \( z_{m,s} \) is equidistributed on \( X_{s,v}^{an} \).

Proof. The properties of being unitary and small are clearly stable under reduction. The genericity follows from Theorem 4.8 below on the equidistribution of Moriwaki small points on the fibers. The equidistribution follows from the work of Yuan [Yu] when \( s \) is a closed point, and Faber [Fab] and Gubler [Gu3] when \( v \) is trivial on \( \mathbb{Q} \).

Equidistribution on fibers of Moriwaki small points

Let \( (X, K) \) be as above. Namely, \( K \) is a field finitely generated over \( \mathbb{Q} \) of transcendental degree \( d \), and \( X \) is a projective variety over \( K \) of dimension \( n \).

Let \( \mathcal{L} \in \text{Pic}(X)_{\text{int}} \) be an arithmetic line bundle. Let \( U \to V \) be an open arithmetic model of \( X/K \) such that \( \mathcal{L} \) is in \( \widehat{\text{Pic}}(U)_{\text{adlc}} \). Then for any closed point \( s \in V_{\mathbb{Q}} \), the base change \( \mathcal{L}_s \) gives an adelic line bundle on \( X_s := U_s \) in the sense of [Zh2].

Let \( B \) be a projective model of \( V \), and \( \mathcal{H} \) be a nef hermitian line bundle on \( B \) with \( \mathcal{H}^d > 0 \) and \( \mathcal{H}^{d+1} = 0 \). Then \( \mathcal{B} = (B, \mathcal{H}) \) gives a polarization of \( K \) in the sense of Moriwaki [Mo2].

Theorem 4.8. Assume that \( \mathcal{L} \) is semipositive and that the generic fiber \( \mathcal{L}_K \) is ample on \( X \). For any closed point \( s \in V_{\mathbb{Q}} \), there is a finite set \( \Omega_s \) of finite places of \( \mathbb{Q} \) satisfying the following property. For any generic sequence \( \{x_m\} \) of \( h^1_{\mathcal{L}} \)-small unitary 0-cycles of \( X \), the specialization sequence \( \{x_{m,s}\} \) on the fiber \( X_s \) is equidistributed on the analytic space \( X_{s,v}^{an} \), for all finite places \( v \) of \( \mathbb{Q} \) outside \( \Omega_s \), with respect to the equilibrium measure of \( (X_s, \mathcal{L}_s) \). In particular, the sequence \( \{x_{m,s}\} \) is generic on \( X_s \).

Proof. As the above argument, it suffices to assume that every \( x_m \) comes from a single closed point of \( X \). Note that \( x_{m,s} \) may still come from more than one closed point on \( X_s \). The genericity still means that as a sequence of unitary 0-cycles.

To obtain the equidistribution, we will apply the variational principle of Szpiro-Ullmo-Zhang [SUZ]. For more details of the variational principle, we refer to [Yu, §3]. Let \( (S, g_S) \) be an arithmetic 1-cycle on \( B \) linearly equivalent
to $\mathcal{H}^d$. Assume that the generic fiber $s := S_Q$ is contained in $V_Q$. Denote by $\Omega_S$ the set of finite places of $\mathbb{Q}$ over which $S$ has vertical components. Then $\Omega_S$ is the image of the support of the vertical part $S - \pi$ in $\text{Spec}(\mathbb{Z})$.

Let $\phi$ be a continuous function on $X_{\mathbb{Q}, v}$, it corresponds to an adelic line bundle $\mathcal{O}(\phi)$ on $X_{\mathbb{Q}}$ over $\mathbb{Q}$ in the sense of Zhang [Zh2]. It has trivial generic fiber, and trivial metric outside $v$. If $\phi$ is a model function, $\mathcal{O}(\phi)$ can be obtained by a single integral model of $(X_{\mathbb{Q}}, \mathcal{O}_Q)$. In general, it is a limit since any continuous function is a uniform limit of model functions. Conventionally, we write $M(\phi) = M_s + \mathcal{O}(\phi)$ for any arithmetic line bundle $M$. For more details, we refer to [Yu, §3.3].

**Lemma 4.9.** Let $(S, s, \Omega_S)$ be as above, and let $\phi$ be a continuous function on $X_{\mathbb{Q}, v}$ for a finite place $v \notin \Omega_S$. Then for any closed subvariety $Z$ of $X_{\mathbb{K}}$,

$$ h_{\mathcal{E}(\phi)}(Z) - h_{\mathcal{E}}(Z) = h_{\mathcal{L}_{s}(\phi)}(Z|_s) - h_{\mathcal{L}_{s}}(Z|_s) $$

Here $\phi|_{X_{\mathbb{Q}, Q}}$ is the restriction, and the specialization $Z|_s$ is viewed as a closed subvariety of the $\mathbb{Q}$-variety $X_s$.

Note that $\mathcal{L}_{s}$ and thus $\mathcal{L}_{s}(\phi)$ are usual adelic line bundles over the $\mathbb{Q}$-variety $X_s$ in the sense of [Zh2]. We remark that $s$ is in general a linear combination $\sum_k a_k s_k$ of closed points $s_i$ in $V_Q$. Then $h_{\mathcal{L}_s}$ is understood to be $\sum_i a_i h_{\mathcal{L}_{s_i}}$, and so is $h_{\mathcal{L}_{s}(\phi)}$. Many intersections and integrations below are also understood in this sense.

**Proof.** We can assume that $Z_{\text{gal}} = Z$ since both sides depends only on $Z_{\text{gal}}$. Denote $r = \dim Z$. By definition,

$$ h_{\mathcal{E}(\phi)}(Z) - h_{\mathcal{E}}(Z) = \frac{1}{(r + 1) \deg_L(Z)} \sum_{i=1}^{r+1} \binom{r + 1}{i} \mathcal{O}(\phi)|_Z^i \cdot \mathcal{E}_{Z|_s}^{r+1-i} \cdot \mathcal{H}^d, $$

$$ h_{\mathcal{L}_{s}(\phi)}(Z|_s) - h_{\mathcal{L}_{s}}(Z|_s) = \frac{1}{(r + 1) \deg_L(Z)} \sum_{i=1}^{r+1} \binom{r + 1}{i} \mathcal{O}(\phi)|_{Z|_s}^i \cdot \mathcal{L}_{s|_s}^{r+1-i}. $$

It suffices to show that for each $\mathcal{M} \in \text{Pic}(Z|_s)$, the difference

$$ \delta(\mathcal{M}) := \mathcal{O}(\phi)^i \cdot \mathcal{M}_{s}^{r+1-i} \cdot \mathcal{H}^d - \mathcal{O}(\phi)^i \cdot \mathcal{M}_{s}^{r+1-i} $$

vanishes for all $i = 1, \cdots, r + 1$. 22
By approximation, we can assume that $\overline{\mathcal{M}}$ is an arithmetic line bundle on an integral model $\mathcal{Z} \to \mathcal{B}$ of $Z$. The intersections in
\[
\delta(\overline{\mathcal{M}}) = \mathcal{O}(\phi)^i \cdot \overline{\mathcal{M}}^{r+1-i} \cdot (\mathcal{S}, g_{\mathcal{S}}) - \mathcal{O}(\overline{\phi})^i \cdot \overline{\mathcal{M}}_s^{r+1-i}
\]
are taken over $\mathcal{Z}$. It is easy to obtain
\[
\delta(\overline{\mathcal{M}}) = \mathcal{O}(\phi)^i \cdot \overline{\mathcal{M}}^{r+1-i}|_{\mathcal{S}-\overline{s}}.
\]
It vanishes since $\phi$ is supported over $v \notin \Omega_{\mathcal{S}}$ and the vertical cycle $\mathcal{S} - \overline{s}$ has no component lying over $v$.

By definition,
\[
\mathcal{M} = \mathcal{O}(\phi)^i \cdot \overline{\mathcal{M}}^{r+1-i},
\]
are taken over $\mathcal{Z}$. It is easy to obtain
\[
\delta(\overline{\mathcal{M}}) = \mathcal{O}(\phi)^i \cdot \overline{\mathcal{M}}^{r+1-i}|_{\mathcal{S}-\overline{s}}.
\]
It vanishes since $\phi$ is supported over $v \notin \Omega_{\mathcal{S}}$ and the vertical cycle $\mathcal{S} - \overline{s}$ has no component lying over $v$.

For simplicity, we first consider the case that $s = S_{\mathbb{Q}}$ is a single point. Let $\epsilon > 0$ be a rational number. The lemma gives
\[
\frac{1}{\deg_{L}(Z_{\text{gal}})} \int_{\mathcal{Z}^{|\text{an}}_{S, Q_v}} \overline{\phi} \cdot c_1(\overline{\mathcal{L}}_s)^{\dim_{\mathbb{Z}}} + O(\epsilon^2).
\]
Here $\mathcal{Z}^{|\text{an}}_{S, Q_v}$ denotes the canonical measure of $\overline{\mathcal{L}}_s$ on $\overline{Z}^{|\text{an}}_{S, Q_v}$ introduced by Chambert-Loir [Ch]. The error term $O(\epsilon^2)$ depends on $Z$, but vanishes if $Z$ is a closed point.

Set $Z$ to be the point $x_m$ or the whole variety $X$. We obtain
\[
\frac{1}{\deg_{L}(Z_{\text{gal}})} \int_{\mathcal{Z}^{|\text{an}}_{S, Q_v}} \overline{\phi} \cdot c_1(\overline{\mathcal{L}}_s)^{\dim_{\mathbb{Z}}} + O(\epsilon^2).
\]
Here $\mu_{\pi_{x_m}|_{S, v}}$ denotes the canonical measure of $\overline{\mathcal{L}}_s$ on $\overline{Z}^{|\text{an}}_{S, Q_v}$ associated to the Galois orbit of $x_m$ on $\mathcal{X}^{|\text{an}}_{S, Q_v}$, and $d\mu_{x_m, v} = \frac{1}{\deg_{L}(\mathcal{X}_s)} c_1(\overline{\mathcal{L}}_s)^{\dim_{\mathbb{Z}}}$. The Chambert-Loir measure of with respect to $\overline{\mathcal{L}}_s$.

We have the fundamental inequality
\[
\sup_{Z} \inf_{x \in \mathcal{X}(K)-Z(K)} \frac{1}{\deg_{L}(Z_{\text{gal}})} \int_{\mathcal{Z}^{|\text{an}}_{S, Q_v}} \overline{\phi} \cdot c_1(\overline{\mathcal{L}}_s)^{\dim_{\mathbb{Z}}} + O(\epsilon^2).
\]
It is a generalized version of [Mo2, Corollary 5.2]. We omit the proof but leave a few remarks. Moriwaki’s result is based on the arithmetic Hilbert-Samuel formula due to Gillet-Soule, Bismut-Vasserot and Zhang. The key

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for our case is the arithmetic bigness result in [Yu, Theorem 2.2, Lemma 3.3], where a Hilberts-Samuel type estimation is obtained based on the arithmetic ampleness of Zhang [Zh1].

Now the standard argument of Szpiro-Ullmo-Zhang gives

$$\lim_{m \to \infty} \int_{X_{s,v}^{an}} \phi \mu_{X_{s,v}^m} = \int_{X_{s,v}^{an}} \phi \mu_{s,v}.$$

It is true for $\phi = \phi|_{X_{s,v}^{an}}$ as long as $\phi$ is continuous on $X_{s,v}^{an}$. This actually covers all continuous functions on $X_{s,v}^{an}$ since $X_{s,v}^{an}$ is closed in $X_{s,v}^{an}$. Therefore, we have proved the weak convergence $d\mu_{X_{s,v}^m} \to d\mu_{s,v}$ on the analytic space $X_{s,v}^{an}$.

The weak convergence is true under the assumption that $(S, g_S)$ represents $H^d$ and the generic fiber $s = S_Q$ is a single point contained in $V_Q$. In the general case that $s = \sum_i a_i s_i$ is a linear combination of closed points $s_i$ in $V_Q$, everything above still makes sense by linearity. Then the weak convergence $\mu_{X_{s,v}^m} \to d\mu_{s,v}$ implies the weak convergence $\mu_{X_{s_i,v}^m} \to d\mu_{s_i,v}$ for all $s_i$. For any closed point $u$ of $V_Q$, we can find a representative $(S, g_S)$ of $H^d$, such that the supported of the 0-cycle $S_Q$ contains $u$. Then we have the equidistribution $\mu_{X_{s,v}^m} \to d\mu_{s,v}$ on the fiber of $u$ for all finite places $v \not\in \Omega_S$.

It proves the result.

\[\square\]

## 5 Heights for dynamical systems

### Invariant metrics for dynamical systems

Let $K$ be a finitely generated field over $\mathbb{Q}$ of transcendental degree $d$. Let $(X, f, L)$ be a polarized algebraic dynamical system of dimension $n$ over $K$. Fix an isomorphism $f^* L = q L$ where $q > 1$ by assumption.

Fix an arithmetic variety $B$ over $\mathbb{Z}$ with function field $K$. Choose any arithmetic model $\pi : X \to B$ over $B$ and any hermitian line bundle $L = (L, \| \cdot \|)$ over $X$ such that $(X_B, L_B) = (X, L)$.

For each positive integer $m$, consider the composition $X \xrightarrow{f_m} X \xrightarrow{\pi_m} B$. Denote its normalization by $f_m : X_m \to X$, and the induced map to $B$ by $\pi_m : X_m \to B$. Denote $L_m = q^{-1} f_m^* L$, which lies in $\text{Pic}(X_m)_\mathbb{Q}$. The sequence $\{(X_m, L_m)\}_{m \geq 1}$ is an adelic structure in the sense of [Mo3].
In the following, we will show that the sequence \( \{(X_m, L_m)\}_{m \geq 1} \) converges to a line bundle \( L_f \) in \( \hat{\text{Pic}}(X)_{\text{int}} \). There is an open subscheme \( V \) of \( B \) such that \( U := X_V \) is flat over \( V \) and that \( f : X \to X \) extends to a morphism \( f_V : U \to U \) with \( f_V^* L_V = q L_V \). By the construction, we have \( X_{m,V} = X_V \) and \( L_{m,V} = L_V \). We can assume that \( D := B - V \) is an effective Cartier divisor of \( B \) by enlarging \( V \) if necessary.

**Theorem 5.1.** The sequence \( L_f = \{(X_m, L_m)\}_{m \geq 1} \) is convergent in \( \hat{\text{Pic}}(X)_{\text{int}} \). Furthermore, the limit is semipositive and depends only on the generic fiber \((X, f, L, K)\).

**Proof.** We only prove the existence of the limit, since the independence of the integral models can be proved similarly. Recall that \( X_{m,V} = X_V \) and \( L_{m,V} = L_V \). But these isomorphisms are not given by the morphism \( f_m : X_m \to X \).

Let \( \tilde{\pi}_m : \tilde{X}_m \to B \) be an arithmetic model of \( X \) which dominates both \( X_m \) and \( X_{m+1} \). More precisely, \( \tilde{\pi}_m \) factors through a birational morphism \( \phi_m : \tilde{X}_m \to X_m \) (resp. \( \phi'_m : \tilde{X}_m \to X_{m+1} \)) such that \( \phi_{m,V} \) (resp. \( \phi'_{m,V} \)) is an isomorphism to \( X_{m,V} \) (resp. \( X_{m+1,V} \)). The construction works for \( m = 0 \) by the convention \( X_0 = X \).

We first consider the relation between \((X, \overline{L})\) and \((\overline{L}_1, X_1)\). The difference \( \phi_0^* \overline{L} - \phi'_0^* \overline{L}_1 \) is an arithmetic \( \mathbb{Q} \)-line bundle on \( \tilde{X}_0 \). It is trivial on \( \tilde{X}_{0,V} \), and thus represented by an arithmetic divisor supported on \( \tilde{\pi}_0^* |D| \). Then there exists \( r > 0 \) such that

\[
\phi_0^* \overline{L} - \phi'_0^* \overline{L}_1 \in B(r, \hat{\text{Pic}}(U))
\]

Now we consider \( \overline{L}_m - \overline{L}_{m+1} \) for general \( m \). Without loss of generality, we can assume that \( \tilde{X}_m \) dominates \( \tilde{X}_0 \) via a birational morphism \( \tau_m : \tilde{X}_m \to \tilde{X}_0 \).

Then it is easy to see that

\[
\phi_m^* \overline{L}_m - \phi'_m^* \overline{L}_{m+1} = \frac{1}{q^m} \tau_m^* (\phi_0^* \overline{L} - \phi'_0^* \overline{L}_1).
\]

It follows that

\[
\phi_m^* \overline{L}_m - \phi'_m^* \overline{L}_{m+1} \in B\left(\frac{r}{q^m}, \hat{\text{Pic}}(U)\right)
\]

In terms of the partial order in \( \hat{\text{Pic}}(X)_{\mathbb{Q}} \), it is just

\[
\overline{L}_m - \overline{L}_{m+1} \in B\left(\frac{r}{q^m}, \hat{\text{Pic}}(U)\right)
\]

It follows that \( \{\overline{L}_m\}_m \) is a Cauchy sequence. \( \square \)
Remark 5.2. The arithmetic line bundle $\overline{L}_f$ is invariant under the pull-back $f^*: \hat{\text{Pic}}(X)_{\text{int}} \to \hat{\text{Pic}}(X)_{\text{int}}$ in the sense that $f^*\overline{L}_f = q\overline{L}_f$ as in the number field case.

Canonical height

Let $K$ and $(X, f, L)$ be as above. It gives an $f$-invariant line bundle $\overline{L}_f$ in $\hat{\text{Pic}}(X)_{\text{int}}$. For any closed $\overline{K}$-subvariety $Z$ of $X$, define the canonical height function of $Z$ as

$$h_f(Z) := h_{\text{T}_f}(Z) \in \hat{\text{Pic}}(K)_{\text{int}}.$$ 

We can also define the canonical height by Tate’s limiting argument:

$$h_f(Z) = \lim_{m \to \infty} \frac{1}{q^m} h_{(\chi, Z)}(f^m(Z)).$$

Here $(\chi, \overline{L})$ is any initial model of $(X, L)$ as in the construction of $\overline{L}_f$ above. Then one can check that it is convergent in $\hat{\text{Pic}}(K)_{\text{int}}$ and compatible with the previous definition.

Proposition 5.3. Let $Z$ be a closed subvariety of $X$ over $\overline{K}$. Then:

(a) The height $h_f(Z) \geq 0$ under the partial order in $\hat{\text{Pic}}(K)_{\text{int}}$.

(b) The height is $f$-invariant in the sense that $h_f(f(Z)) = q \cdot h_f(Z)$.

(c) The height $h_f(Z) = 0$ in $\hat{\text{Pic}}(K)_{\text{int}}$ if $Z$ is preperiodic under $f$. The inverse is also true if $Z$ is a point.

Proof. Since $\overline{L}_f$ is semipositive, the height $h_f(Z) \geq 0$ under the partial order. The formula $h_f(f(Z)) = q \cdot h_f(Z)$ follows from the projection formula and the invariance of $\overline{L}_f$. Thus $h_f(Z) = 0$ if $Z$ is preperiodic under $f$. The second statement of (c) follows from the Northcott property.

Conjecture 5.4. Let $(X, f, L)$ be a dynamical triple defined over a finitely generated field $K$ over $\mathbb{Q}$. Let $v$ be a place of $K$. Let $x_m$ be a generic sequence of preperiodic points. Then the Galois orbit of $x_m$ is equidistributed with respect to the equilibrium measure $d\mu_{\text{T}_f,v}$ on $X^\text{an}_v$.
References


