Calabi–Yau theorem and algebraic dynamics

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Contents

1 Introduction 1
   1.1 Calabi–Yau Theorem . . . . . . . . . . . . . . . . . . . . . . 2
   1.2 Algebraic dynamics . . . . . . . . . . . . . . . . . . . . . . . 3
   1.3 Dynamical Manin–Mumford conjecture . . . . . . . . . . . . 5

2 Calabi–Yau Theorem 7
   2.1 Hodge index theorem . . . . . . . . . . . . . . . . . . . . . . 7
   2.2 Smooth case . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
   2.3 Continuous case . . . . . . . . . . . . . . . . . . . . . . . . . 11

3 Algebraic dynamics 13
   3.1 Number field case . . . . . . . . . . . . . . . . . . . . . . . 14
   3.2 General case . . . . . . . . . . . . . . . . . . . . . . . . . . . 19

1 Introduction

The aim of this paper is to prove the uniqueness part of the Calabi–Yau theorem for metrized line bundles over non-archimedean analytic spaces, and apply it to endomorphisms with the same polarization and the same set of preperiodic points over a complex projective variety. The proof uses Arakelov theory (cf. [Ar, GS]) and Berkovich’s non-archimedean analytic spaces (cf. [Be]) even though the results on dynamical systems can be purely stated over complex numbers. In the following, we will describe our results and main ideas in details.
1.1 Calabi–Yau Theorem

Let $K$ be either $\mathbb{C}$ or a complete discrete valuation field, $X$ be a projective variety over $K$, and $L$ be an ample line bundle over $X$ endowed with a continuous semipositive $K$-metric $\|\cdot\|$ which is Galois invariant. Then $(L, \|\cdot\|)$ induces a canonical semipositive measure $c_1(L, \|\cdot\|)_{\dim X}$ on the analytic space $X^\text{an}$. We explain it as follows.

If $K = \mathbb{C}$, then $X^\text{an}$ is just the complex analytic space $X(\overline{K})$. The measure $c_1(L, \|\cdot\|)_{\dim X}$ is just the determinant of the Chern form $c_1(L, \|\cdot\|)$ which is locally defined by

$$c_1(L, \|\cdot\|) = \frac{\partial \bar{\partial}}{\pi i} \log \|\cdot\|$$

in complex analysis.

If $K$ is non-archimedean, $X^\text{an}$ is the Berkovich space associated to the variety $X$ over $K$. It is a Hausdorff, compact and path-connected topological space. Furthermore, it naturally includes the set of closed point of $X$. We refer to [Be] for more details on the space. The metric $\|\cdot\|$ being semipositive, in the sense of [Zh2], means that it is a uniform limit of metrics induced by ample integral models of $L$. The canonical measure $c_1(L, \|\cdot\|)_{\dim X}$ is constructed by Chambert-Loir [Ch2] using intersection theory.

**Theorem 1.1.** Let $L$ be an ample line bundle over $X$, and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semipositive metrics on $L$. Then

$$c_1(L, \|\cdot\|_1)_{\dim X} = c_1(L, \|\cdot\|_2)_{\dim X}$$

if and only if $\frac{\|\cdot\|_1}{\|\cdot\|_2}$ is a constant.

The history of the theorem in the complex case is as follows. In the 1950s, Calabi [Ca1, Ca2] made the following famous conjecture: *Let $L$ be an ample line bundle on a complex projective variety $X$, and $\omega$ be a positive smooth volume form on $X$ such that $\int_X \omega = \deg_L(X)$. Then there exists a positive smooth metric $\|\cdot\|$ on $L$ such that $c_1(L, \|\cdot\|)_{\dim X} = \omega$.* Calabi also proved that the metric is unique up to scalar multiples. The existence of the metric is much deeper, and was finally solved by S. T. Yau in the seminal paper [Ya] in 1977. Now the whole results are called the Calabi-Yau theorem.

Theorem 1.1 includes the non-archimdeean analogue of the uniqueness part of the Calabi-Yau theorem. The “if” part of the theorem is trivial by
definition. For archimedean $K$, the positive smooth case is due to Calabi as we mentioned above, and the continuous semipositive case is due to Kolodziej [Ko]. Afterwards Blocki [Bl] provided a very simple proof of Kolodziej’s result. We will prove the non-archimedean case using Blocki’s idea.

Gubler points out that Theorem 1.1 is actually true for any complete valuation field $K$ using his intersection theory developed in [Gu1]. In the general case, the base valuation ring $O_K$ is non-noetherian and the classical intersection theory is not applicable. Gubler’s results allow us to extend Chambert-Loir’s definition of canonical measures and our proof of the theorem. To illustrate the main idea of our proof, we will only restrict to the discrete valuation case in this paper.

### 1.2 Algebraic dynamics

Let $X$ be a projective variety over a field $K$ of characteristic zero, and $L \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be an ample class in the sense that it is a positive linear combination of usual ample line bundles. Denote

$$\mathcal{H}(L) := \{ f : X \to X \mid f^*L = qL \text{ for some } q \in \mathbb{Q}, \ q > 1 \}.$$ 

Here we write the tensor product of line bundles additively. Apparently $\mathcal{H}(L)$ is a semigroup in the sense that it is closed under composition of morphisms.

For any $f \in \mathcal{H}(L)$, the triple $(X, f, L)$ gives a polarized algebraic dynamical system in the usual sense. Let $\text{Prep}(f)$ denote the set of preperiodic points, i.e.,

$$\text{Prep}(f) := \{ x \in X(K) \mid f^m(x) = f^n(x) \text{ for some } m, n \in \mathbb{N}, \ m \neq n \}.$$ 

A well-known result of Fakhruddin [Fak] asserts that $\text{Prep}(f)$ is always Zariski dense in $X$.

#### Main result

**Theorem 1.2.** Let $X$ be a projective variety over a field $K$ of characteristic zero, and $L \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be an ample class on $X$. For any $f, g \in \mathcal{H}(L)$, the following are equivalent:

(a) $\text{Prep}(f) = \text{Prep}(g)$;

(b) $g\text{Prep}(f) = \text{Prep}(f)$;
(c) \( \text{Prep}(f) \cap \text{Prep}(g) \) is Zariski dense in \( X \).

Remark 1.3. When \( X = \mathbb{P}^1 \), the theorem is independently proved by M. Baker and L. DeMarco [BD] during the preparation of this paper. Their treatment for the number field case is the same as ours, while the method for the general case is quite different.

Remark 1.4. Assuming that \( K \) is a number field, the following are some works related to the above theorem:

1. A. Mimar [Mi] proved the theorem in the case that \( X = \mathbb{P}^1 \).
2. If \( f \) is a Lattès map on \( \mathbb{P}^1 \) or a power map on \( \mathbb{P}^d \) induced by \( (\mathbb{G}_m)^d \), the theorem is implied by the explicit description of \( g \) by S. Kawaguchi and J. H. Silverman [KS].
3. In the case that \( X = \mathbb{P}^1 \), C. Petsche, L. Szpiro, and T. Tucker [PST] found a further equivalent statement in terms of heights and intersections.

Variants and consequences

In a private communication, Mazur points out that one direction of Theorem 1.2 can be generalized as follows:

Theorem 1.5. Let \( X \) be a projective variety over a field \( K \) of characteristic zero, and \( L \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \) be an ample class on \( X \). Let \( f, g \in \mathcal{H}(L) \), and denote by \( Y \) the Zariski closure of \( \text{Prep}(f) \cap \text{Prep}(g) \) in \( X \). Then

\[
\text{Prep}(f) \cap Y(K) = \text{Prep}(g) \cap Y(K).
\]

On the other hand, in the situation of Theorem 1.2, if \( K \) is just \( \mathbb{C} \) or a valuation field, our proof gives the following result:

Theorem 1.6. Let \( K \) be either \( \mathbb{C} \) or \( \mathbb{C}_p \) for some prime number \( p \). Let \( X \) be a projective variety over \( K \), \( L \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \) be an ample class on \( X \), and \( f, g \in \mathcal{H}(L) \) be two polarized endomorphisms. If \( \text{Prep}(f) \cap \text{Prep}(g) \) is Zariski dense in \( X \), then \( d\mu_f = d\mu_g \).

Here \( d\mu_f \) denotes the equilibrium measure of \( (X, f, L) \) on \( X^{an} \). It can be obtained from any initial “smooth” measure on \( X^{an} \) by Tate’s limiting argument. It satisfies \( f^*d\mu_f = q^{\dim X}d\mu_f \) and \( f_*d\mu_f = d\mu_f \).
For any subset $P$ of $X(\overline{K})$, denote
\[ \mathcal{H}(L, P) := \{ g \in \mathcal{H}(L) \mid \text{Prep}(g) = P \} \]

We say that that $P$ is a special set of $X$ polarized by $L$ if $\mathcal{H}(L, P)$ is non-empty. Then we have the following consequence:

**Corollary 1.7.** Let $P$ be a special set polarized by $L$. Then the set $\mathcal{H}(L, P)$ is a semigroup, i.e., $g \circ h \in \mathcal{H}(L, P)$ for any $g, h \in \mathcal{H}(L, P)$.

In fact, by Theorem 1.2, we can write:
\[ \mathcal{H}(L, P) = \{ g \in \mathcal{H}(L) \mid gP = P \} \]

Then it is easy to see that $\mathcal{H}(L, P)$ is a semigroup.

Now we explain some ingredients in our proof of Theorem 1.2. We first show the result assuming that $K$ is a number field. The hard part is to show that (c) implies (a). We use Theorem 1.1 and the dynamical equidistribution theorem (cf. Theorem 3.1). The dynamical equidistribution theorem used here was proved by Yuan [Yu]. It generalized the equidistribution results of Szpiro-Ullmo-Zhang [SUZ], Bilu [Bi] and Chambert-Loir [Ch1, Ch2]. The first result of this type originated in [SUZ] which covered the case of abelian varieties.

For general $K$, the Lefshetz principle allows us to assume that $K$ is finitely generated over $\mathbb{Q}$. The main tool is the theory of canonical heights over finitely generated fields introduced by Moriwaki [Mo1]. We do not apply Moriwaki’s theory directly, but use a refinement of the theory in [YZ] by the authors of this paper. One is led to consider the arithmetic model $\mathcal{X} \to \mathcal{B}$ of $X \to \text{Spec}(K)$, which is the relative version of the number field case. The result we need to prove is essentially equivalent to the corresponding results on a dense family of fibres of $\mathcal{X} \to \mathcal{B}$. The fibres are defined over number fields, and the previous case applies if the reduction of $\text{Prep}(f) \cap \text{Prep}(g)$ is still Zariski dense. The density is also proved in [YZ].

### 1.3 Dynamical Manin–Mumford conjecture

The second author of this paper proposed in [Zh2] a dynamical analogue of the classical Manin–Mumford conjecture for abelian varieties. Namely, given a dynamical trip $(X, L, f)$ over a field $K$, a closed subvariety $Y$ of $X$
is preperiodic under $f$ if and only if the set $Y(\mathbb{K}) \cap \text{Prep}(f)$ is Zariski dense in $Y$. Recently, Ghocia and Tucker found the following counter example of this conjecture:

**Proposition 1.8** (Ghocia and Tucker, [GTZ]). Let $E$ be an elliptic curve with complex multiplication by an order $R$ in imaginary quadratic field $K$. Let $f$ be an endomorphism on $E \times E$ defined by multiplications by two nonzero elements $\alpha$ and $\beta$ in $R$ with equal norm $N(\alpha) = N(\beta)$. Then $f$ is polarized by any symmetric and ample line bundle 

$$\text{Prep}(f) = E_{\text{tor}} \times E_{\text{tor}}.$$ 

Moreover, the diagonal $\Delta_E$ in $E \times E$ is not preperiodic under $f$ if $\alpha/\beta$ is not a root of unity.

Notice that the diagonal is preperiodic for multiplication by $(2, 2)$. Thus the proposition shows that it is possible that two endomorphisms of a projective variety polarized by the same ample line bundle with the same set of preperiodic points may have different set of preperiodic subvarieties. We would like to propose the following revision of the dynamical Manin–Mumford conjecture:

**Conjecture 1.9.** Let $X$ be a projective variety over any field $K$, $L$ be an ample line bundle on $X$, and $P$ be a special set of $X$ polarized by $L$. Let $Y$ be a proper closed subvariety of $X$ such that $Y(\mathbb{K}) \cap P$ is dense in $Y$. Then there are two endomorphisms $f, g \in \mathcal{H}(L, P)$, and a proper $g$-periodic closed subvariety $Z$ such that $f(Y) \subset Z$.

If we apply the conjecture successively, we can find a finite sequence of subvarieties

$$X := X_0 \supset X_1 \supset X_2 \cdots \supset X_s$$

and endomorphisms $f_i, g_i \in \mathcal{H}(X_i, P \cap X_i(\mathbb{K}))$ for $i = 0, 1, \ldots, s$ such that

1. $X_i$ is $g_i$-periodic, which implies that $P \cap X_i(\mathbb{K})$ is a special set of $X_i$ polarized by $L|_{X_i}$;
2. $f_i \circ f_{i-1} \cdots f_0(Y) \subset X_{i+1}$ for any $i = 0, 1, \ldots, s - 1$;
3. $f_{s-1} \circ f_{s-2} \cdots f_0(Y) = X_s$. 

6
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2 Calabi–Yau Theorem

The goal of this section is to prove Theorem 1.1 in the non-archimedean case. The idea is parallel to that of Blocki [Bl], where the counterpart of Cauchy-Schwartz inequality is Hodge index theorem for vertical divisors on arithmetic varieties. We first show the Hodge index theorem, and then prove the result in the smooth case. In the end, we see that the continuous case is proved similarly.

Throughout this section, we let $K$ be the fraction field of a complete discrete valuation ring $O_K$. Denote by $\varpi$ a fixed uniformizer, and by $\kappa$ its residue field.

2.1 Hodge index theorem

By an arithmetic variety $\mathcal{X}$ over $O_K$ we mean a flat projective scheme $\mathcal{X}$ over $O_K$ which is irreducible and reduced. Denote by $\text{Ramp}(\mathcal{X})$ (resp. $\text{RNef}(\mathcal{X}), \text{Vert}(\mathcal{X})$) the group of relatively ample line bundles (resp. relatively nef line bundles, vertical divisors) on $\mathcal{X}$.

To be precise, we explain the terms above. A line bundle on $\mathcal{X}$ is relatively ample (resp. relatively nef) if it is ample (resp. nef) on both the generic fibre and the special fibre of $\mathcal{X}$. By a vertical divisor of $\mathcal{X}$, we mean an $\mathbb{R}$-linear combination of the irreducible components of the special fibre of $\mathcal{X}$. A vertical divisor is called principal if it is a scalar multiple of $\text{div}(\varpi)$.
Theorem 2.1 (Hodge index theorem). Let \( X \) be an arithmetic variety over \( O_K \) of relative dimension \( n \). Then the following are true:

(a) If \( L_1, \cdots, L_{n-1} \in \text{RNef}(X) \) and \( D \in \text{Vert}(X) \), then
\[
D^2 \cdot L_1 \cdot L_2 \cdots L_{n-1} \leq 0.
\]

(b) If \( L_1, \cdots, L_{n-1} \in \text{RAmp}(X) \) and \( D \in \text{Vert}(X) \), then
\[
D^2 \cdot L_1 \cdot L_2 \cdots L_{n-1} = 0 \iff D \text{ principal}.
\]

(c) If \( L_1, \cdots, L_{n-1} \in \text{RNef}(X) \) and \( D_1, D_2 \in \text{Vert}(X) \), then
\[
(D_1 \cdot D_2 \cdot L_1 \cdot L_2 \cdots L_{n-1})^2 \leq (D_1^2 \cdot L_1 \cdot L_2 \cdots L_{n-1})(D_2^2 \cdot L_1 \cdot L_2 \cdots L_{n-1}).
\]

The results are well-known for \( n = 1 \) and more or less for general \( n \). The proof for general \( n \) is not more difficult than the case \( n = 1 \). We include it here for convenience.

Enumerate the irreducible components of the special fibre by \( V_1, \cdots, V_r \). Then \( \text{div}(\varpi) = \sum_{i=1}^r a_i V_i \) with multiplicity \( a_i > 0 \). For convenience, denote \( E_i = a_i V_i \). We first show (a). Write \( D = \sum_{i=1}^r b_i E_i \) with some \( b_i \in \mathbb{R} \). We have
\[
D^2 \cdot L_1 \cdots L_{n-1} = \sum_{i,j=1}^r b_i b_j E_i \cdot E_j \cdot L_1 \cdots L_{n-1}.
\]
Note that
\[
\sum_{j=1}^r b_j^2 E_i \cdot E_j \cdot L_1 \cdots L_{n-1} = 0, \quad \forall i
\]
since \( \sum_j E_j \) is principal. We obtain
\[
D^2 \cdot L_1 \cdots L_{n-1}
= -\frac{1}{2} \sum_{i,j=1}^r (b_i - b_j)^2 E_i \cdot E_j \cdot L_1 \cdots L_{n-1}
= -\frac{1}{2} \sum_{i \neq j} (b_i - b_j)^2 E_i \cdot E_j \cdot L_1 \cdots L_{n-1}.
\]
Hence we get the inequality in (a) since \( E_i \cdot E_j \) is effective or zero for \( i \neq j \).
As for (b), still use the above square identity. We have

\[ E_i \cdot E_j \cdot L_1 \cdots L_{n-1} > 0, \quad i \neq j \]

as long as \( E_i \cap E_j \) is nonempty. For such \( i, j \), we have \( b_i = b_j \). It gives the equality of all \( b_i \) since the whole special fibre is connected.

It remains to derive (c), which is another form of (a). Let \( x, y \in \mathbb{R} \) be variables. The quadratic form

\[
(xD_1 + yD_2)^2 = x^2 D_1^2 \cdot L_1 \cdots L_{n-1} + 2xy D_1 \cdot D_2 \cdot L_1 \cdots L_{n-1} + y^2 D_2^2 \cdot L_1 \cdots L_{n-1}
\]

is negative semi-definite. It follows that the discriminant is negative or zero, which gives the inequality.

2.2 Smooth case

Let \( X \) be an arithmetic variety of relative dimension \( n \) over \( O_K \). We are going to show:

**Theorem 2.2.** Let \( L_1, L_2 \in \text{RNef}(X) \) such that the generic fibre \( (L_1)_K = (L_2)_K \). Assume that

\[ D \cdot L_1^n = D \cdot L_2^n \]

where \( D = L_1 - L_2 \) is naturally viewed as a vertical divisor. Then

(a) For any \( L \in \text{RNef}(X) \) with generic fibre \( L_K = (L_1)_K \), we have

\[ D^2 \cdot L^{n-1} = 0. \]

(b) If there exists an \( L \in \text{RAmp}(X) \) with generic fibre \( L_K = (L_1)_K \), then \( D \) is principal.

**Proof.** The second statement follows from Theorem 2.1 (b). Next we prove the first statement. Write

\[ L_1 = L + D_1, \quad L_2 = L + D_2, \quad D_1, D_2 \in \text{Vert}(X). \]

Then \( D = D_1 - D_2 \). We first factorize

\[ 0 = D \cdot L_1^n - D \cdot L_2^n = \sum_{i=0}^{n-1} D^2 \cdot L_1^i \cdot L_2^{n-1-i} \]
By Theorem 2.1, each term on the right-hand side is negative or zero. Hence,

\[ D^2 \cdot L_1^i \cdot L_2^{n-1-i} = 0, \quad i = 0, \ldots, n - 1. \]

More generally, we claim that

\[ D^2 \cdot L_1^i \cdot L_2^j \cdot L_3^k = 0, \quad \forall i, j, k \geq 0, \quad i + j + k = n - 1. \] (1)

If it is true, then the case \( i = j = 0, k = n - 1 \) gives what we want.

Next we prove (1) by induction on \( k \). The case \( k = 0 \) is what we have obtained at the beginning. Assuming (1), we need to show the case for \( k + 1 \). By Theorem 2.1 (c), for any \( i + j = n - 1 - k \),

\[ D \cdot D_1 \cdot L_1^i \cdot L_2^j \cdot L_3^k = 0. \] (2)

If further \( i \geq 1 \), then we also have

\[ D \cdot D_1 \cdot L_1^{i-1} \cdot L_2^{j+1} \cdot L_3^k = 0. \] (3)

The difference between (2) and (3) gives

\[ D^2 \cdot D_1 \cdot L_1^{i-1} \cdot L_2^j \cdot L_3^k = 0. \]

Since \( D_1 = L_1 - L \), the above becomes

\[ D^2 \cdot L_1^i \cdot L_2^j \cdot L_3^k - D^2 \cdot L_1^{i-1} \cdot L_2^j \cdot L_3^{k+1} = 0. \]

By (1), we get

\[ D^2 \cdot L_1^{i-1} \cdot L_2^j \cdot L_3^{k+1} = 0. \]

It finishes the induction.

\[ \square \]

Remark 2.3. (a) A similar proof gives \( D^2 \cdot M_1 \cdots M_{n-1} = 0 \) for any \( M_1, \ldots, M_{n-1} \) in \( \text{RNef}(\mathcal{X}) \) whose generic fibres are isomorphic to \((L_1)_K\).

(b) The existence of \( L \in \text{RAnn}(\mathcal{X}) \) in (b) implies that \((L_1)_K = (L_2)_K\) is ample, but not vice versa for fixed \( \mathcal{X} \).
2.3 Continuous case

Now we prove Theorem 1.1 in the non-archimedean case. For convenience, we clarify a few notions we will refer to. An arithmetic variety $\mathcal{X}$ over $O_K$ that extends $X$ is called an integral model of $X$. A line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}_K = L$ is called an integral model of $L$. For convenience, we allow $\mathcal{L}$ to be a $\mathbb{R}$-line bundle, that is a usual line bundle extending $L$ plus a vertical Cartier divisor with $\mathbb{R}$-coefficients.

Shilov boundaries

Any integral model $\mathcal{X}$ of $X$ induces a surjective reduction map

$$r : X^{\text{an}} \to |\mathcal{X}_\kappa|.$$  

Here $|\mathcal{X}_\kappa|$ denotes the underlying topological space of $\mathcal{X}_\kappa$ as a scheme.

Let $V$ be an irreducible component of $\mathcal{X}_\kappa$. Then the generic fibre of $V$ has a unique preimage $X^{\text{an}}$ under $r$. We call this preimage the Shilov boundary of $V$. Here is another description. The irreducible component $V$ is a divisor on $\mathcal{X}$, and thus induces a valuation from the local ring $\mathcal{O}_{\mathcal{X}, V_j}$ to $\mathbb{Z}$. By exponentiation, it gives a semi-norm on $\mathcal{O}_{\mathcal{X}, V_j}$. Locally, this semi-norm defines a point in the affinoids.

Vary the model $(\mathcal{X}, V)$. We get infinitely many Shilov boundaries in $X^{\text{an}}$. We are going to show that they are dense in $X^{\text{an}}$.

Before that, we recall a result on model functions mainly due to Gubler [Gu1]. Let $(\mathcal{X}, \mathcal{M})$ be an integral model of $(X, \mathcal{O}_X)$. It induces a metric $\| \cdot \|_\mathcal{M}$ on $\mathcal{O}_X$. Then $\log \|1\|_\mathcal{M}$ gives a function on $|X|$. Moreover, it extends to a continuous function on $X^{\text{an}}$. Such a function on $X^{\text{an}}$ is called a model function. Then [Yu, Lemma 3.5] asserts that the vector space of model functions is uniformly dense in the space of continuous functions on $X^{\text{an}}$. This result is a combination of Gubler [Gu1, Theorem 7.12] and some algebraicity arguments in [Yu].

**Lemma 2.4.** The set of Shilov boundaries is dense in $X^{\text{an}}$.

**Proof.** It suffices to show that if a continuous function $f : X^{\text{an}} \to \mathbb{R}$ vanishes on all Shilov boundaries, then $f$ vanishes everywhere.

By the density of model functions, for any $\epsilon > 0$, we have a model function $f_\epsilon$ such that $\|f - f_\epsilon\|_{\sup} < \epsilon$. It follows that $|f_\epsilon| < \epsilon$ on Shilov boundaries.
Let $f_\epsilon$ be induced by the integral model $(X, M)$. Then $M$ is associated to the vertical divisor

$$D = \sum_V f_\epsilon(v)V.$$  

Here the summation is over all irreducible components $V$ of the special fibre $X_\kappa$, and $v$ denotes the Shilov boundary associated to $V$.

The we see that $\epsilon \text{ div}(\varpi) \pm D$ is effective. It implies that $\epsilon \pm f_\epsilon$ is positive everywhere on $X^{\text{an}}$. In another word, $\|f_\epsilon\|_{\text{sup}} < \epsilon$. It follows that $\|f\|_{\text{sup}} < 2\epsilon$ on $X^{\text{an}}$. Then we must have $f = 0$ everywhere.

Proof of the general case

Now we go back to Theorem 1.1. We will prove the following stronger result as in the smooth case:

**Theorem 2.5.** Let $L$ be an ample line bundle over $X$, and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semipositive metrics on $L$. View $f = -\log(\|\cdot\|_1/\|\cdot\|_2)$ as a continuous function on $X^{\text{an}}$. Then

$$\int_{X^{\text{an}}} f \ c_1(L, \|\cdot\|_1)^{\dim X} = \int_{X^{\text{an}}} f \ c_1(L, \|\cdot\|_2)^{\dim X}$$  

if and only if $f$ is a constant.

Denote $n = \dim X$ and the metrized line bundles by $L_1 = (L, \|\cdot\|_1)$ and $L_2 = (L, \|\cdot\|_2)$. Denote $D = L_1 - L_2$. It is the trivial line bundle with a continuous metric, and can also be viewed as a “vertical divisor”. It is standard to write $D = O(f)$. Then the equality in the theorem can also be written as $D \cdot L_1^n = D \cdot L_2^n$.

Note that Theorem 2.1 (a) (c) are true in the semipositive case by taking limit. Let $L = (L, \|\cdot\|)$ be any line bundle $L$ endowed with any semipositive metric. Then exactly the same proof of Theorem 2.2 gives

$$D^2 \cdot L^{n-1} = 0.$$  

We want to show that $f$ is a constant from this. We can decode more information. Let $E$ be the trivial line bundle endowed with any continuous metric. Then the continuous version of Theorem 2.1 (c) gives

$$D \cdot E \cdot L^{n-1} = 0.$$  

12
Assume further $\mathcal{L}$ and $E$ are induced by $\mathcal{L}_0 \in \text{Ramp}(\mathcal{X})$ and $E_0 \in \text{Vert}(\mathcal{X})$ over an integral model $\mathcal{X}$ of $X$. Then we have a “projection formula”:

$$D \cdot E \cdot \mathcal{L}^{n-1} = D_0 \cdot E_0 \cdot \mathcal{L}_0^{n-1}$$

(4)

where

$$D_0 = \sum_{j=1}^{r} f(v_j)V_j$$

is the “push forward” to $\mathcal{X}$ of $D$. The summation is over all irreducible components $V_j$ of the special fibre $\mathcal{X}_\kappa$ of $\mathcal{X}$, and the point $v_j \in X^{an}$ is the Shilov boundary associated to $V_j$.

The equality (4) follows from the definition of canonical measures in [Ch2]. We explain it in a few words. The function $f$ is a limit of model functions $\{f_m\}$ where each $f_m$ is induced by a vertical divisor $D_m$ on an integral model $\mathcal{X}_m$ of $X$. We can assume that there exists a morphism $\pi_m : \mathcal{X}_m \to \mathcal{X}$ that is the identity on the generic fibre. Then the projection formula gives

$$D_m \cdot \pi^*_m E_0 \cdot \pi^*_m (\mathcal{L}_0^{n-1}) = \pi^*_m D_m \cdot E_0 \cdot \mathcal{L}_0^{n-1}.$$ 

It is easy to check that $\pi^*_m D_m = \sum_{j=1}^{r} f_m(v_j)V_j$. Take limit $m \to \infty$. We obtain (4).

Once we have (4), then $D_0 \cdot E_0 \cdot \mathcal{L}_0^{n-1} = 0$. The case $E_0 = D_0$ gives $D_0^2 \cdot \mathcal{L}_0^{n-1} = 0$. It implies that $f(v_1) = f(v_2) = \cdots = f(v_r)$ by Theorem 2.1 (b) since $\mathcal{L}_0$ is relatively ample on $\mathcal{X}$. Now we vary the model $(\mathcal{X}, \mathcal{L}_0)$. Since any two models are dominated by a third one, we see that $f$ is constant on all Shilov boundaries. Then $f$ must be constant by the density result proved in Lemma 2.4.

## 3 Algebraic dynamics

In this section we are going to prove Theorem 1.2. By Lefshetz principle, we can assume that $K$ is finitely generated over $\mathbb{Q}$. The hard part is to show that (c) implies (a).

We first show the result assuming that $K$ is a number field. We use the dynamical equidistribution theorem (cf. Theorem 3.1) and the Calabi–Yau theorem. By (c), we can construct a generic sequence in $\text{Prep}(f) \cap \text{Prep}(g).$
Then the Dirac measures of the Galois orbits of the sequence converge to both the $f$-equilibrium measure and the $g$-equilibrium measure. It forces these two equilibrium measures to be equal. By Theorem 1.1, it implies that the $f$-invariant adelic metric of $L$ is proportional to the $g$-invariant one. Then it is easy to conclude that $f$ and $g$ defines the same canonical height. The identity on the preperiodic points are obtained by setting canonical heights to be zero.

In the case that $K$ is finitely generated fields over $\mathbb{Q}$, the main tool is the theory of canonical heights over $K$ introduced by Moriwaki [Mo1]. To show that $f$ and $g$ define the same canonical height, we relate the canonical heights over $K$ to canonical heights over number fields by a specialization so that we can apply the result we have already shown in the number field case.

### 3.1 Number field case

Here we prove Theorem 1.2 assuming that $K$ is a number field. Before the proof, we recall the dynamical equidistribution of small points.

#### Equidistribution of small points

Let $(X, f, L)$ be a polarized algebraic dynamical system over a number field $K$. Fix an isomorphism $f^*L = L^{\otimes q}$ with $q > 1$.

By Zhang [Zh2], we have a unique canonical $f$-invariant adelic metric $\{ \| \cdot \|_{f,v} \}_{v \in M_K}$ on $L$ such that the adelic metrized line bundle $L_f = (L, \{ \| \cdot \|_{f,v} \}_{v \in M_K})$ gives a polarization of $L$ in the sense that the isomorphism $f^*L = L^{\otimes q}$ induces an isometry $f^*L_f = L_f^{\otimes q}$. Here we denote by $M_K$ the set of places of $K$.

The canonical height is defined to be

$$h_f(x) := -\frac{1}{\deg(x)} \sum_{v \in M_K} \sum_{y \in O(x)} \log \|s(y)\|_{f,v}, \quad x \in X(\overline{K}).$$

Here $O(x) = \text{Gal}(\overline{K}/K)x$ is the Galois orbit of $x$ which can be viewed as a subset of $X(\overline{K}_v)$ for every $v$. The height $h_f$ is the class of height functions of $L$. We further have $h_f(x) \geq 0$ for all $x \in X(\overline{K})$, and the equality holds if and only if $x \in \text{Prep}(f)$.
Recall that for each $v \in M_K$, we have an analytic space $X^K_v$ associated to $X^K_v$, and a canonical measure $c_1(L, \| \cdot \|_{f,v})^{\dim X}$ on $X^K_v$. The canonical measure is invariant under both the push-forward and the pull-back of $f$ up to scalar multiples. Denote the equilibrium measure
\[
d\mu_{f,v} := \frac{1}{\deg_L(X)} c_1(L, \| \cdot \|_{f,v})^{\dim X}.
\]
It has total volume 1 on $X^K_v$.

For any $x \in X(K)$, the Galois orbit $O(x) \subset X(K_v)$ descends to a finite subset $O_v(x)$ of the underlying space $|X^K_v|$ of the scheme $X^K_v$. View $O_v(x)$ as a finite subset of $X^K_v$ by the inclusion $|X^K_v| \subset X^K_v$. Define the probability measure associated to the Galois orbit of $x$ by
\[
\mu_{v,x} := \frac{1}{\deg(x)} \sum_{z \in O_v(x)} \deg(z) \delta_z
\]
where $\delta_z$ is the Dirac measure of $z$ in $X^K_v$, and $\deg(z)$ is the degree of the residue field of $z$ over $K_v$.

**Theorem 3.1** (Equidistribution of small points). Let $\{x_m\}$ be an infinite sequence of $X(K)$ satisfying:

1. It is generic in $X$, i.e., any infinite subsequence is Zariski dense in $X$;
2. It is $h_f$-small, i.e., $h_f(x_m) \to 0$ as $m \to \infty$.

Then for any $v \in M_K$, the measure $\mu_{v,x_m}$ converges weakly to $d\mu_{f,v}$ on $X^K_v$.

The theorem was first proved for abelian varieties in the work of Szpiro-Ullmo-Zhang [SUZ]. Then Bilu [Bi] proved the torus case, and Chambert-Loir [Ch1] proved the semi-abelian case. All those works considered only archimedean $v$. Then Chambert-Loir [Ch2] introduced canonical measures on Berkovich spaces and generalized the work of [SUZ] to non-archimedean $v$. Finally, the general dynamical case was proved by Yuan [Yu].

The function field analogue of the theorem was obtained by Faber [Fa] and Gubler [Gu2] independently in slightly different settings.
Now we are ready to prove:

**Theorem 3.2.** Let $X$ be a projective variety over a number field $K$, and $L$ be an ample line bundle on $X$. For any $f, g \in \mathcal{H}(L)$, the following are equivalent:

(a) $\text{Prep}(f) = \text{Prep}(g);
(b) g\text{Prep}(f) = \text{Prep}(f);
(c) \text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense in $X$;
(d) $L_f = \mathcal{O}_X \otimes L_g$ for some $\mathcal{O} \in \widehat{\text{Pic}}^0(K);
(e) h_f(x) = h_g(x)$ for any $x \in X(K)$.

**Remark 3.3.** Assuming $X = \mathbb{P}^n$, S. Kawaguchi and J. H. Silverman [KS] proves that (c) $\iff$ (d).

Before the proof, we explain (d) in more details. The group $\widehat{\text{Pic}}(K)$ (resp. $\widehat{\text{Pic}}(X)$) denotes the group of adelic line bundles on $K$ (resp. $X$) in the sense of Zhang [Zh2], and $\widehat{\text{Pic}}^0(K)$ denotes the subgroup of elements with arithmetic degree zero. View $\widehat{\text{Pic}}(K)$ as a subgroup of $\widehat{\text{Pic}}(X)$ via the pull-back of the structure morphism $X \to \text{Spec}(K)$. Thus an element of $\widehat{\text{Pic}}(K)$ is an adelic line bundle $(\mathcal{O}_X, \{\| \cdot \|_v\})$ where $\|1\|_v = c_v$ is constant for any $v$. The element lies in $\widehat{\text{Pic}}^0(K)$ if and only if $\prod_v c_v = 1$.

Now we show (d) $\implies$ (e) $\implies$ (a) $\implies$ (b) $\implies$ (c).

First, (d) $\implies$ (e) follows from the formula

$$h_f(x) = -\frac{1}{\deg(x)} \sum_{v \in M_K} \sum_{y \in \mathcal{O}(x)} \log \|s(y)\|_{f,v}, \quad x \in X(K).$$

Second, (e) $\implies$ (a) since preperiodic points are exactly points with canonical height equal to zero.

Third, it is trivial that (a) $\implies$ (b) and (a) $\implies$ (c) since $\text{Prep}(f)$ is Zariski dense.

Finally, it is also not hard to show (b) $\implies$ (c). For any integer $d > 0$, denote

$$\text{Prep}(f, d) := \{x \in \text{Prep}(f) \mid \deg(x) < d\}. $$

16
By Northcott’s property, $\text{Prep}(f, d)$ is a finite set since its points have trivial canonical heights. Assuming (b), then $g$ fixes the set $\text{Prep}(f)$. By definition,

$$\text{Prep}(f) = \bigcup_{d>0} \text{Prep}(f, d).$$

Since $g$ is also defined over $K$, it fixes the set $\text{Prep}(f, d)$. By the finiteness, we obtain that

$$\text{Prep}(f, d) \subset \text{Prep}(g), \quad \forall d.$$ 

Hence,

$$\text{Prep}(f) \subset \text{Prep}(g).$$

Then (c) is true since $\text{Prep}(f)$ is Zariski dense in $X$.

**From (c) to (d)**

It remains to show that (c) implies (d).

Assume that $\text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense in $X$. We can choose a generic sequence $\{x_m\}$ in $\text{Prep}(f) \cap \text{Prep}(g)$. The sequence is small with respect to both $f$ and $g$ since the canonical heights $h_f(x_m) = h_g(x_m) = 0$ for each $m$.

By Theorem 3.1, for any place $v \in M_K$, we have

$$\mu_{v,x_m} \to d\mu_{f,v}, \quad \mu_{v,x_m} \to d\mu_{g,v}.$$ 

It follows that $d\mu_{f,v} = d\mu_{g,v}$ as measures on $X_v^{an}$. Then

$$c_1(L, \| \cdot \|_{f,v})^{\dim X} = c_1(L, \| \cdot \|_{g,v})^{\dim X}.$$ 

By the arithmetic Calabi–Yau in Theorem 1.1, we obtain constants $c_v > 0$ such that

$$\| \cdot \|_{f,v} = c_v \| \cdot \|_{g,v}.$$ 

Here $c_v = 1$ for almost all $v$. To finish the proof, we only need to check that the product $c = \prod_v c_v$ is 1.

In fact, by the height formula we have

$$h_f(x) = h_g(x) - \log c, \quad \forall x \in X(\overline{K}).$$

Then $c = 1$ by considering any $x$ in $\text{Prep}(f) \cap \text{Prep}(g)$. It finishes the proof.
Theorem 1.5

The same method can prove Theorem 1.5 in the number field case. In fact, Theorem 3.1 is still valid for generic and small sequences on $Y$ by [Yu]. More precisely, if $\{x_m\}$ is an $h_f$-small and generic sequence in $Y(\overline{K})$, then the measure $\mu_{v,x_m}$ converges weakly to the probability measure $\frac{1}{\deg_L(Y)} c_1(L, \|f\|_f)^{\dim Y}$ on $Y_v^{\text{an}}$ for any place $v \in M_K$. Then we apply Theorem 1.1 on $Y$ to obtain $h_f(x) = h_g(x)$ for any $x \in Y(\overline{K})$.

Alternative approach

Here we sketch a slightly different proof of the implication $(c) \Rightarrow (d)$ in Theorem 1.2. It generalizes Chambert-Loir’s argument in [Ch3, §3.4] from the 1-dimensional case to the general case. We will use the following equidistribution result.

Theorem 3.4 ([CT], Lemme 6.1). Let $\{x_m\}$ be a generic and $h_f$-small sequence of $X(\overline{K})$. Then for any integrable adelic line bundle $\mathcal{M}$ on $X$,

$$
\lim_{m \to \infty} h_{\mathcal{M}}(x_m) = \frac{\mathcal{L}_f^n \cdot \mathcal{M}}{\deg_L(X)}.
$$

The result is proved by the variational principle on $\mathcal{L}_f + \epsilon \mathcal{M}$, where the key is still the bigness theorem in [Yu].

Go back to the implication $(c) \Rightarrow (d)$. Assume that $\text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense in $X$. We can choose a generic sequence $\{x_m\}$ in $\text{Prep}(f) \cap \text{Prep}(g)$. Apply the theorem above to $\mathcal{M} = \mathcal{L}_g$, we obtain $\mathcal{L}_f^n \cdot \mathcal{L}_g = 0$. Similarly, $\mathcal{L}_g^n \cdot \mathcal{L}_f = 0$. We also have $\mathcal{L}_f^{n+1} = \mathcal{L}_g^{n+1} = 0$. Therefore,

$$(\mathcal{L}_f - \mathcal{L}_g) \cdot (\mathcal{L}_f^n - \mathcal{L}_g^n) = 0.
$$

For each $v$, denote by $\mathcal{L}_{f,v}$ and $\mathcal{L}_{g,v}$ the corresponding metrized line bundles on $X_{K_v}$. The above becomes

$$
\sum_v (\mathcal{L}_{f,v} - \mathcal{L}_{g,v}) \cdot (\mathcal{L}_{f,v}^n - \mathcal{L}_{g,v}^n) = 0.
$$

By the continuous version of the Hodge index theorem in Theorem 2.1,

$$(\mathcal{L}_{f,v} - \mathcal{L}_{g,v}) \cdot (\mathcal{L}_{f,v}^n - \mathcal{L}_{g,v}^n) = \sum_{i+j=n-1} (\mathcal{L}_{f,v} - \mathcal{L}_{g,v})^2 \cdot \mathcal{L}_{f,v}^i \cdot \mathcal{L}_{g,v}^j \leq 0.$$
It follows that for each \( v \),
\[
(L_{f,v} - L_{g,v}) \cdot (L_{f,v}^n - L_{g,v}^n) = 0.
\]
It is exactly
\[
\int_{X_{an}^n} \phi_v c_1(L_{f,v})^n = \int_{X_{an}^n} \phi_v c_1(L_{g,v})^n
\]
where \( \phi_v = -\log(\| \cdot \|_{f,v}/\| \cdot \|_{g,v}) \). Apply Theorem 2.5, which is also true in the archimedean case by [Bl]. We conclude that \( \phi_v \) is constant for every \( v \). The rest of the proof is the same.

This proof “avoids” the use of Berkovich spaces by the strong equidistribution result in Theorem 3.4, and uses the strong form Theorem 2.5 of the Calabi-Yau theorem. However, its spirit is very close to the original proof.

### 3.2 General case

Now we prove Theorem 1.2 for a general base field \( K \). Let \((X, f, g, L)\) be defined over \( K \) as in the theorem. By Lefshetz principle, we can assume that \( K \) is finitely generated over \( \mathbb{Q} \).

The tool is the theory of canonical heights over finitely generated fields in [YZ], which was inspired by the work of Moriwaki [Mo1, Mo2] and refines Moriwaki’s work. We will use the notations and results of [YZ] freely.

Let \( K \) be a finitely generated over \( \mathbb{Q} \), and let \((X, f, L)\) be a dynamical triple defined over \( K \). In [YZ], we have introduced an \( f \)-invariant arithmetic line bundle \( \mathcal{L}_f \in \hat{\text{Pic}}(X)_{\text{int}} \) and a canonical height functions
\[
\mathfrak{h}_f : X(K) \to \hat{\text{Pic}}(K)_{\text{int}}.
\]
In terms of arithmetic intersections, the canonical height of any \( x \in X(K) \) is just
\[
\mathfrak{h}_f(x) = \frac{1}{\text{deg}(x)} \mathcal{L}_f|_{x_{\text{gal}}} \in \hat{\text{Pic}}(K)_{\text{int}}.
\]
Here \( x_{\text{gal}} \) denotes the close point of \( X \) containing the image of \( x \).

**Easy directions**

Now we are ready to prove:
Theorem 3.5. Let $X$ be a projective variety over a field $K$ which is finitely generated over $\mathbb{Q}$, and $L$ be an ample line bundle on $X$. For any $f, g \in \mathcal{H}(L)$, the following are equivalent:

(a) $\text{Prep}(f) = \text{Prep}(g)$;

(b) $g\text{Prep}(f) = \text{Prep}(f)$;

(c) $\text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense in $X$;

(d) $h_f(x) = h_g(x)$ for any $x \in X(K)$.

As in the number field case, it is immediate to have $(d) \Rightarrow (a) \Rightarrow (b)$. Similar arguments also show that $(b) \Rightarrow (c)$. In fact, the set

$$\text{Prep}(f, d) := \{x \in \text{Prep}(f) \mid \deg(x) < d\}$$

is still finite by Northcott’s property on $h_f$.

From (c) to (d)

Assume that $\text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense in $X$. We need to show $h_f(x) = h_g(x)$ for every $x \in X(K)$. Let $\mathcal{L}_f$ and $\mathcal{L}_g$ be the objects in $\widehat{\text{Pic}}(X)_{\text{ms}}$ defined by $f$ and $g$.

Let $U \to V$ be a flat integral model over $\mathbb{Z}$ for $X \to \text{Spec}K$ such that $(X, f, g, L)$ extends to $(U, f_U, g_U, L_U)$. Here we require that $L_U$ is relatively ample over $V$ and polarizes the $U$-morphisms $f_U, g_U : U \to U$.

Let $x_0$ be a point in $\text{Prep}(f) \cap \text{Prep}(g)$ and let $L_{x_0} \simeq \mathcal{O}_{x_0}$ be a rigidification. Since $x_0$ has height 0 with respect to both $f$ and $g$, we can normalize the metrics on $\mathcal{L}_f$ and $\mathcal{L}_g$ such that the above rigidification induces isometry

$$\mathcal{L}_f|_{x_0} \simeq \mathcal{O}_{x_0} \simeq \mathcal{L}_g|_{x_0}.$$

By [YZ, Theorem 4.7], $\text{Prep}(f_s) \cap \text{Prep}(g_s)$ is Zariski dense in the fibre $U_s$ for each closed point $s \in V$. Restricted to $\mathcal{L}_s$, the metric $\| \cdot \|_f$ is the same as the $f_s$-invariant metric $\| \cdot \|_{f_s}$ of $\mathcal{L}_s$ on $\mathcal{X}_s$ for every place $v$ of $\mathbb{Q}$. Similar result hold for $\| \cdot \|_g$ and $\| \cdot \|_{g_s}$. By the result in the case of number fields again, we see that $\| \cdot \|_{f_s}/\| \cdot \|_{g_s}$ is a constant on $\mathcal{X}_s$ thus equals 1 by our rigidification.
Let $E$ be the divisor in $\widehat{\text{Div}}(X)_{\text{ms}}$ defined by the section of $L_f \otimes L_g^{-1}$ which is the identity at the generic fiber. The the Green’s function of $E$ at any place $v$ of $\mathbb{Q}$ whose value is just
\[ \log(\| \cdot \|_{f,v}/\| \cdot \|_{g,v}) = 0. \]

Then $E = 0$ by Proposition 1.4 in [YZ], and thus $L_f \simeq L_g$.

**Proof of Theorem 1.6**

The proof of Theorem 1.5 for general fields can be obtain similarly. Here we sketch a proof of Theorem 1.6.

Let $K$ and $(X, f, g, L)$ be as in the theorem. Assume that $(X, f, g, L)$ descends to a subfield $K_0$ of $K$ which is finitely generated over $\mathbb{Q}$. Let $\mathcal{U} \to \mathcal{V}$ be a flat integral model over $\mathbb{Z}$ for $X \to \text{Spec} K_0$ such that $(X, f, g, L)$ extends to $(\mathcal{U}, f_\mathcal{U}, g_\mathcal{U}, L_\mathcal{U})$ as above.

In the above proof, we already see that $L_f \simeq L_g$ after the rigidification. It implies $L_{f_s} \simeq L_{g_s}$ for any point $s \in \mathcal{V}(K)$. Note that $L_{f_s}$ has only one metric since $K$ is local. Let $\eta_0 : \text{Spec} K_0 \to \mathcal{V}$ be the generic point of $\mathcal{V}$. It gives a morphism $\eta : \text{Spec} K \to \mathcal{V}$ by composing with $\text{Spec} K \to \text{Spec} K_0$. Then $\eta \in \mathcal{V}(K)$ is just a $K$-point, and we have $L_{f_\eta} \simeq L_{g_\eta}$. These are exactly the invariant $K$-metrics of the dynamical systems $(X, f, L)$ and $(X, g, L)$ over $K$.

**References**


