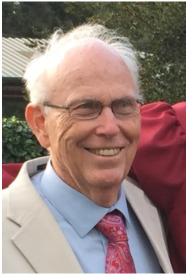
NONLINEAR PDE's
and POTENTIAL THEORY

Work with Reese Harvey



SOME HISTORY

Calibrated Geometries

Let Ψ be an exterior p-form on $\mathbb{R}^n$ (or on a riemannian manifold) with

$$d\Psi = 0$$

$$\Psi\big|_P \leq d\,\mathrm{vol}_P \qquad \text{for all oriented } p\text{-planes } P$$

SOME HISTORY

Calibrated Geometries

Let Ψ be an exterior p-form on $\mathbb{R}^n$ (or on a riemannian manifold) with

$$d\Psi = 0$$

$$\Psi\big|_P \leq d\,\mathrm{vol}_P \qquad \text{for all oriented } p\text{-planes } P$$

Define $G(\Psi) \equiv \{P : \Psi\big|_P = d\,\mathrm{vol}_P\}$

SOME HISTORY

Calibrated Geometries

Let Ψ be an exterior p-form on $\mathbb{R}^n$ (or on a riemannian manifold) with

$$d\Psi = 0$$

$$\Psi\big|_P \leq d\,\mathrm{vol}_P \qquad \text{for all oriented } p\text{-planes } P$$

Define $G(\Psi) \equiv \{P : \Psi\big|_P = d\,\mathrm{vol}_P\}$

An oriented p-dimensional submanifold M is **calibrated** by Ψ if

$$T_x M \in G(\Psi) \quad \forall\, x \in M.$$

SOME HISTORY

Calibrated Geometries

Let Ψ be an exterior p-form on $\mathbb{R}^n$ (or on a riemannian manifold) with

$$d\Psi \;=\; 0$$

$$\Psi\big|_P \;\leq\; d\,\mathrm{vol}_P \qquad \text{for all oriented } p\text{-planes } P$$

Define $\quad G(\Psi) \;\equiv\; \{P : \Psi\big|_P \;=\; d\,\mathrm{vol}_P\}$

An oriented p-dimensional submanifold M is **calibrated** by Ψ if

$$T_xM \in G(\Psi) \quad \forall\, x \in M.$$

LEMMA. *If M is calibrated by Ψ, then M is homologically volume minimizing.*

SOME HISTORY

Calibrated Geometries

Let Ψ be an exterior p-form on $\mathbb{R}^n$ (or on a riemannian manifold) with

$$d\Psi \;=\; 0$$

$$\Psi\big|_P \;\leq\; d\,\mathrm{vol}_P \qquad \text{for all oriented } p\text{-planes } P$$

Define $G(\Psi) \;\equiv\; \{P : \Psi\big|_P \;=\; d\,\mathrm{vol}_P\}$

An oriented p-dimensional submanifold M is **calibrated** by Ψ if
$T_x M \in G(\Psi) \quad \forall\, x \in M.$

LEMMA. *If M is calibrated by Ψ, then M is homologically volume minimizing.*

Proof: M' another such manifold with $\partial M' = \partial M$ (possibly $= \emptyset$)
and $[M - M'] = 0 \in H_p$

$$\mathrm{vol}(M) \;=\; \int_M \Psi \;=\; \int_{M'} \Psi \;\leq\; \mathrm{vol}(M')$$

Ex: $\Psi = \omega = \sum_{k=1}^{n} \frac{i}{2} dz_k \wedge d\bar{z}_k$ on $\mathbb{C}^n$

Ex: $\Psi = \omega = \sum_{k=1}^{n} \frac{i}{2} dz_k \wedge d\bar{z}_k$ on $\mathbb{C}^n$

(also on any Kähler manifold).

Ex: $\Psi = \omega = \sum_{k=1}^{n} \frac{i}{2} dz_k \wedge d\bar{z}_k$ on $\mathbb{C}^n$

(also on any Kähler manifold).

Ex: $\Psi = \frac{1}{p!}\omega^p$

Ex: $\Psi = \omega = \sum_{k=1}^{n} \frac{i}{2} dz_k \wedge d\bar{z}_k$ on $\mathbb{C}^n$

(also on any Kähler manifold).

Ex: $\Psi = \frac{1}{p!}\omega^p$

Reese and I went on a search for other such forms.

Ex: $\Psi = \omega = \sum_{k=1}^{n} \frac{i}{2} dz_k \wedge d\bar{z}_k$ on $\mathbb{C}^n$

(also on any Kähler manifold).

Ex: $\Psi = \frac{1}{p!}\omega^p$

Reese and I went on a search for other such forms.

Some Nice Cases

Constant coefficient forms in $\mathbb{R}^n$

Ex: $\Psi = \omega = \sum_{k=1}^{n} \frac{i}{2} dz_k \wedge d\bar{z}_k$ on $\mathbb{C}^n$

(also on any Kähler manifold).

Ex: $\Psi = \frac{1}{p!}\omega^p$

Reese and I went on a search for other such forms.

Some Nice Cases

Constant coefficient forms in $\mathbb{R}^n$

Parallel forms on Riemannian manifolds

We Found:

Special Lagrangian in $\mathbb{C}^n$ or on Calabi-Yau manifolds

We Found:

Special Lagrangian in $\mathbb{C}^n$ or on Calabi-Yau manifolds

Associative and Coassociative Φ and $*\Phi$ on $\mathrm{Im}\,\mathbb{O} = \mathbb{R}^7$ or on G_2-manifolds.

We Found:

Special Lagrangian in $\mathbb{C}^n$ or on Calabi-Yau manifolds

Associative and Coassociative Φ and $*\Phi$ on $\mathrm{Im}\,\mathbb{O} = \mathbb{R}^7$ or on G_2-manifolds.

Cayley calibration Θ on $\mathbb{O} = \mathbb{R}^8$ or on Spin$_7$-manifolds.

We Found:

Special Lagrangian in $\mathbb{C}^n$ or on Calabi-Yau manifolds

Associative and Coassociative Φ and $*\Phi$ on $\mathrm{Im}\,\mathbb{O} = \mathbb{R}^7$ or on G_2-manifolds.

Cayley calibration Θ on $\mathbb{O} = \mathbb{R}^8$ or on Spin_7-manifolds.

QUESTION:

Are there analogues of holomorphic or pluriharmonic functions on such manifolds?

ANSWER: NO.

ANSWER: **NO.**

But there are **ALWAYS ANALOGUES OF PLURISUBHARMONIC FUNCTIONS!**

ANSWER: **NO.**

But there are **ALWAYS ANALOGUES OF PLURISUBHARMONIC FUNCTIONS!**

$$G \equiv G(\Psi) \subset \mathrm{Grass}_p(\mathbb{R}^n)$$

Def. A C^2-function f is **G-plurisubharmonic** if at each point in its domain

$$\mathrm{tr}\left\{D^2 f\big|_P\right\} \geq 0 \qquad \forall\, P \in G$$

ANSWER: **NO.**

But there are **ALWAYS ANALOGUES OF PLURISUBHARMONIC FUNCTIONS!**

$$G \equiv G(\Psi) \subset \mathrm{Grass}_p(\mathbb{R}^n)$$

Def. A C^2-function f is ***G*-plurisubharmonic** if at each point in its domain

$$\mathrm{tr}\left\{D^2 f\big|_P\right\} \geq 0 \qquad \forall P \in G$$

or equivalently

$$D^2 f \in F(G)$$

ANSWER: **NO.**

But there are **ALWAYS ANALOGUES OF PLURISUBHARMONIC FUNCTIONS!**

$$G \equiv G(\Psi) \subset \mathrm{Grass}_p(\mathbb{R}^n)$$

Def. A C^2-function f is ***G*-plurisubharmonic** if at each point in its domain

$$\mathrm{tr}\left\{D^2 f\big|_P\right\} \geq 0 \qquad \forall P \in G$$

or equivalently

$$D^2 f \in F(G)$$

$$F(G) \equiv \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \mathrm{tr}\{A\big|_P\} \geq 0 \ \forall P \in G\}$$

Def. An **upper semi-continuous** function u on $\Omega^{\text{open}} \subset \mathbb{R}^n$ is **G-psh** if

$$u\big|_P \text{ is subharmonic for all affine } G\text{-planes } P$$

Def. An **upper semi-continuous** function u on $\Omega^{\text{open}} \subset \mathbb{R}^n$ is **G-psh** if

$$u\big|_P \text{ is subharmonic for all affine } G\text{-planes } P$$

or

Equivalent Definition (Viscosity Theory).

Def. An **upper semi-continuous** function u on $\Omega^{\text{open}} \subset \mathbb{R}^n$ is **G-psh** if

$$u\big|_P \text{ is subharmonic for all affine } G\text{-planes } P$$

or

Equivalent Definition (Viscosity Theory). A **test function** for u at $x \in \Omega$ is a C^2-function φ defined near x such that

$$\varphi \ \geq \ u \qquad \text{and} \qquad \varphi(x) = u(x)$$

Def. An **upper semi-continuous** function u on $\Omega^{\text{open}} \subset \mathbb{R}^n$ is **G-psh** if

$$u\big|_P \text{ is subharmonic for all affine } G\text{-planes } P$$

or

Equivalent Definition (Viscosity Theory). A **test function** for u at $x \in \Omega$ is a C^2-function φ defined near x such that

$$\varphi \ \geq \ u \qquad \text{and} \qquad \varphi(x) = u(x)$$

Then u is **G-psh** if for each test function φ at a point $x \in \Omega$

$$D^2\varphi(x) \in F(G).$$

Def. An **upper semi-continuous** function u on $\Omega^{\text{open}} \subset \mathbb{R}^n$ is **G-psh** if

$$u\big|_P \text{ is subharmonic for all affine } G\text{-planes } P$$

or

Equivalent Definition (Viscosity Theory). A **test function** for u at $x \in \Omega$ is a C^2-function φ defined near x such that

$$\varphi \geq u \qquad \text{and} \qquad \varphi(x) = u(x)$$

Then u is **G-psh** if for each test function φ at a point $x \in \Omega$

$$D^2\varphi(x) \in F(G).$$

(If there does not exist a test function at x, this is automatically satisfied.)

Def. An **upper semi-continuous** function u on $\Omega^{\text{open}} \subset \mathbb{R}^n$ is ***G*-psh** if

$$u\big|_P \text{ is subharmonic for all affine } G\text{-planes } P$$

or

Equivalent Definition (Viscosity Theory). A **test function** for u at $x \in \Omega$ is a C^2-function φ defined near x such that

$$\varphi \geq u \qquad \text{and} \qquad \varphi(x) = u(x)$$

Then u is ***G*-psh** if for each test function φ at a point $x \in \Omega$

$$D^2\varphi(x) \in F(G).$$

(If there does not exist a test function at x, this is automatically satisfied.)

($\Leftarrow$ uses our Restriction Theorem)

In fact, the set PSH(G) of G-psh functions on Ω gives us a

"G-POTENTIAL" OR "G-PLURIPOTENTIAL" THEORY

In fact, the set PSH(G) of G-psh functions on Ω gives us a

“G-POTENTIAL” OR “G-PLURIPOTENTIAL” THEORY

PSH(G) has many of the standard properties of subharmonic or plurisubharmonic functions.

In fact, the set PSH(G) of G-psh functions on Ω gives us a

"G-POTENTIAL" OR "G-PLURIPOTENTIAL" THEORY

PSH(G) has many of the standard properties of subharmonic or plurisubharmonic functions.

PSH(G) is a convex cone.

PSH(G) is closed under:

- decreasing limits
- uniform limits
- taking the maximum of two functions
- taking the USC-regularization of the upper envelope of a locally bounded family of functions

A MAIN POINT: Theorems in potential theory or pluripotential theory lead to natural conjectures in G-potential theory

A MAIN POINT: Theorems in potential theory or pluripotential theory lead to natural conjectures in G-potential theory

Some Examples:

1. G-**convex hull** of $K \subset \Omega$ is

$$\widehat{K} \equiv \{x \in \Omega : u(x) \leq \sup_K u \quad \forall\, u \in \mathrm{PSH}(G)\}$$

Def. Ω is G-**convex** if $\quad K \subset\subset \Omega \quad \Rightarrow \quad \widehat{K} \subset\subset \Omega$

Thm. Ω is G-convex $\Rightarrow \exists$ a G-psh exhaustion of Ω.

A MAIN POINT: Theorems in potential theory or pluripotential theory lead to natural conjectures in G-potential theory

Some Examples:

1. G-**convex hull** of $K \subset \Omega$ is

$$\widehat{K} \equiv \{x \in \Omega : u(x) \leq \sup_K u \quad \forall\, u \in \mathrm{PSH}(G)\}$$

Def. Ω is G-**convex** if $\quad K \subset\subset \Omega \quad \Rightarrow \quad \widehat{K} \subset\subset \Omega$

Thm. Ω is G-convex $\Rightarrow$ $\exists$ a G-psh exhaustion of Ω.

2. G-convexity of hypersurfaces

A MAIN POINT: Theorems in potential theory or pluripotential theory lead to natural conjectures in *G*-potential theory

Some Examples:

1. *G*-**convex hull** of $K \subset \Omega$ is

$$\widehat{K} \equiv \{x \in \Omega : u(x) \le \sup_K u \quad \forall\, u \in \mathrm{PSH}(G)\}$$

Def. Ω is *G*-**convex** if $\quad K \subset\subset \Omega \quad \Rightarrow \quad \widehat{K} \subset\subset \Omega$

Thm. Ω is *G*-convex $\Rightarrow$ $\exists$ a *G*-psh exhaustion of Ω.

2. *G*-convexity of hypersurfaces

3. Andreotti-Frankel type theorems

A MAIN POINT: Theorems in potential theory or pluripotential theory lead to natural conjectures in *G*-potential theory

Some Examples:

1. *G*-**convex hull** of $K \subset \Omega$ is

$$\widehat{K} \equiv \{x \in \Omega : u(x) \leq \sup_K u \quad \forall\, u \in \mathrm{PSH}(G)\}$$

Def. Ω is *G*-**convex** if $\quad K \subset\subset \Omega \quad \Rightarrow \quad \widehat{K} \subset\subset \Omega$

Thm. Ω is *G*-convex $\Rightarrow$ $\exists$ a *G*-psh exhaustion of Ω.

2. *G*-convexity of hypersurfaces

3. Andreotti-Frankel type theorems

4. Totally real submanifolds and Grauert's Theorem

A MAIN POINT: Theorems in potential theory or pluripotential theory lead to natural conjectures in *G*-potential theory

Some Examples:

1. *G*-**convex hull** of $K \subset \Omega$ is

$$\widehat{K} \equiv \{x \in \Omega : u(x) \leq \sup_K u \quad \forall\, u \in \mathrm{PSH}(G)\}$$

Def. Ω is *G*-**convex** if $\quad K \subset\subset \Omega \quad \Rightarrow \quad \widehat{K} \subset\subset \Omega$

Thm. Ω is *G*-convex $\Rightarrow$ $\exists$ a *G*-psh exhaustion of Ω.

2. *G*-convexity of hypersurfaces

3. Andreotti-Frankel type theorems

4. Totally real submanifolds and Grauert's Theorem

5. The Dirichlet Problem

$F(G)$ is a **convex cone**, with 0 as vertex, which contains

$$\mathcal{P} \ \equiv \ \{A \geq 0\},$$

$F(G)$ is a **convex cone**, with 0 as vertex, which contains

$$\mathcal{P} \equiv \{A \geq 0\}, \text{ i.e.,}$$

$$\mathcal{P} \subset F(G) \subset \mathrm{Sym}^2(\mathbb{R}^n)$$

$F(G)$ is a **convex cone**, with 0 as vertex, which contains

$$\mathcal{P} \equiv \{A \geq 0\}, \text{ i.e.,}$$

$$\mathcal{P} \subset F(G) \subset \mathrm{Sym}^2(\mathbb{R}^n)$$

MAJOR POINT

$\partial F(G)$ **can be viewed as a differential equation**

We **solved the Dirichlet problem** for this equation on any domain Ω with strictly G-convex boundary for all continuous boundary values $\varphi \in C(\partial\Omega)$

We **solved the Dirichlet problem** for this equation on any domain Ω with strictly G-convex boundary for all continuous boundary values $\varphi \in C(\partial\Omega)$

The solutions are called ***G*-harmonic**.

$\partial\Omega$ is **strictly *G*-convex** $\iff$ $\forall\, G$-planes $P \subset T(\partial\Omega)$, $\operatorname{tr}\left\{\mathrm{II}\big|_P\right\} > 0$

We **solved the Dirichlet problem** for this equation on any domain Ω with strictly G-convex boundary for all continuous boundary values $\varphi \in C(\partial\Omega)$

The solutions are called **G-harmonic**.

$$\partial\Omega \text{ is } \textbf{strictly } G\textbf{-convex} \quad \iff \quad \forall\, G\text{-planes } P \subset T(\partial\Omega),\ \mathrm{tr}\,\left\{\mathrm{II}\big|_P\right\} > 0$$

Interestingly, for Ψ the Associative, Coassociative or Caley calibration, there appears to be **no polynomial function** Q on $\mathrm{Sym}^2(\mathbb{R}^n)$ with

$$\partial F\{G(\Psi)\} \ \subset \mathrm{Zeros}(Q)$$

We **solved the Dirichlet problem** for this equation on any domain Ω with strictly G-convex boundary for all continuous boundary values $\varphi \in C(\partial\Omega)$

The solutions are called ***G*-harmonic**.

$\partial\Omega$ is **strictly *G*-convex** $\iff$ $\forall\, G$-planes $P \subset T(\partial\Omega)$, $\mathrm{tr}\,\left\{\mathrm{II}\big|_P\right\} > 0$

Interestingly, for Ψ the Associative, Coassociative or Caley calibration, there appears to be **no polynomial function** Q on $\mathrm{Sym}^2(\mathbb{R}^n)$ with

$$\partial F\{G(\Psi)\} \subset \mathrm{Zeros}(Q)$$

So there is **no classical differential operator for this equation.**

We realized that this approach was much more general.

Let $\mathbb{G} \subset \mathrm{Grass}_p(\mathbb{R}^n)$ be any closed set.

Then the **whole discussion holds** for $F(\mathbb{G})$

We realized that this approach was much more general.

Let $\mathbb{G} \subset \mathrm{Grass}_p(\mathbb{R}^n)$ be any closed set.

Then the **whole discussion holds** for $F(\mathbb{G})$

($\mathbb{G}$ could be a Cantor-type set, a finite set, etc.)

Important Example. $\mathbb{G} = \mathrm{Grass}_p(\mathbb{R}^n)$

Important Example. $\mathbb{G} = \mathrm{Grass}_p(\mathbb{R}^n)$

u is $\mathbb{G}$-subharmonic $\iff$ $\mathrm{tr}\left\{(D^2 u)_x\big|_P\right\} \geq 0$ $\forall$ p-planes P at all points x

Important Example. $\mathbb{G} = \mathrm{Grass}_p(\mathbb{R}^n)$

u is $\mathbb{G}$-subharmonic $\iff$ $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} \geq 0$ $\forall\, p$-planes P at all points x

u is $\mathbb{G}$-harmonic $\iff$ in addition $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} = 0$ for some P at each x

Important Example. $\mathbb{G} = \mathrm{Grass}_p(\mathbb{R}^n)$

u is $\mathbb{G}$-subharmonic $\iff$ $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} \geq 0$ $\forall\, p$-planes P at all points x

u is $\mathbb{G}$-harmonic $\iff$ in addition $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} = 0$ for some P at each x

$$\iff \quad \prod_{i_1<\cdots<i_p}(\lambda_{i_1} + \cdots + \lambda_{i_p}) \equiv Q(u) = 0$$

Q is the ***p*-fold sum operator**.

Important Example. $\mathbb{G} = \mathrm{Grass}_p(\mathbb{R}^n)$

u is $\mathbb{G}$-subharmonic $\iff$ $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} \geq 0$ $\forall\, p$-planes P at all points x

u is $\mathbb{G}$-harmonic $\iff$ in addition $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} = 0$ for some P at each x

$$\iff \quad \prod_{i_1<\cdots<i_p}(\lambda_{i_1} + \cdots + \lambda_{i_p}) \equiv Q(u) = 0$$

Q is the ***p*-fold sum operator**.

For $p = 1$ this is the Monge-Ampère operator.

Important Example. $\mathbb{G} = \mathrm{Grass}_p(\mathbb{R}^n)$

u is $\mathbb{G}$-subharmonic $\iff$ $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} \geq 0$ $\forall\, p$-planes P at all points x

u is $\mathbb{G}$-harmonic $\iff$ in addition $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} = 0$ for some P at each x

$$\iff \prod_{i_1<\cdots<i_p}(\lambda_{i_1} + \cdots + \lambda_{i_p}) \equiv Q(u) = 0$$

Q is the ***p*-fold sum operator**.

For $p = 1$ this is the Monge-Ampère operator.

The potential theory is very interesting.

Important Example. $\mathbb{G} = \mathrm{Grass}_p(\mathbb{R}^n)$

u is $\mathbb{G}$-subharmonic $\iff$ $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} \geq 0$ $\forall$ p-planes P at all points x

u is $\mathbb{G}$-harmonic $\iff$ in addition $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} = 0$ for some P at each x

$$\iff \prod_{i_1<\cdots<i_p}(\lambda_{i_1} + \cdots + \lambda_{i_p}) \equiv Q(u) = 0$$

Q is the ***p*-fold sum operator**.

For $p = 1$ this is the Monge-Ampère operator.

The potential theory is very interesting.

Note that $\widetilde{Q} \equiv$ the minimal $\lambda_{i_1} + \cdots + \lambda_{i_p}$ (the truncated Laplacian) is an alternative operator for this potential theory.

Important Example. $\mathbb{G} = \mathrm{Grass}_p(\mathbb{R}^n)$

u is $\mathbb{G}$-subharmonic $\iff$ $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} \geq 0$ $\forall$ p-planes P at all points x

u is $\mathbb{G}$-harmonic $\iff$ in addition $\mathrm{tr}\left\{(D^2u)_x\big|_P\right\} = 0$ for some P at each x

$$\iff \prod_{i_1<\cdots<i_p}(\lambda_{i_1} + \cdots + \lambda_{i_p}) \equiv Q(u) = 0$$

Q is the ***p*-fold sum operator**.

For $p = 1$ this is the Monge-Ampère operator.

The potential theory is very interesting.

Note that $\widetilde{Q} \equiv$ the minimal $\lambda_{i_1} + \cdots + \lambda_{i_p}$ (the truncated Laplacian) is an alternative operator for this potential theory.

(Here u is C^2)

Important Example. $\mathbb{G} = LAG \subset \mathrm{Grass}_n(\mathbb{C}^n)$

Important Example. $\mathbb{G} = LAG \subset \mathrm{Grass}_n(\mathbb{C}^n)$

$$u \text{ is } \mathbb{G}\text{-subharmonic} \iff \mathrm{tr}\left\{D^2 u\big|_P\right\} \geq 0 \quad \forall \text{ Lagrangian } P$$

Important Example. $\mathbb{G} = LAG \subset \mathrm{Grass}_n(\mathbb{C}^n)$

$$u \text{ is } \mathbb{G}\text{-subharmonic} \iff \mathrm{tr}\left\{D^2 u\big|_P\right\} \geq 0 \quad \forall \text{ Lagrangian } P$$

$$u \text{ is } \mathbb{G}\text{-harmonic} \iff \text{in addition } \mathrm{tr}\left\{D^2 u\big|_P\right\} = 0 \quad \text{for some } P \text{ at each } x$$

Important Example. $\mathbb{G} = LAG \subset \mathrm{Grass}_n(\mathbb{C}^n)$

$$u \text{ is } \mathbb{G}\text{-subharmonic} \iff \mathrm{tr}\left\{D^2 u\big|_P\right\} \geq 0 \quad \forall \text{ Lagrangian } P$$

$$u \text{ is } \mathbb{G}\text{-harmonic} \iff \text{in addition } \mathrm{tr}\left\{D^2 u\big|_P\right\} = 0 \quad \text{for some } P \text{ at each } x$$

$\iff$ A **Lagrangian Monge-Ampère operator vanishes.**

(u is C^2.)

Subequations and Duality Theory

Definition. A **subequation** is a closed subset $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ s.t.

$$F + \mathcal{P} \subset F.$$

Subequations and Duality Theory

Definition. A **subequation** is a closed subset $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ s.t.

$$F + \mathcal{P} \subset F.$$

Definition. The **dual subequation** is

$$\widetilde{F} = -(\sim \mathrm{Int} F) = \sim (-\mathrm{Int} F)$$

Subequations and Duality Theory

Definition. A **subequation** is a closed subset $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ s.t.

$$F + \mathcal{P} \subset F.$$

Definition. The **dual subequation** is

$$\widetilde{F} = -(\sim \mathrm{Int}F) = \sim (-\mathrm{Int}F)$$

$$\widetilde{\widetilde{F}} = F$$

Subequations and Duality Theory

Definition. A **subequation** is a closed subset $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ s.t.

$$F + \mathcal{P} \subset F.$$

Definition. The **dual subequation** is

$$\widetilde{F} = -(\sim \mathrm{Int} F) = \sim (-\mathrm{Int} F)$$

$$\widetilde{\widetilde{F}} = F$$

$$F + \mathcal{P} \subset F \quad \iff \quad \widetilde{F} + \mathcal{P} \subset \widetilde{F}$$

Subequations and Duality Theory

Definition. A **subequation** is a closed subset $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ s.t.

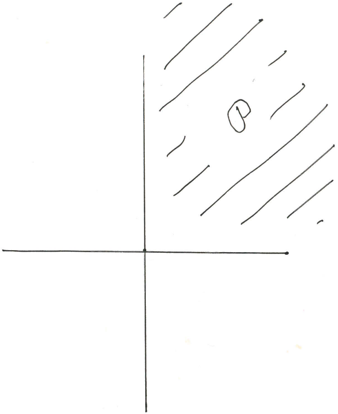
$$F + \mathcal{P} \subset F.$$

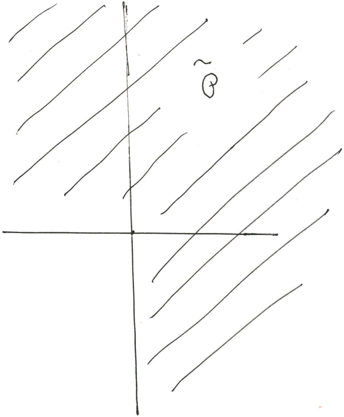
Definition. The **dual subequation** is

$$\widetilde{F} = -(\sim \mathrm{Int} F) = \sim (-\mathrm{Int} F)$$

$$\widetilde{\widetilde{F}} = F$$

$$F + \mathcal{P} \subset F \quad \iff \quad \widetilde{F} + \mathcal{P} \subset \widetilde{F}$$


P
P̃



Definition. An u.s.c. function u is **F-subharmonic**

if for every test function φ at a point x for u

$$(D^2\varphi)_x \in F$$

Definition. An u.s.c. function u is **F-subharmonic**

if for every test function φ at a point x for u

$$(D^2\varphi)_x \in F$$

Definition. An function u is **F-harmonic** if

$$u \text{ is } F\text{-subharmonic} \qquad \text{and} \qquad -u \text{ is } \widetilde{F}\text{-subharmonic.}$$

Definition. An u.s.c. function u is ***F*-subharmonic**

if for every test function φ at a point x for u

$$(D^2\varphi)_x \in F$$

Definition. An function u is ***F*-harmonic** if

u is F-subharmonic and $-u$ is $\widetilde{F}$-subharmonic.

$$u \text{ is } C^2 \text{ and } F\text{-harmonic} \quad \Rightarrow \quad (D^2 u)_x \in \partial F \quad \forall\, x$$

Note: F-subharmonics give a Potential Theory

Note: F-subharmonics give a Potential Theory

Note: There may be many operators Q with $\partial F \subset \{Q = 0\}$.

$$u \text{ is } F\text{-subharmonic} \quad \iff \quad u \text{ is a subsolution}$$

$$-u \text{ is } \widetilde{F}\text{-subharmonic} \quad \iff \quad u \text{ is a supersolution}$$

$$u \text{ is } F\text{-subharmonic} \quad \iff \quad u \text{ is a subsolution}$$

$$-u \text{ is } \widetilde{F}\text{-subharmonic} \quad \iff \quad u \text{ is a supersolution}$$

Definition. $\overrightarrow{F}$ = all rays from 0 which are eventually in F.

$$u \text{ is } F\text{-subharmonic} \quad \iff \quad u \text{ is a subsolution}$$

$$-u \text{ is } \widetilde{F}\text{-subharmonic} \quad \iff \quad u \text{ is a supersolution}$$

Definition. $\overrightarrow{F}$ = all rays from 0 which are eventually in F.

Definition. $\partial\Omega$ is **strictly** $\overrightarrow{F}$ **convex** if at each $x \in \partial\Omega$

$$\mathrm{II}_x = B\big|_{T_x(\partial\Omega)} \text{ for some } B \in \mathrm{Int}\,\overrightarrow{F}.$$

u is F-subharmonic $\iff$ u is a subsolution

$-u$ is $\widetilde{F}$-subharmonic $\iff$ u is a supersolution

Definition. $\overrightarrow{F}$ = all rays from 0 which are eventually in F.

Definition. $\partial\Omega$ is **strictly** $\overrightarrow{F}$ **convex** if at each $x \in \partial\Omega$
$$\mathrm{II}_x = B\big|_{T_x(\partial\Omega)} \text{ for some } B \in \mathrm{Int}\,\overrightarrow{F}.$$

THEOREM. (The Dirichlet Problem) Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$ which is both $\overrightarrow{F}$ and $\overrightarrow{\widetilde{F}}$ strictly convex.

u is F-subharmonic $\iff$ u is a subsolution

$-u$ is $\widetilde{F}$-subharmonic $\iff$ u is a supersolution

Definition. $\overrightarrow{F}$ = all rays from 0 which are eventually in F.

Definition. $\partial\Omega$ is **strictly** $\overrightarrow{F}$ **convex** if at each $x \in \partial\Omega$

$$\mathrm{II}_x = B\big|_{T_x(\partial\Omega)} \text{ for some } B \in \mathrm{Int}\,\overrightarrow{F}.$$

THEOREM. (The Dirichlet Problem) Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$ which is both $\overrightarrow{F}$ and $\overrightarrow{\widetilde{F}}$ strictly convex.

Then for each $\varphi \in C(\partial\Omega)$

there **exists** a **unique** F-harmonic function u on Ω,

with a continuous extension to $\overline{\Omega}$ satisfying

$$u\big|_{\partial\Omega} = \varphi.$$

Examples: For the following operators Q

$$\mathrm{Int}\, F \;=\; \text{the connected component containing } I \text{ of } \{A : Q(A) > 0\},$$
$$\text{or the subset of this where } Q(A) \geq c > 0$$

Monge-Ampère equations over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Examples: For the following operators Q

$$\mathrm{Int}\, F \;=\; \text{the connected component containing } I \text{ of } \{A : Q(A) > 0\},$$
$$\text{or the subset of this where } Q(A) \geq c > 0$$

Monge-Ampère equations over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Any $\sigma_k(D^2 u)$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Examples: For the following operators Q

$$\mathrm{Int}\, F = \text{the connected component containing } I \text{ of } \{A : Q(A) > 0\},$$
$$\text{or the subset of this where } Q(A) \geq c > 0$$

Monge-Ampère equations over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Any $\sigma_k(D^2 u)$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The examples above from p-convex geometry and Lagrangian geometry.

Examples: For the following operators Q

$$\mathrm{Int}\, F \ = \ \text{the connected component containing } I \text{ of } \{A : Q(A) > 0\},$$
$$\text{or the subset of this where } Q(A) \geq c > 0$$

Monge-Ampère equations over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Any $\sigma_k(D^2 u)$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The examples above from p-convex geometry and Lagrangian geometry.

In fact, any Gårding-Dirichlet polynomial operator $\mathfrak{g}(D^2 u)$.

Examples: For the following operators Q

$$\mathrm{Int}\, F \;=\; \text{the connected component containing } I \text{ of } \{A : Q(A) > 0\},$$
$$\text{or the subset of this where } Q(A) \geq c > 0$$

Monge-Ampère equations over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Any $\sigma_k(D^2u)$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The examples above from p-convex geometry and Lagrangian geometry.

In fact, any Gårding-Dirichlet polynomial operator $\mathfrak{g}(D^2u)$.

Branches of the homogeneous equations above.

Examples: For the following operators Q

$$\mathrm{Int}\, F \;=\; \text{the connected component containing } I \text{ of } \{A : Q(A) > 0\},$$
$$\text{or the subset of this where } Q(A) \geq c > 0$$

Monge-Ampère equations over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Any $\sigma_k(D^2u)$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The examples above from p-convex geometry and Lagrangian geometry.

In fact, any Gårding-Dirichlet polynomial operator $\mathfrak{g}(D^2u)$.

Branches of the homogeneous equations above.

For example, for Monge-Ampère,

$$\lambda_k = 0$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the ev's of the Hessian

Examples: For the following operators Q

$$\mathrm{Int}\, F \;=\; \text{the connected component containing } I \text{ of } \{A : Q(A) > 0\},$$
$$\text{or the subset of this where } Q(A) \geq c > 0$$

Monge-Ampère equations over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Any $\sigma_k(D^2u)$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The examples above from p-convex geometry and Lagrangian geometry.

In fact, any Gårding-Dirichlet polynomial operator $\mathfrak{g}(D^2u)$.

Branches of the homogeneous equations above.

For example, for Monge-Ampère,

$$\lambda_k = 0$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the ev's of the Hessian

The special Lagrangian potential equation with constant RHS.

Examples: For the following operators Q

$$\mathrm{Int}\, F \;=\; \text{the connected component containing } I \text{ of } \{A : Q(A) > 0\},$$
$$\text{or the subset of this where } Q(A) \geq c > 0$$

Monge-Ampère equations over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Any $\sigma_k(D^2 u)$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The examples above from p-convex geometry and Lagrangian geometry.

In fact, any Gårding-Dirichlet polynomial operator $\mathfrak{g}(D^2 u)$.

Branches of the homogeneous equations above.

For example, for Monge-Ampère,

$$\lambda_k = 0$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the ev's of the Hessian

The special Lagrangian potential equation with constant RHS.

functions of the principal curvatures of the graph

Examples: For the following operators Q

$$\mathrm{Int}\, F \ = \ \text{the connected component containing } I \text{ of } \{A : Q(A) > 0\},$$
$$\text{or the subset of this where } Q(A) \geq c > 0$$

Monge-Ampère equations over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Any $\sigma_k(D^2 u)$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The examples above from p-convex geometry and Lagrangian geometry.

In fact, any Gårding-Dirichlet polynomial operator $\mathfrak{g}(D^2 u)$.

Branches of the homogeneous equations above.

For example, for Monge-Ampère,

$$\lambda_k = 0$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the ev's of the Hessian

The special Lagrangian potential equation with constant RHS.

functions of the principal curvatures of the graph

Many many equations with no smooth operator

Incidentally, **For the first four classes of operators mentioned above, there was a remarkable paper by**

Bin Guo, Duong H. Phong, and Freid Tong

Annals of Math. **198** (2023), 393-418

which established L^∞-estimates for these operators in complex geometry.

Incidentally, **For the first four classes of operators mentioned above, there was a remarkable paper by**

Bin Guo, Duong H. Phong, and Freid Tong

Annals of Math. **198** (2023), 393-418

which established L^∞-estimates for these operators in complex geometry.

It was based on a fundamental new idea of Chen and Cheng in their ground-breaking work

Xiuxong Chen and Jungrui Cheng,

J. Amer. Math. Soc. **34** no.4 (2021), 909-1009

Incidentally, **For the first four classes of operators mentioned above, there was a remarkable paper by**

Bin Guo, Duong H. Phong, and Freid Tong

Annals of Math. **198** (2023), 393-418

which established L^∞-estimates for these operators in complex geometry.

It was based on a fundamental new idea of Chen and Cheng in their ground-breaking work

Xiuxong Chen and Jungrui Cheng,

J. Amer. Math. Soc. **34** no.4 (2021), 909-1009

[GPT] gave a purely PDE proof of the Yau's Theorem on the Calabi Conjecture.

Incidentally, **For the first four classes of operators mentioned above, there was a remarkable paper by**

Bin Guo, Duong H. Phong, and Freid Tong

Annals of Math. **198** (2023), 393-418

which established L^∞-estimates for these operators in complex geometry.

It was based on a fundamental new idea of Chen and Cheng in their ground-breaking work

Xiuxong Chen and Jungrui Cheng,

J. Amer. Math. Soc. **34** no.4 (2021), 909-1009

[GPT] gave a purely PDE proof of the Yau's Theorem on the Calabi Conjecture.

It has also engendered very much research recently

not a little part of it by Phong in collaboration with others, particularly Bin Guo.

A Much More General Theory

A Much More General Theory

X a manifold

$J^2(X) \longrightarrow X \;=\;$ the bundle of 2-jets of functions

A Much More General Theory

X a manifold

$$J^2(X) \longrightarrow X \;=\; \text{the bundle of 2-jets of functions}$$
$$= \text{the bundle of Taylor polynomials of degree 2}$$

A Much More General Theory

X a manifold

$$J^2(X) \longrightarrow X \;=\; \text{the bundle of 2-jets of functions}$$
$$= \text{the bundle of Taylor polynomials of degree 2}$$
$$= \mathbb{R} \oplus J^2_{\mathrm{red}}(X)$$

A Much More General Theory

X a manifold

$$
\begin{aligned}
J^2(X) \longrightarrow X \;&=\; \text{the bundle of 2-jets of functions} \\
&= \text{the bundle of Taylor polynomials of degree 2} \\
&= \mathbb{R} \oplus J^2_{\mathrm{red}}(X) \\
&= \mathbb{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X) \quad \text{if riemannian}
\end{aligned}
$$

A Much More General Theory

X a manifold

$$
\begin{aligned}
J^2(X) \longrightarrow X \;&=\; \text{the bundle of 2-jets of functions} \\
&= \text{the bundle of Taylor polynomials of degree 2} \\
&= \mathbb{R} \oplus J^2_{\mathrm{red}}(X) \\
&= \mathbb{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X) \quad \text{if riemannian}
\end{aligned}
$$

In local coordinate neighborhood U

$$J^2(U) \;=\; \mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n)$$

A Much More General Theory

X a manifold

$$
\begin{aligned}
J^2(X) \longrightarrow X \;&=\; \text{the bundle of 2-jets of functions} \\
&= \text{the bundle of Taylor polynomials of degree 2} \\
&= \mathbb{R} \oplus J^2_{\mathrm{red}}(X) \\
&= \mathbb{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X) \quad \text{if riemannian}
\end{aligned}
$$

In local coordinate neighborhood U

$$J^2(U) \;=\; \mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n)$$

$$J^2\varphi(x) \;=\; (\varphi(x), D\varphi(x), D^2\varphi(x))$$

Definition. A **subequation** is a closed subset

$$F \subset J^2(X)$$

s.t.

$$(1) \quad F + \mathcal{P} \subset F$$

where $\mathcal{P} \equiv$ 2-jets of functions with minimum value zero at x

Definition. A **subequation** is a closed subset

$$F \subset J^2(X)$$

s.t.

$$(1) \quad F + \mathcal{P} \subset F$$

where $\mathcal{P} \equiv$ 2-jets of functions with minimum value zero at x

$$(2) \quad F + \mathcal{N} \subset F$$

where $\mathcal{N} \equiv$ 2-jets of $\varphi \equiv$ constant ≤ 0

Definition. A **subequation** is a closed subset

$$F \subset J^2(X)$$

s.t.

$$(1) \quad F + \mathcal{P} \subset F$$

where $\mathcal{P} \equiv$ 2-jets of functions with minimum value zero at x

$$(2) \quad F + \mathcal{N} \subset F$$

where $\mathcal{N} \equiv$ 2-jets of $\varphi \equiv$ constant ≤ 0

Plus mild topological conditions
which hold in all interesting cases.

As before

$$\widetilde{F} \ = \ -(\sim \mathrm{Int} F)$$

As before

$$\widetilde{F} \ = \ -(\sim \mathrm{Int} F)$$

Definition. $u \in USC(X)$ is **F-subharmonic** if

$$\forall \text{ test functions } \varphi \text{ at a point } x, \ J^2_x(\varphi) \in F_x$$

As before

$$\widetilde{F} = -(\sim \mathrm{Int} F)$$

Definition. $u \in USC(X)$ is **F-subharmonic** if

$$\forall \text{ test functions } \varphi \text{ at a point } x, \; J_x^2(\varphi) \in F_x$$

Definition. u is **F-harmonic** if

$$u \text{ is } F\text{-subharmonic} \quad \text{and} \quad -u \text{ is } \widetilde{F}\text{-subharmonic}$$

As before

$$\widetilde{F} \;=\; -(\sim \mathrm{Int} F)$$

Definition. $u \in USC(X)$ is **F-subharmonic** if

$$\forall \text{ test functions } \varphi \text{ at a point } x,\ J^2_x(\varphi) \in F_x$$

Definition. u is **F-harmonic** if

$$u \text{ is } F\text{-subharmonic} \qquad \text{and} \qquad -u \text{ is } \widetilde{F}\text{-subharmonic}$$

Again there is an F potential theory.

THE MONOTONICITY METHOD

THE MONOTONICITY METHOD

A **Monotonicity Cone** is a convex cone subequation $M \subset J^2(X)$ s.t.

$$F + M \ \subset \ F$$

THE MONOTONICITY METHOD

A **Monotonicity Cone** is a convex cone subequation $M \subset J^2(X)$ s.t.

$$F + M \;\subset\; F$$

which is equivalent to

$$F + \widetilde{F} \;\subset\; \widetilde{M}$$

THE MONOTONICITY METHOD

A **Monotonicity Cone** is a convex cone subequation $M \subset J^2(X)$ s.t.

$$F + M \;\subset\; F$$

which is equivalent to

$$F + \widetilde{F} \;\subset\; \widetilde{M}$$

This is involved in establishing **comparison**.

THE MONOTONICITY METHOD

A **Monotonicity Cone** is a convex cone subequation $M \subset J^2(X)$ s.t.

$$F + M \subset F$$

which is equivalent to

$$F + \widetilde{F} \subset \widetilde{M}$$

This is involved in establishing **comparison**.

If u is F-sub and $-v$ is $\widetilde{F}$-sub, the function $u - v$ satisfies the the Zero Maximum Principle, i.e.,

$$u - v \leq 0 \text{ on } \partial\Omega \quad \Rightarrow \quad u - v \leq 0 \text{ on } \Omega.$$

To prove the **Dirichlet Problem** for a domain $\Omega \subset X$ on a manifold we require **the existence of a global strictly M-subharmonic C^2-function** on $\overline{\Omega}$.

To prove the **Dirichlet Problem** for a domain $\Omega \subset X$ on a manifold we require **the existence of a global strictly M-subharmonic C^2-function** on $\overline{\Omega}$.

It is clear that one need something of this sort to give control over Ω. Just giving the boundary values and the strict boundary convexity is not enough!

To prove the **Dirichlet Problem** for a domain $\Omega \subset X$ on a manifold we require **the existence of a global strictly M-subharmonic C^2-function** on $\overline{\Omega}$.

It is clear that one need something of this sort to give control over Ω. Just giving the boundary values and the strict boundary convexity is not enough!

This is the only global assumption. It provides an important approximation.

To prove the **Dirichlet Problem** for a domain $\Omega \subset X$ on a manifold we require **the existence of a global strictly M-subharmonic C^2-function** on $\overline{\Omega}$.

It is clear that one need something of this sort to give control over Ω. Just giving the boundary values and the strict boundary convexity is not enough!

This is the only global assumption. It provides an important approximation.

For pure second order subequations $\mathcal{P}$ (also $\mathcal{P}_{\mathbb{C}}$, $\mathcal{P}_{\mathbb{H}}$ in those cases)
is always a monotonicity cone.

To prove the **Dirichlet Problem** for a domain $\Omega \subset X$ on a manifold we require **the existence of a global strictly M-subharmonic C^2-function** on $\overline{\Omega}$.

It is clear that one need something of this sort to give control over Ω. Just giving the boundary values and the strict boundary convexity is not enough!

This is the only global assumption. It provides an important approximation.

For pure second order subequations $\mathcal{P}$ (also $\mathcal{P}_{\mathbb{C}}$, $\mathcal{P}_{\mathbb{H}}$ in those cases) is always a monotonicity cone.

In the classical complex case, this plus strict boundary convexity, implies that Ω is Stein.

Any O(n) invariant subequation

$$F_0 \subset J_0^2(\mathbb{R}^n) = \mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n)$$

gives a subequation on **every riemannian manifold.**

Any O(n) invariant subequation

$$F_0 \subset J_0^2(\mathbb{R}^n) = \mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n)$$

gives a subequation on **every riemannian manifold.**

One of Many Results

THEOREM. Let F be such a subequation. Suppose there exists a strictly M-subharmonic function on X where M is a monotonicity cone for F.

Then for every domain $\Omega \subset\subset X$ whose boundary is strictly F and $\widetilde{F}$ convex, both existence and uniqueness for the Dirichlet Problem for F-harmonics holds for all $\varphi \in C(\partial\Omega)$.

Suppose $F_0 \subset J_0^2(\mathbb{R}^n)$ is a G-invariant subeqn for a **compact Lie group** G.

Suppose $F_0 \subset J_0^2(\mathbb{R}^n)$ is a G-invariant subeqn for a **compact Lie group** G.

Then F_0 defines a subequation on every manifold with a **topological** reduction of it structure group to G

(Cover by tangent frames with transition functions valued in G.)

Suppose $F_0 \subset J_0^2(\mathbb{R}^n)$ is a G-invariant subeqn for a **compact Lie group** G.

Then F_0 defines a subequation on every manifold with a **topological** reduction of it structure group to G

(Cover by tangent frames with transition functions valued in G.)

The Same Theorem holds in this case.

We were also able to prove the **inhomogeneous Dirichlet Problem**

for the complex Monge-Ampère equation

on **almost** complex manifolds

with **RHS** ≥ 0.

FOR THE PROOF

Recall that we want **comparison**.

If u is F-sub and $-v$ is $\widetilde{F}$-sub, the function $u - v$ satisfies the the Zero Maximum Principle, i.e.,

$$u - v \leq 0 \ \text{ on } \ \partial\Omega \quad \Rightarrow \quad u - v \leq 0 \ \text{ on } \ \Omega.$$

FOR THE PROOF

Recall that we want **comparison**.

If u is F-sub and $-v$ is $\widetilde{F}$-sub, the function $u - v$ satisfies the the Zero Maximum Principle, i.e.,

$$u - v \leq 0 \ \text{ on } \ \partial\Omega \quad \Rightarrow \quad u - v \leq 0 \ \text{ on } \ \Omega.$$

Assume that the monotonicity assumption holds

FOR THE PROOF

Recall that we want **comparison**.

If u is F-sub and $-v$ is $\widetilde{F}$-sub, the function $u - v$ satisfies the the Zero Maximum Principle, i.e.,

$$u - v \leq 0 \ \text{ on } \ \partial\Omega \quad \Rightarrow \quad u - v \leq 0 \ \text{ on } \ \Omega.$$

Assume that the monotonicity assumption holds

Then we found the hypothesis of **weak comparison:**

u is F_c-sub for $c > 0$ and $-v$ is $\widetilde{F}$-sub, where

$$F_c \ \equiv \ \{A \in F : \mathrm{dist}(A, \partial F) \geq c\}.$$

(or the opposite)

FOR THE PROOF

Recall that we want **comparison**.

If u is F-sub and $-v$ is $\widetilde{F}$-sub, the function $u - v$ satisfies the the Zero Maximum Principle, i.e.,

$$u - v \leq 0 \ \text{ on } \ \partial\Omega \quad \Rightarrow \quad u - v \leq 0 \ \text{ on } \ \Omega.$$

Assume that the monotonicity assumption holds

Then we found the hypothesis of **weak comparison:**

u is F_c-sub for $c > 0$ and $-v$ is $\widetilde{F}$-sub, where

$$F_c \ \equiv \ \{A \in F : \mathrm{dist}(A, \partial F) \geq c\}.$$

(or the opposite) which **implies comparison**.

FOR THE PROOF

Recall that we want **comparison**.

If u is F-sub and $-v$ is $\widetilde{F}$-sub, the function $u - v$ satisfies the the Zero Maximum Principle, i.e.,

$$u - v \leq 0 \ \text{ on } \ \partial\Omega \quad \Rightarrow \quad u - v \leq 0 \ \text{ on } \ \Omega.$$

Assume that the monotonicity assumption holds

Then we found the hypothesis of **weak comparison:**

u is F_c-sub for $c > 0$ and $-v$ is $\widetilde{F}$-sub, where

$$F_c \ \equiv \ \{A \in F : \mathrm{dist}(A, \partial F) \geq c\}.$$

(or the opposite) which **implies comparison**.

This hypothsis has the property that

LOCAL $\Rightarrow$ **GLOBAL**

So

The monotonicity assumption and local weak comparison

$\Rightarrow$ **comparison**

An Important Technique

AFFINE JET EQUIVALENCE

Given local coordinates $(x_1, ..., x_n)$ on $U^{\text{open}} \subset \mathbb{R}^n$

$$J^2(U) \;=\; U \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n)$$

$$J^2_x(u) \;=\; (u(x), D_x u, D^2_x u)$$

Given local coordinates $(x_1, ..., x_n)$ on $U^{\text{open}} \subset \mathbb{R}^n$

$$J^2(U) \;=\; U \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n)$$

$$J^2_x(u) \;=\; (u(x), D_x u, D^2_x u)$$

An **AUTOMORPHISM**

$$\Phi : J^2(U) \longrightarrow J^2(U)$$

is given by

$$\boxed{\Phi(u, Du, D^2u) \;=\; (u,\; g\,Du,\; h\,(D^2u)\,h^t + L(Du))}$$

$$g_x, h_x \in \mathrm{GL}(n) \qquad \text{and} \qquad L_x : \mathbb{R}^n \longrightarrow \mathrm{Sym}^2(\mathbb{R}^n)$$

- There is an invariant definition which works globally

- There is an invariant definition which works globally
- These form a group

- There is an invariant definition which works globally
- These form a group
- They are **radical** – they are much more than a change of coordinates.
  If u is function, $\Phi(J^2 u)$ is essentially never the 2-jet of a function.

- There is an invariant definition which works globally
- These form a group
- They are **radical** – they are much more than a change of coordinates.
  If u is function, $\Phi(J^2 u)$ is essentially never the 2-jet of a function.
- Covers the expression of Hess(u) in local coordinates

- There is an invariant definition which works globally
- These form a group
- They are **radical** – they are much more than a change of coordinates.
  If u is function, $\Phi(J^2 u)$ is essentially never the 2-jet of a function.
- Covers the expression of Hess(u) in local coordinates

An **AFFINE AUTOMORPHISM**

$$\Psi = \Phi + \sigma$$

Φ an automorphism and $\sigma \in \Gamma(J^2(U))$

- There is an invariant definition which works globally
- These form a group
- They are **radical** – they are much more than a change of coordinates.
  If u is function, $\Phi(J^2 u)$ is essentially never the 2-jet of a function.
- Covers the expression of Hess(u) in local coordinates

An **AFFINE AUTOMORPHISM**

$$\Psi \ = \ \Phi + \sigma$$

$$\Phi \text{ an automorphism} \qquad \text{and} \qquad \sigma \in \Gamma(J^2(U))$$

- This can convert homogeneous equations into inhomogeneous equations.

THEOREM. Suppose F is a subequation on X which is **locally** affinely jet equivalent to a constant coefficient subequation. Then weak comparison holds for F.

THEOREM. Suppose F is a subequation on X which is **locally** affinely jet equivalent to a constant coefficient subequation. Then weak comparison holds for F.

Weak comparison

and the existence of a strict M subharmonic function

$\Rightarrow$ **Comparison holds**

THEOREM. Suppose F is a subequation on X which is **locally** affinely jet equivalent to a constant coefficient subequation. Then weak comparison holds for F.

Weak comparison

and the existence of a strict M subharmonic function

$\Rightarrow$ **Comparison holds**

$\Rightarrow$ **Uniqueness of solutions to the Dirichlet problem**

THEOREM. Suppose F is a subequation on X which is **locally** affinely jet equivalent to a constant coefficient subequation. Then weak comparison holds for F.

Weak comparison

and the existence of a strict M subharmonic function

$\Rightarrow$ **Comparison holds**

$\Rightarrow$ **Uniqueness of solutions to the Dirichlet problem**

and, with boundary convexity, solutions to the Dirichlet problem exist.