Columbia Lectures on the stability of Kerr

Sergiu Klainerman

October 25, 2023

Abstract. The main goals of these lectures are:

1. Provide a comprehensive introduction to the proof of the nonlinear stability of slowly rotating Kerr black holes established recently in the sequence of works [K-S:Kerr], [GKS-2022], [K-S:GCM1], [K-S:GCM2] and [Shen], and briefed in [K-S:review]

2. Discuss the geometric formalism based on non-integrable null horizontal structures used in these works. Derive the main Teukolsky and generalized Regge- Wheeler equations. These follow the material 1 of Part 1 in [GKS-2022].

3. Discuss the proof of the basic hyperbolic estimates, Morawetz and r^p-weighted, following Part 2 of [GKS-2022].

4. Discuss open problems related to these topics.

Contents

Part I

Introduction and Geometric Formalism

Chapter 1

General Introduction

This a brief introduction to the sequence of works [K-S:Kerr], [GKS-2022], [K-S:GCM1], [K-S:GCM2] and [Shen] which establish the nonlinear stability of Kerr black holes with small angular momentum. This chapter is essentially the review paper [K-S:review] with a few additions.

1.1 Kerr stability conjecture

1.1.1 Kerr spacetime

Let $(\mathcal{K}(a,m), \mathbf{g}_{a,m})$ denote the family of Kerr spacetimes depending on the parameters m (mass) and a (with $J = am$ angular momentum). In Boyer-Lindquist coordinates the Kerr metric is given by

$$\mathbf{g}_{a,m} = -\frac{q^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{|q|^2}{\Delta}(dr)^2 + |q|^2(d\theta)^2, \tag{1.1.1}$$

where

$$\begin{cases} \Delta = r^2 + a^2 - 2mr, \qquad q = r + ia\cos\theta, \\ \Sigma^2 = (r^2+a^2)|q|^2 + 2mra^2(\sin\theta)^2 = (r^2+a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases} \tag{1.1.2}$$

The asymptotically flat[1] metrics $\mathbf{g}_{a,m}$ verify the Einstein vacuum equations (EVE)

$$\mathbf{Ric}(\mathbf{g}) = 0, \tag{1.1.3}$$

[1] That is they approach the Minkowski metric for large r.

are stationary and axially symmetric[2], possess well-defined event horizon $r = r_+$ (the largest root of $\Delta(r) = 0$), domain of outer communication $r > r_+$ and smooth future null infinity $\mathcal{I}^+$ where $r = +\infty$. The metric can be extended smoothly inside the black hole region, see Figure 5.1. The boundary $r = r_-$ (the smallest root of $\Delta(r) = 0$) inside the black hole region is a Cauchy horizon across which predictability fails[3].

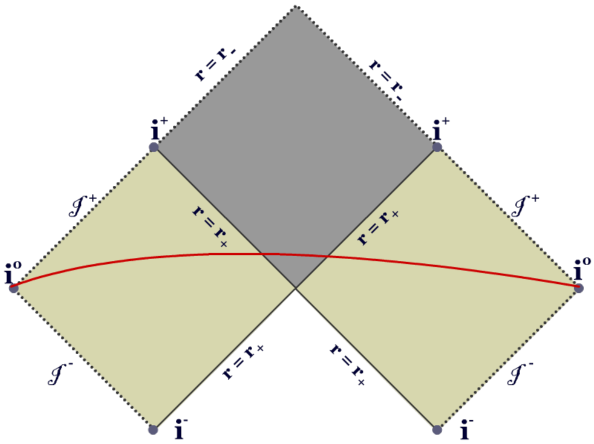


Figure 1.1: Penrose diagram of Kerr for $0 < |a| < m$. The surface $r = r_+$, the larger root of $\Delta = 0$, is the event horizon of the black hole, $r > r_+$ the domain of outer communication, $\mathcal{I}^+$ is the future null infinity, corresponding to $r = +\infty$.

Here are some of the most important properties of $\mathcal{K}(a, m)$:

- $\mathcal{K}(a, m)$ possesses a canonical family of null pairs, called *principal null pairs*, of the form $(\lambda e_4, \lambda^{-1} e_3)$, with $\lambda > 0$ an arbitrary scalar function, and

$$e_4 = \frac{r^2 + a^2}{|q|^2}\partial_t + \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi, \qquad e_3 = \frac{r^2 + a^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\phi. \tag{1.1.4}$$

- The horizontal structure, perpendicular to e_3, e_4, denoted $\mathcal{H}$, is spanned by the vectors

$$e_1 = \frac{1}{|q|}\partial_\theta, \qquad e_2 = \frac{a \sin\theta}{|q|}\partial_t + \frac{1}{|q| \sin\theta}\partial_\phi. \tag{1.1.5}$$

  The distribution generated by $\mathcal{H}$ is non-integrable for $a \neq 0$.

[2]That is $\mathcal{K}(a, m)$ possess two Killing vectorfields: the stationary vectorfield $\mathbf{T} = \partial_t$, which is time-like in the asymptotic region, away from the horizon, and the axial symmetric Killing field $\mathbf{Z} = \partial_\phi$.

[3]Infinitely many smooth extensions are possible beyond the boundary.

- The horizontal structure $(e_3, e_4, \mathcal{H})$ has the remarkable property that all components of the Riemann curvature tensor $\mathbf{R}$, decomposed relative to them, vanish with the exception of those which can be deduced from[4] $\mathbf{R}(e_a, e_3, e_b, e_4)$.

- $\mathcal{K}(a, m)$ possesses the Killing vectorfields $\mathbf{T}, \mathbf{Z}$ which, in BL coordinates, are given by $\mathbf{T} = \partial_t, \mathbf{Z} = \partial_\phi$.

- In addition to the symmetries generated by $\mathbf{T}, \mathbf{Z}$, $\mathcal{K}(a, m)$ possesses also a non-trivial Killing tensor[5], i.e. a symmetric 2-tensor $\mathbf{C}_{\alpha\beta}$ verifying the property $\mathbf{D}_{(\gamma}\mathbf{C}_{\alpha\beta)} = 0$. The tensor carries the name of its discoverer B. Carter, see [Carter], who made use of it to show that the geodesic flow in Kerr is integrable. Its presence, in addition to $\mathbf{T}$ and $\mathbf{Z}$, as a higher order symmetry, is at the heart of what Chandrasekhar, see [Chand3], called the most striking feature of Kerr, "the separability of all the standard equations of mathematical physics in Kerr geometry".

- The Carter tensor can be used to define the Carter operator

$$\mathcal{C} = \mathbf{D}_\alpha(\mathbf{C}^{\alpha\beta}\mathbf{D}_\beta), \tag{1.1.6}$$

  a second order operator which commutes with $\square_{a,m}$. This property plays a crucial role in the proof of our stability result, Theorem 1.1.1, more precisely in Part II of [GKS-2022].

1.1.2 Kerr stability conjecture

The discovery of black holes, first as explicit solutions of EVE and later as possible explanations of astrophysical phenomena[6], has not only revolutionized our understanding of the universe, it also gave mathematicians a monumental task: to test the physical reality of these solutions. This may seem nonsensical since physics tests the reality of its objects by experiments and observations and, as such, needs mathematics to formulate the

[4]One can in fact complexify the curvature tensor by setting $\mathbf{C} = \mathcal{R} + i\,{}^*\mathcal{R}$ so that ${}^*\mathbf{C} = -i\mathbf{C}$. All null components of $\mathbf{C}$ vanish except $\mathbf{C}(e_3, e_4, e_3, e_4) = -\frac{2m}{\rho^3}$.

[5]Given by the expression $\mathbf{C} = -a^2\cos^2\theta\mathbf{g} + O$, $O = |q|^2\big(e_1 \otimes e_1 + e_2 \otimes e_2\big)$.

[6]According to Chandrasekhar "Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent that they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described exactly by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single two parameter family of solutions for their description, they are the simplest as well."

theory and make quantitative predictions, not to test it. The problem, in this case, is that black holes are by definition non-observable and thus no direct experiments are possible. Astrophysicists ascertain the presence of such objects through indirect observations[7] and numerical experiments, but both are limited in scope to the range of possible observations or the specific initial conditions in which numerical simulations are conducted. One can rigorously check that the Kerr solutions have vanishing Ricci curvature, that is, their mathematical reality is undeniable. But to be real in a physical sense, they have to satisfy certain properties which, as it turns out, can be neatly formulated in unambiguous mathematical language. Chief among them[8] is the problem of stability, that is, to show that if the precise initial data corresponding to Kerr are perturbed a bit, the basic features of the corresponding solutions do not change much[9].

Conjecture (Stability of Kerr conjecture). *Vacuum, asymptotically flat, initial data sets, sufficiently close to $Kerr(a, m)$, $|a|/m < 1$, initial data, have maximal developments with complete future null infinity and with domain of outer communication*[10] *which approaches (globally) a nearby Kerr solution.*

1.1.3 Resolution of the conjecture for slowly rotating black holes

Statement of the main result

The goal of this article is to give a short introduction to our recent result in which we settle the conjecture in the case of slowly rotating Kerr black holes.

Theorem 1.1.1. *The future globally hyperbolic development of a general, asymptotically flat, initial data set, sufficiently close (in a suitable topology) to a $Kerr(a_0, m_0)$ initial data*

[7] The physical reality of these objects was recently put to test by LIGO-Viergo which is supposed to have detected the gravitational waves generated in the final stage of the in-spiraling of two black holes. Rainer Weiss, Barry C. Barish and Kip S. Thorne received the 2017 Nobel prize for their "decisive contributions" in this respect. The 2020 Nobel prize in Physics was awarded to R. Genzel and A. Ghez for providing observational evidence for the presence of super massive black holes in the center of our galaxy, and to R. Penrose for his theoretical foundational work: his concept of a trapped surface and the proof of his famous singularity theorem.

[8] Other such properties concern the rigidity of the Kerr family, see [IK-review] for a current survey, or the dynamical formations of black holes from regular configurations, see the [Chr-BH], [Kl-Rod1] and the introduction to [?] for an up to date account of more recent results.

[9] If the Kerr family would be unstable under perturbations, black holes would be nothing more than mathematical artifacts.

[10] This presupposes the existence of an event horizon. Note that the existence of such an event horizon can only be established a posteriori, upon the completion of the proof of the conjecture.

set, for sufficiently small $|a_0|/m_0$, has a complete future null infinity $\mathcal{I}^+$ and converges in its causal past $\mathcal{J}^{-1}(\mathcal{I}^+)$ to another nearby Kerr spacetime $Kerr(a_f, m_f)$ with parameters (a_f, m_f) close to the initial ones (a_0, m_0).

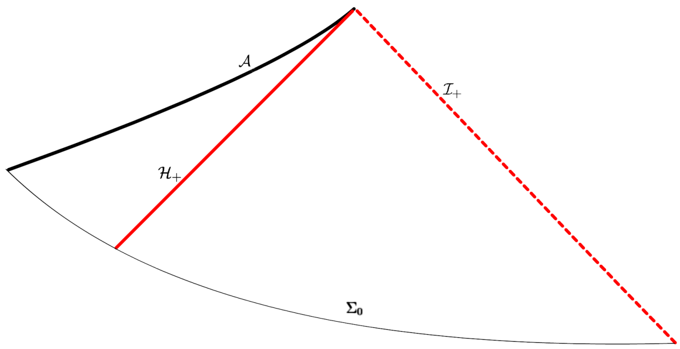


Figure 1.2: The Penrose diagram of the final space-time in Theorem 1.1.1 with initial hypersurface Σ_0, future space-like boundary $\mathcal{A}$, and $\mathcal{I}^+$ the complete future null infinity. The hypersurface $\mathcal{H}_+$ is the future event horizon of the final Kerr.

The precise version of the result, all the main features of the architecture of its proof, as well as detailed proofs for most of the main steps are to be found in [K-S:Kerr]. The full proof relies also on our joint work [GKS-2022] with E. Giorgi, our papers [K-S:GCM1], [K-S:GCM2] on GCM spheres, and the extension [Shen] to GCM hypersurfaces by D. Shen.

Brief comments on the proof

We will discuss the main ideas of the proof in more details in section 1.4. It pays however to give already a graphic sense of the main building blocks of our approach, which we call *general covariant modulated* (GCM), admissible spacetimes.

The main features of these finite spacetimes $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ with future boundaries $\mathcal{A} \cup {}^{(top)}\Sigma \cup \Sigma_*$ and past boundaries $\underline{\mathcal{B}}_1 \cup \underline{\mathcal{B}}_2$ are as follows:

- The capstone of the entire construction is the sphere S_*, on the future boundary Σ_* of ${}^{(ext)}\mathcal{M}$, which verifies a set of specific extrinsic and intrinsic conditions denoted by the acronym GCM.
- The spacelike hypersurface Σ_*, initialized at S_*, verifies a set of additional GCM conditions.

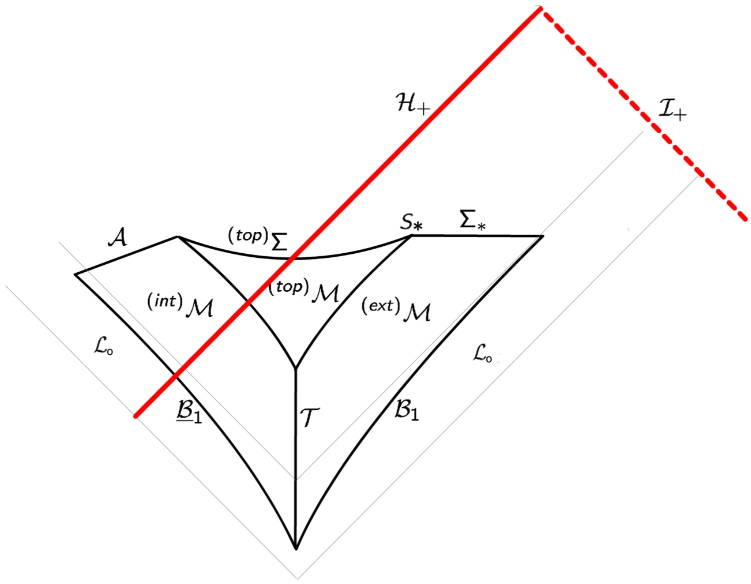


Figure 1.3: The Penrose diagram of a finite GCM admissible space-time $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$. The future boundary Σ_* initiates at the GCM sphere S_*. The past boundary of $\mathcal{M}$, $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$, is included in the initial layer $\mathcal{L}_0$, in which the spacetime is assumed given.

- Once Σ_* is specified the whole GCM admissible spacetime $\mathcal{M}$ is determined by a more conventional construction, based on geometric transport type equations[11].
- The construction, which also allows us to specify adapted null frames[12], is made possible by the covariance properties of the Einstein vacuum equations.
- The past boundary $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$ of $\mathcal{M}$, which is itself to be constructed, is included in the initial layer $\mathcal{L}_0$ in which the spacetime is assumed to be known[13], i.e. a small vacuum perturbation of a Kerr solution.

The proof of Theorem 1.1.1 is centered around a limiting argument for a continuous family of such spacetimes $\mathcal{M}$ together with a set of bootstrap assumptions (BA) for the

[11]More precisely ${}^{(ext)}\mathcal{M}$ can be determined from Σ_* by a specified outgoing foliation terminating in the timelike boundary $\mathcal{T}$, ${}^{(int)}\mathcal{M}$ is determined from $\mathcal{T}$ by a specified incoming one, and ${}^{(top)}\mathcal{M}$ is a complement of ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ which makes $\mathcal{M}$ a causal domain.

[12]In our work we prefer to talk about horizontal structures, see the brief discussion in section 1.4.3. Another important novelty in the proof of Theorem 1.1.1 is that it relies on non-integrable horizontal structures, see section 1.4.3.

[13]The passage form the initial data, specified on the initial spacelike hypersurface Σ_0, to the initial layer spacetime $\mathcal{L}_0$, can be justified by arguments similar to those of [Kl-Ni1] [Kl-Ni2], based on the methods introduced in [Ch-Kl].

connection and curvature coefficients, relative to the adapted frames. Assuming that a given finite, GCM admissible, spacetime $\mathcal{M}$ saturates BA we reach a contradiction as follows:

- First improve BA for some of the components of the curvature tensor with respect to the frame. These verify equations (called Teukolsky equations) which decouple, up to terms quadratic in the perturbation, and are treated by wave equations methods.
- Use the information provided by these curvature coefficients together with the gauge choice on $\mathcal{M}$, induced by the GCM condition on Σ_*, to improve BA for all other Ricci and curvature components.
- Use these improved estimates to extend $\mathcal{M}$ to a strictly larger spacetime $\mathcal{M}'$ and then construct a new GCM sphere S'_*, a new boundary Σ'_* which initiates on S'_*, and a new GCM admissible spacetime $\mathcal{M}'$, with Σ'_* as boundary, strictly larger than $\mathcal{M}$.

Remark 1.1.2. *The critical new feature of this argument is the fact that the new GCM sphere S'_* has to be constructed as a co-dimension 2 sphere in $\mathcal{M}'$ with no reference to the initial conditions*[14]*. This construction appears first in [K-S:Schw] in a polarized situation. The general construction appears in the GCM papers [K-S:GCM1], [K-S:GCM2]. The construction of Σ'_* from S'_* appears first in [K-S:Schw] in the polarized case. The general construction used in our work is due to D. Shen [Shen].*

1.2 Linear and nonlinear stability

1.2.1 Notions of nonlinear stability

Consider a stationary solution ϕ_0 of a nonlinear evolution equation

$$\mathcal{N}[\phi] = 0. \tag{1.2.1}$$

There are two distinct notions of stability, *orbital stability*, according to which small perturbations of ϕ_0 lead to solutions ϕ which remain close to ϕ_0 for all time, and *asymptotic stability (AS)* according to which the perturbed solutions converge as $t \to \infty$ to ϕ_0. In the case where ϕ_0 is non trivial, there is a third notion, which we call *asymptotic orbital stability (AOS)*, to describe the fact that the perturbed solutions may converge to a

[14]See a more detailed discussion in section 1.4.3.

different stationary solution. This happens if ϕ_0 belongs to a multi-parameter smooth family of stationary solutions, or by applying a gauge transform to ϕ_0 which keeps the equation invariant[15].

For quasilinear equations[16], such as EVE, a proof of stability means necessarily AS or AOS stability. Both require a detailed understanding of the *decay properties* of the linearized equation, i.e.

$$\mathcal{L}[\phi_0]\psi = 0, \tag{1.2.2}$$

with $\mathcal{L}[\phi_0]$ the Fréchet derivative $\mathcal{N}'[\phi_0]$. This is, essentially, a linear hyperbolic system with variable coefficients which, typically, presents instabilities[17].

In the exceptional situation, when stability can ultimately be established, one can tie all the instability modes to the following properties of the nonlinear equation:

M1. If ϕ_λ is a family of stationary solutions, near ϕ_0, verifying $\mathcal{N}[\phi_\lambda] = 0$. Then $\psi_0 = (\frac{d}{d\lambda}\phi_\lambda)_{\lambda=0}$ verifies $\mathcal{N}'[\phi_0]\psi_0 = 0$, i.e. ψ_0 is a nontrivial, stationary, bound state of the linearized equations (1.2.2).

M2. If Φ_λ is a smooth family of diffeomorphisms of the background manifold, $\Phi_0 = I$, such that $\mathcal{N}[\Phi_\lambda^*(\phi_0)] = 0$. Then $\Psi_0 = \left(\frac{d}{d\lambda}\Phi_\lambda^*(\phi_0)\right)_{\lambda=0}$ verifies $\mathcal{N}'[\phi_0]\Psi_0 = 0$, i.e. Ψ_0 is also a stationary bound state of the linearized equation (1.2.2).

These linear instabilities are responsible for the fact that a small perturbation of the fixed stationary solution ϕ_0 may not converge to ϕ_0 but to another nearby stationary solution[18].

To prove the asymptotic convergence of ϕ to a final state ϕ_f, different form ϕ_0, we need to establish sufficiently strong rates of decay[19] for $\phi - \phi_f$. Rates of decay however are strongly *coordinate dependent*, i.e. dependent on the choice of the diffeomorphism (or gauge) Φ in which decay is measured. Thus, to prove a nonlinear stability result we need to know both the final state ϕ_f and the coordinate system Φ_f in which sufficient decay, and thus convergence to ϕ_f, can be established. The difficulty here is that neither ϕ_f nor Φ_f can be determined a-priori (from the initial perturbation), they have to emerge

[15] In the case of Kerr, both cases are present as we shall see below.

[16] Orbital stability can be established directly (i.e. without establishing the stronger version) only in rare occasions, such as for hamiltonian equations with weak nonlinearities.

[17] In unstable situations (1.2.2) may have exponentially growing solutions, see for example [DKSW].

[18] The methodology of tracking this asymptotic final state, in general different from ϕ_0, is usually referred to as modulation. See for example [Ma-Me],[Me-R] for how modulation theory can be used to deal with some examples of scalar nonlinear dispersive equations.

[19] To control the nonlinear terms of the equation.

dynamically in the process of convergence. Moreover, in all examples involving nonlinear wave equations in $1+3$ dimensions, the nonlinear terms have to also cooperate, that is an appropriate version of the so called null condition has to be verified.

To summarize, given a nonlinear system $\mathcal{N}[\phi]=0$ which possesses both a smooth family of stationary solutions ϕ_λ and a smooth family of invariant diffeomorphisms a proof of the nonlinear stability of ϕ_0 requires the following ingredients:

- The only non-decaying modes of the linearized equation $\mathcal{L}(\phi_0)\psi=0$ are those due to the items M1–M2 above. In particular there are no exponentially growing modes.
- A dynamical construction of both the final state ϕ_f and the final gauge Φ_f in which convergence to the final state takes place.
- The nonlinear terms in the equation

$$\mathcal{L}[\phi_f]\psi = N(\psi)$$

obtained by expanding the equation $\mathcal{N}[\phi]$ near ϕ_f, in the gauge given by the diffeomorphism Φ_f, has to verify an appropriate version of the null condition.

1.2.2 The case of the Kerr family

The issue of the stability of the Kerr family has been at the center of attention of GR physics and mathematical relativity for more than half a century, ever since their discovery by Kerr in [Kerr]. In this case we have to deal not only with a 2-parameter family of solutions, corresponding to the parameters (a, m), but also with the entire group of diffeomorphisms[20] of $\mathcal{M}$. In what follows we try to discuss the main difficulties of the problem. In doing that it helps to compare these to those arising in the simplest case when $a=m=0$, i.e. stability of Minkowski.

1.2.3 Stability of Minkowski space

Until very recently the only space-time for which full nonlinear stability had been established was the Minkowski space, see [Ch-Kl]. The proof is based on some important PDE advances of late last century:

[20]Indeed, according to the covariant properties of the Einstein vacuum equations we cannot distinguish between $\mathbf{g}$ and $\Phi^*\mathbf{g}$, for any diffeomorphism Φ of $\mathcal{M}$.

(i) Robust approach, based on the vectorfield method, to derive quantitative decay based on generalized energy estimates and commutation with (approximate) Killing and conformal Killing vectorfields.

(ii) The *null condition* identifying the deep mechanism for nonlinear stability, i.e. the specific structure of the nonlinear terms which enables stability despite the low decay of the perturbations.

(iii) Elaborate bootstrap argument according to which one makes educated assumptions about the behavior of solutions to nonlinear wave equations and then proceeds, by a long sequence of a-priori estimates, to show that they are in fact satisfied. This amounts to a *conceptual linearization*, i.e. a method by which the equations become, essentially, linear[21] without actually linearizing them.

The main innovation in the proof in [Ch-Kl] is the choice of an appropriate gauge condition, readjusted dynamically through the convergence process, by a continuity argument, which allows one to separate the curvature estimates, treated by hyperbolic methods, from the estimates for the connection coefficients. The key point is that these latter verify transport or elliptic equations in which the curvature terms appear as sources. Thus both the curvature components and connection coefficients can be controlled by a bootstrap argument. The gauge condition is based on the constructive choice of a maximal time function t and two outgoing optical functions ${}^{(int)}u$[22] and ${}^{(ext)}u$[23] covering the interior and exterior parts of the spacetime.

Another novelty of [Ch-Kl] was the reliance on null frames adapted to the S-foliations induced by the level surfaces of t and u. These define integrable horizontal structures (in the language of part I of [GKS-2022]), by contrast with the non-integrable ones used in the proof of Theorem 1.1.1 and discussed in section 1.4.3. The functions t, u and this integrable horizontal structure can be used to define approximate Killing vectorfields used in estimating the curvature.

[21]Note that in the context of EVE, and other quasilinear hyperbolic systems, this differs substantially from the usual notion of linearization around a fixed background.

[22]The interior optical function is initialized on a timelike geodesic from the initial hypersurface.

[23]The exterior optical function ${}^{(ext)}u$ is initialized on the last slice $t = t_*$, by the construction of a foliation (inverse lapse foliation) initialized at space-like infinity. It is thus readjusted dynamically as $t_* \to \infty$.

1.2.4 Main difficulties

There are a few major obstacles in passing from the stability of Minkowski to that of Kerr:

1. The first one was already discussed in section 1.2.1 in the general context of the stability of a stationary solution ϕ_0. In the case when ϕ_0 is trivial there are no nontrivial bound states for the linearized problem and thus we expect that the final state does actually coincide with ϕ_0. This is precisely the case for the special member of the Kerr family $a = m = 0$, i.e. the Minkowski space[24] $(\mathbb{R}^{1+3}, m)$. On the other hand, in perturbations of Kerr, general covariance affects the entire construction of the spacetime. In the proof of Theorem 1.1.1 the crucial concept of a GCM admissible spacetime is meant to deal with both finding the final parameters and the gauge in which convergence to the final state takes place.

2. A fundamental insight in the stability of the Minkowski space was that the Bianchi identities decouple at first order from the null structure equations which allows one to control curvature first, as a Maxwell type system (see [Ck-Kl0]), and then proceed with the rest of the solution. This cannot work for perturbations of Kerr due to the fact that some of the null components[25] of the curvature tensor are non-trivial in Kerr.

3. Even if one succeeds in tackling the above mentioned issues, there are still major obstacles in understanding the decay properties of the solution. Indeed, when one considers the simplest, relevant, linear equation on a fixed Kerr background, i.e. the scalar wave equation $\square_{a,m}\psi = 0$, one encounters serious difficulties to prove decay. Below is a very short description of these:

    - *The problem of trapped null geodesics.* This concerns the existence of null geodesics[26] neither crossing the event horizon nor escaping to null infinity, along which solutions can concentrate for arbitrary long times. This leads to degenerate energy-Morawetz estimates which require a very delicate analysis.
    - *The trapping properties of the horizon.* The horizon itself is ruled by null geodesics, which do not communicate with null infinity and can thus concen-

[24] Note however that even though the linearized system around Minkowski does not contain instabilities, the proof of the nonlinear stability of the Minkowski space in [Ch-Kl] takes into account (in a fundamental way!) general covariance. Indeed the presence of the ADM mass affects the causal structure of the far, asymptotic, region of the perturbed space-time.

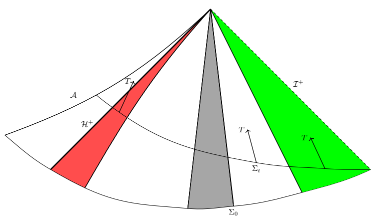
[25] With respect to the *principal null directions of Kerr,* i.e a distinguished null pair which diagonalizes the full curvature tensor, the middle component $P = \rho + i\ {}^*\rho$ is nontrivial.

[26] In the Schwarzschild case, these geodesics are associated with the celebrated photon sphere $r = 3m$.

trate energy. This problem was solved by understanding the so called red-shift effect associated to the event horizon, which counteracts this type of trapping.

- *The problem of superradiance.* This is the failure of the stationary Killing field $\mathbf{T} = \partial_t$ to be everywhere timelike in the domain of outer communication[27], and thus, of the associated conserved energy to be positive. Note that this problem is absent in Schwarzschild and, in general, for axially symmetric solutions of EVE. In both cases however there still is a degeneracy along the horizon.
- *Superposition problem.* This is the problem of combining the estimates in the near region, close to the horizon, (including the ergoregion and trapping) with estimates in the asymptotic region, where the spacetime is close to Minkowski.

Figure 1.4: Penrose diagram of the Kerr exterior to the future of a spacelike hypersurface. Note that the ergoregion, in red, and the trapping region in blue are separated only if $|a|/m$ is sufficiently small.



4. Though, as seen above, the analysis of the scalar wave equation in Kerr presents formidable difficulties, it is itself just a vastly simplified model problem. A more realistic equation is the so called spin 2 wave equation, or Teukolsky equation, which presents many new challenges[28].

5. The full linearized system, whatever its formulation, presents many additional difficulties due to its complex tensorial structure and the huge gauge covariance of the equations[29]. The crucial breakthrough in this regard is the observation, due to Teukolsky [Teuk], that the extreme components of the linearized curvature tensor are both gauge invariant (see below in section 1.2.5) and verify decoupled spin 2 equations, that is the Teukolsky equations mentioned above.

[27]The stationary Killing vectorfield $\mathbf{T}$ is timelike only outside of the so-called ergoregion.

[28]Unlike the scalar wave equation $\square_{a,m}\psi = 0$, which is conservative, the Teukolsky equation is not, and we thus lack the most basic ingredient in controlling the solutions of the equation, i.e. energy estimates.

[29]As mentioned earlier, rates of decay are heavily dependent on a proper choice of gauge, thus affecting the issue of convergence.

6. A crucial simplification of the linear theory, by comparison to the full nonlinear case, is that one can separate the treatment of the gauge invariant extreme curvature components form all the other gauge invariant quantities. In the nonlinear case this separation is no longer true, all quantities need to be treated simultaneously. Moreover, methods based on separation of variables, developed to treat scalar and and spin 2 wave equations in Kerr, are incompatible with the nonlinear setting which requires, instead, robust methods to derive decay.

1.2.5 Linear stability

Linear stability for the vacuum equations is formulated in the following way. Given the Einstein tensor $\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\alpha\beta}$ and a stationary solution $\mathbf{g}_0$, i.e. a fixed Kerr metric, one has to solve the system of equations

$$\mathbf{G}'(\mathbf{g}_0)\,\delta\mathbf{g} = 0. \tag{1.2.3}$$

The covariant properties of the Einstein equations, i.e. the equivalence between a solution $\mathbf{g}$ and $\Phi^*(\mathbf{g})$, leads us to identify $\delta\mathbf{g}$ with $\delta\mathbf{g} + \mathcal{L}_X(\mathbf{g}_0)$ for arbitrary vectorfields X in $\mathcal{M}$, i.e.

$$\delta\mathbf{g} \equiv \delta\mathbf{g} + \mathcal{L}_X(\mathbf{g}_0). \tag{1.2.4}$$

- 2-parameter family $Kerr(a, m)$.

$$\mathbf{G}(\mathbf{g}_{a,m}) = 0 \Rightarrow \mathbf{G}'(\mathbf{g}_{a,m})\Big(\frac{d}{da}\mathbf{g}_{a,m}, \frac{d}{dm}\mathbf{g}_{a,m}\Big) = 0.$$

- General covariance

$$\mathbf{G}(\Phi^*_\lambda\mathbf{g}_{a,m}) = 0 \Rightarrow G'(\mathbf{g}_{a,m})\Big(\frac{d}{d\lambda}\Phi^*_\lambda\mathbf{g}_{a,m}\Big|_{\lambda=0}\Big) = 0.$$

We can now attempt to formulate a version of linear stability for (1.2.3), loosely related to the nonlinear stability of Kerr conjecture, as follows.

Definition 1.2.1. *By linear stability of the Kerr metric* $\mathbf{g}_0$ *we understand a result which achieves the following:*

Given an appropriate initial data for a perturbation $\delta\mathbf{g}$*, find a vectorfield* X *such that, after projecting away the bound states generated by the parameters* a, m*, according to M1–M2 in section 1.2.1, a solution of the form* $\delta\mathbf{g} + \mathcal{L}_X\mathbf{g}_0$ *to* (1.2.3)*, decays, relative to an appropriate null frame of* $\mathcal{K}(a_0, m_0)$*, sufficiently fast in time.*

Remark 1.2.2. *The definition above is necessarily vague. What is the meaning of sufficiently fast? In fact various components of the metric* $\delta\mathbf{g}$*, relative to the canonical null frame of* $\mathcal{K}(a, m)$*, are expected to decay at different slow polynomial rates, some of which are not even integrable. Unlike in the nonlinear context, where one needs precise rates of decay for components of the curvature tensor and Ricci coefficients, as well as their derivatives, to be able to control the nonlinear terms, in linear theory any type of nontrivial control of solutions may be regarded as satisfactory*[30] *Thus linear stability, as formulated above, can only be regarded as a vastly simplified model problem. Nevertheless the study of linear stability of the Kerr family has turned out to be useful in various ways, as we shall see below.*

Historically, the following versions of linear stability have been considered.

(a) *Metric Perturbations.* At the level of the metric itself, i.e. as above in (1.2.3).

(b) *Curvature Perturbations.* Via the Newman-Penrose (NP) formalism, based on null frames.

The strategy followed in both cases[31] is:

- Find components of the metric (in case (a)) or curvature tensor (in case (b)), invariant with respect to linearized gauge transformations 1.2.4, which verify decoupled wave equations. The main insight of this type was the discovery, by Teukolsky [Teuk], in the context of (b) above, that the extreme components of the linearized curvature tensor verify both these properties.

- Analyze these components by showing one of the following:

[30] Thus, for example, in their well known linear stability result around Schwarzschild [DHR], the authors derive satisfactory results (compatible with what is needed in nonlinear theory) for components of the curvature tensor, and some Ricci coefficients, but not all. Similar comments apply to [HKW] and [Johnson].

[31] In the article we refer mainly to the curvature perturbation approach.

  - There are no exponentially growing modes. This is known as mode stability.
  - Boundedness for all time.
  - Decay (sufficiently fast) in time.

- Find a linearized gauge condition, i.e. a vectorfield X, such that all remaining (gauge dependent) components (at the metric or curvature level) inherit the property mentioned above: no exponentially growing modes, boundedness, or decay in time. In the physics literature this is known as the problem of reconstruction.

Mode stability

All results on the linear stability of Kerr in the physics literature during the 10-15 years after Roy Kerr's 1963 discovery, often called the "Golden Age of Black Hole Physics", are based on mode decompositions. One makes use of the separability[32] of the linearized equations, more precisely the Teukolsky equations, on a fixed Kerr background, to derive simple ODEs for the corresponding modes. One can then show, by ingenious methods, that these modes cannot exhibit exponential growth. The most complete result of this type is due to Bernard Whiting [Whit] in the case of the scalar wave equation.

The obvious limitation of these results are as follows:

- They are far from even establishing the boundedness of general solutions to the Teukolsky equations, let alone to establish quantitative decay for the general solutions.

- Results based on mode decompositions depend strongly on the specific symmetries of Kerr which cannot be adapted to perturbations of Kerr.

Robust methods to deal with both issues have been developed in the mathematical community, based on the vectorfield method which we discuss below.

Classical vectorfield method

The vectorfield method, as an analytic tool to derive decay, was first developed in connection with the wave equation in Minkowski space. As well known, solutions of the wave equation $\square\psi = 0$ in the Minkowski space $\mathbb{R}^{n+1}$ both conserve energy and decay

[32]See discussion in section 1.1.1.

uniformly in time like $t^{-\frac{n-1}{2}}$. While conservation of energy can be established by a simple integration by parts, and is thus robust to perturbations of the Minkowski metric, decay was first derived either using the Kirchhoff formula or by Fourier methods, which are manifestly not robust. An integrated version of local energy decay, based on an inspired integration by parts argument, was first derived by C. Morawetz [Mor1], [Mor2]. The first derivation of decay based on the commutations properties of $\square$ with Killing and conformal Killing vectorfields of Minkowski space together with energy conservation appear in [Kl-vect1] and [Kl-vect2]. The same method also provides precise information about the decay properties of derivatives of solutions with respect to the standard null frame of Minkowski space, an important motivating factor in the discovery of the null condition [Kl-ICM], [Chr] and [Kl-null].

The crucial feature of the methodology initiated by these papers, to which we refer as the classical vectorfield method, is that it can be easily adapted to perturbations of the Minkowski space. As such the method has had numerous applications to nonlinear wave equations and played an important role in the proof of the nonlinear stability of Minkowski space, as discussed in section 1.2.3. It has also been applied to later versions of the stability of Minkowski in [Kl-Ni1]-[Kl-Ni2], [Lind-Rodn], [Bi], [Lind], [Huneau], [HV2], [Graf], and extensions of it to Einstein equation coupled with various matter fields in [BiZi], [FJS], [Lind-Ta], [BFJT], [Wa], [Lf-Ma], [IP].

New vectorfield method

To derive decay estimates for solutions of wave equations on a Kerr background one has to substantially refine the classical vectorfield method. The new vectorfield method is an extension of the classical method which compensates for the lack of enough Killing and conformal Killing vectorfields in Kerr by introducing, new, cleverly designed, vectorfields whose deformation tensors have coercive properties in different regions of spacetime, not necessarily causal. The method has emerged in the last 20 years in connection to the study of boundedness and decay for the scalar wave equation in Schwarzschild and Kerr, see section 1.3.2 for more details.

1.2.6 Model problems

To solve the stability of Kerr conjecture one has to deal simultaneously with all the difficulties mentioned above. This is, of course, beyond the abilities of mere humans. Instead the problem was tackled in a sequence of steps based on a variety of simplified model problems, in increasing order of difficulty. To start with we can classify model problems based on the following criteria:

1. Whether the result refers to Schwarzschild i.e. $a = 0$, slowly rotating Kerr i.e. $|a| \ll m$ or full non-extremal Kerr $|a| < m$.
2. Whether the result refers to linear or nonlinear stability.
3. Whether the result, in linear theory, refers to scalar wave equation, i.e. spin 0, Teukolsky equation, i.e. spin 2, or the full linearized system.
4. Whether the stability result, in linear theory, is a mode stability result, a boundedness result, one that establishes some version of quantitative decay or one that establishes an optimal version of quantitative decay

1.3 Short survey of model problems

We give below a short outline of the main developments concerning linear and nonlinear model problems for the Kerr stability problem, paying special attention to those which had a measurable influence on our work.

1.3.1 Mode stability results

These are mode stability results, using the method of separation of variables, obtained in the physics community roughly during the period 1963-1990. They rely on what Chandrasekhar called the most striking feature of Kerr i.e. "*the separability of all the standard equations of mathematical physics in Kerr geometry*".

1. *Regge-Wheeler (1957).* Even before the discovery of the Kerr solution physicists were interested in the mode stability of Schwarzschild space, i.e. $\mathcal{K}(0, m)$. The first important result goes back to T. Regge and J.A Wheeler [Re-W], in which they analyzed linear, metric perturbations, of the Schwarzschild metric. They showed that in a suitable gauge, equation (1.2.3) decouples into even-parity and odd-parity perturbations, corresponding to axial and polar perturbations. The most important discovery in that paper is that of the master Regge-Wheeler equation, a wave equation with a favorable potential, verified by an invariant scalar component ϕ of the metric, i.e.
$$\Box_m \phi = V\phi, \qquad V = \frac{4}{r^2}\left(1 - \frac{2m}{r}\right). \tag{1.3.1}$$

where $\square_m$ denotes the wave operator of the Schwarzschild metric of mass m. The R-W study was completed by Vishveshwara [Vishev] and Zerilli [Ze]. A gauge-invariant formulation of *metric perturbations* was then given by Moncrief [Moncr].

2. *Teukolsky (1973).* The curvature perturbation approach, near Schwarzschild, based on the Newman-Penrose (NP) formalism was first undertaken by Bardeen-Press [**?**]. This approach was later extended to the Kerr family by Teukolsky [Teuk], see also [P-T], who made the important discovery that the extreme curvature components, relative to a principal null frame, are gauge invariant and satisfy decoupled, separable, wave equations. The equations, bearing the name of Teukolsky, are roughly of the form

$$\square_{a,m}\psi = \mathcal{L}[\psi] \tag{1.3.2}$$

where $\mathcal{L}[\psi]$ is a first order linear operator in ψ.

3. *Chandrasekhar (1975).* In [Chand2] Chandrasekhar initiated a transformation theory relating the two approaches. He exhibited a transformation which connects the Teukolsky equations to a Regge-Wheeler type equation. In the particular case of Schwarzschild the transformation takes the Teukolsky equation to the Regge-Wheeler equation in (1.3.1). The Chandrasekhar transformation was further elucidated and extended by R. Wald [Wald] and recently by Aksteiner and al [**?**]. Though originally it was meant only to unify the Regge-Wheeler approach with that of Teukolsky, the Chandrasekhar transformation, and various extensions of it, turn out to play an important role in the field.

4. *Whiting (1989).* As mentioned before, the full mode stability, i.e. lack of exponentially growing modes, for the Teukolsky equation on Kerr is due to Whiting[33], see [Whit]. Stronger quantitative versions were proved in [AWPW], [Fins2], [Te].

5. *Reconstruction.* Once we know that the Teukolsky variables, i.e. the extreme components of the curvature tensor verify mode stability, i.e. there are no exponentially growing modes, it still remains to deal with the problem of reconstruction, i.e. to find a gauge relative to which all other components of the curvature and Ricci coefficients enjoy the same property. We refer the reader to Wald [Wald] and the references within for a treatment of this issue in the physics literature.

[33]For the analogous result in the case of the scalar wave equation, see [Fins1]. See also [SR] for a stronger quantitative version which was used in [D-R-SR].

1.3.2 Quantitative decay for the scalar wave equation

As mentioned in section 1.2.5, mode stability is far from establishing even the boundedness of solutions. To achieve that[34] and, more importantly, to derive realistic decay estimates, one needs an entirely different approach based on what we called the "new vectorfield method" in section 1.2.5. The method has emerged in connection to the study of boundedness and decay for the scalar wave equation in $\mathcal{K}(a,m)$,

$$\square_{a,m}\phi = 0. \tag{1.3.3}$$

The starting and most demanding part of the new method, which appeared first in [B-S1], is the derivation of a global, combined, *Energy-Morawetz* estimate which degenerates in the trapping region. Once an Energy-Morawetz estimate is established one can commute with the Killing vectorfields of the background manifold, and the so called red shift vectorfield introduced in [DaRo1], to derive uniform bounds for solutions. The most efficient way to also get decay, and solve the *superposition problem* (see section 1.2.4), originating in [Da-Ro3], is based on the presence of family of r^p*-weighted*, quasi-conformal vectorfields defined in the non-causal, far r region of spacetime[35].

The first Energy-Morawetz type results for scalar wave equation (1.3.3) in Schwarzschild, i.e. $a = 0$, are due to Blue-Soffer [B-S1], [B-S2] and Blue-Sterbenz [B-St], based on a modified version of the classical Morawetz integral energy decay estimate. Further developments appear in the works of Dafermos-Rodnianski, see [DaRo1], [Da-Ro3], and Marzuola-Metcalfe-Tataru-Tohaneanu [Ma-Me-Ta-To]. The vectorfield method can also be extended to derive decay for axially symmetric solutions in Kerr, see [I-Kl] and[36] [St], but it is known to fail for general solutions in Kerr, see Alinhac [Al].

In the absence of axial symmetry the derivation of an Energy-Morawetz estimate in $\mathcal{K}(a,m)$ for $|a/m| \ll 1$ requires a more refined analysis involving both the vectorfield method and either micro-local methods or mode decompositions. The first full quantitative decay[37] result, based on micro-local analysis techniques, is due to Tataru-Tohaneanu [Ta-To]. The derivation of such an estimate in the full sub-extremal case $|a| < m$ is even

[34] The first realistic boundedness result for solutions of the scalar wave equation in Schwarzschild appears in [K-Wald] based on a clever use of the energy method which takes into account the degeneracy of $\mathbf{T}$ at the horizon.

[35] These replace the scaling and inverted time translation vectorfields used in [Kl-vect1] or their corresponding deformations used in [Ch-Kl]. A recent improvement of the method allowing one to derive higher order decay can be found in [AArGa].

[36] In his Princeton PhD thesis Stogin establishes a Morawetz estimate even for the full subextremal case $|a| < m$.

[37] See also [DaRo2] for the first proof of boundedness of solutions, based on mode decompositions.

more subtle and was achieved by Dafermos-Rodnianski-Shlapentokh-Rothman [D-R-SR] by combining the vectorfield method with a full separation of variables approach. A purely physical space proof of the Energy-Morawetz estimate for small $|a/m|$, which avoids both micro-local analysis and mode decompositions, was pioneered by Andersson-Blue in [A-B]. Their method, which extends the classical vectorfield method to include second order operators (in this case the Carter operator, see section 1.1.1), has the usual advantages of the classical vectorfield method, i.e it is robust with respect to perturbations. It is for this reasons that we rely on it in the proof of Theorem 1.1.1, more precisely in part II of [GKS-2022].

1.3.3 Linear stability of Schwarzschild

A first quantitative proof of the linear stability of Schwarzschild spacetime was established[38] by Dafermos-Holzegel-Rodnianski (DHR) in [DHR]. Notable in their analysis is the treatment of the Teukolsky equation in a fixed Schwarzschild background. While the Teukolsky equation is separable, and amenable to mode analysis, it is not variational and thus cannot be treated directly by energy type estimates. As mentioned earlier in section 1.3.1, Chandrasekhar was able to relate the Teukolsky equation to the Regge-Wheeler (RW) equation, which is both variational and coercive (the potential V has a favorable sign). In [DHR] the authors rely on a physical space version of the Chandrasekhar transformation. Once decay estimates for the RW equation have been established, based on the technology developed earlier for the scalar wave equation in Schwarzschild, the authors recover the expected boundedness and decay for solutions to the original Teukolsky equation.

The remaining work in [DHR] is to derive similar control for the other curvature components and the linearized Ricci coefficients associated to the double null foliation. This last step requires carefully chosen gauge conditions, which the authors make within the framework of a double null foliation, initialized both on the initial hypersurface and the background Schwarzschild horizon[39]. This gauge fixing from initial data leads to suboptimal decay estimates for some of the metric coefficients[40] and is thus inapplicable to the nonlinear case. This deficiency was fixed in the PhD thesis of E. Giorgi, in the context of the linear stability of Reissner-Nordström, see [Giorgi], by relying on a linearized

[38]A somewhat weaker version of linear stability of Schwarzschild was subsequently proved in [HKW] by using the original, direct, Regge-Wheeler, Zerilli approach combined with the vectorfield method and adapted gauge choices. See also [Johnson] for an alternate approach of linear stability of Schwarzschild using wave coordinates.

[39]The authors use a a scalar condition for the linearized lapse along the event horizon (part of what the authors call future normalized gauge), itself initialized from initial data, see (212) and (214) in [DHR].

[40]See (250)–(252) and (254) in [DHR].

version of the GCM construction in [K-S:Schw].

1.3.4 Linear stability of Kerr for small angular momentum

The first breakthrough result on the linear stability of Kerr, for $|a|/m \ll 1$, is due to Ma [Ma], see also [DHR-Kerr]. Both results are based on a generalization of the Chandrasekhar transformation to Kerr which takes the Teukolsky equations, verified by the extreme curvature components, to generalized versions of the Regge-Wheeler (gRW) equation. Relying on separation of variables and vectorfield techniques, similar to those developed for the scalar wave equation in slowly rotating Kerr, the authors derive Energy-Morawetz and r^p estimates for the solutions of the gRW equations. Note that these results were recently extended to the full subextremal range, $|a| < m$, in [SR-Te1], [SR-Te2] and [Millet].

The first stability results for the full linearized Einstein vacuum equations near $\mathcal{K}(a, m)$, for $|a|/m \ll 1$, appeared in [ABBMa2019] and [HHV]. The first paper, based on the GHP formalism[41], see [GHP], builds on the results of [Ma] while the second paper is based on an adapted version of the metric formalism and builds on the seminal work of the authors on Kerr-de Sitter [H-V1]. Though the ultimate relevance of these papers to nonlinear stability remains open, they are both remarkable results in so far as they deal with difficulties that looked insurmountable even ten years ago.

1.3.5 Nonlinear model problems

Nonlinear stability of Kerr-de Sitter

There is another important, simplified, nonlinear model problem which has drawn attention in recent years, due mainly to the striking achievement of Hintz and Vasy [H-V1]. This is the problem of stability of Kerr-de Sitter concerning the Einstein vacuum equation with a strictly positive cosmological constant

$$\mathbf{R}_{\alpha\beta} + \Lambda \mathbf{g}_{\alpha\beta} = 0, \qquad \Lambda > 0. \tag{1.3.4}$$

In their work, which relies in part on the important mode stability result of Kodama and Ishibashi [Ko-Is], Hintz and Vasy were able prove the nonlinear stability of the stationary part of Kerr-de Sitter with small angular momentum, the first nonlinear stability result

[41] An adapted spinorial version of the NP formalism.

of any nontrivial stationary solutions for the Einstein equations[42]. It is important to note that, despite the fact that, formally, the Einstein vacuum equation (1.1.3) is the limit[43] of (1.3.4) as $\Lambda \to 0$, the global behavior of the corresponding solutions is radically different[44].

The main simplification in the case of stationary solutions of (1.3.4) is that the expected decay rates of perturbations near Kerr-de Sitter is exponential, while in the case $\Lambda = 0$ the decay is lower degree polynomial[45], with various components of tensorial quantities decaying at different rates, and the slowest decaying rate[46] being no better than t^{-1}. The Hintz-Vasy result was recently revisited in the work of A. Fang [Fang2] [Fang1] where he bridges the gap between the spectral methods of [H-V1] and the vectorfield methods.

Nonlinear stability of Schwarzschild

The first nonlinear stability result of the Schwarzschild space was established in [K-S:Schw]. In its simplest version, the result states the following.

Theorem 1.3.1 (Klainerman-Szeftel [K-S:Schw]). *The future globally hyperbolic development of an* axially symmetric, polarized, *asymptotically flat initial data set, sufficiently close (in a specified topology) to a Schwarzschild initial data set of mass $m_0 > 0$, has a complete future null infinity $\mathcal{I}^+$ and converges in its causal past $\mathcal{J}^{-1}(\mathcal{I}^+)$ to another nearby Schwarzschild solution of mass m_f close to m_0.*

The restriction to axial polarized perturbations is the simplest assumption which insures that the final state is itself Schwarzschild and thus avoids the additional complications of the Kerr stability problem. We refer the reader to the introduction in [K-S:Schw] for a full discussion of the result.

The proof is based on a construction based on GCM admissible spacetimes similar to that

[42] This is also the first general nonlinear stability result in GR establishing asymptotic stability towards a family of solutions, i.e. full quantitative convergence to a final state close, but different from the initial one.

[43] To pass to the limit requires one to understand all global in time solutions of (1.3.4) with $\Lambda = 1$, not only those which are small perturbations of Kerr-de Sitter, treated by [H-V1].

[44] Major differences between formally close equations occur in many other contexts. For example, the incompressible Euler equations are formally the limit of the Navier-Stokes equations as the viscosity tends to zero. Yet, at fixed viscosity, the global properties of the Navier-Stokes equations are radically different from that of the Euler equations.

[45] While there is exponential decay in the stationary part treated in [H-V1], note that lower degree polynomial decay is expected in connection to the stability of the complementary causal region (called cosmological or expanding) of the full Kerr-de Sitter space, see e.g. [Vo].

[46] Responsible for carrying gravitational waves at large distances so that they are detectable.

briefly discussed in section 1.1.3 in the context of slowly rotating Kerr. There are however several important simplifications to be noted:

- The assumption of polarization makes the constructions of the GCM spheres S_* and spacelike hypersurface Σ_* significantly simpler, see Chapter 9 in [K-S:Schw], by comparison to the general case treated in [K-S:GCM1], [K-S:GCM2] and [Shen].
- The spacetime has only two components $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ and the null horizontal structures, defined on each component, are integrable.
- As in the case of the scalar wave equation on Schwarzschild space the main spin-2 Teukolsky wave equations can be treated (via the passage to the Regge-Wheeler equation) by a vectorfield approach. This is no longer true in Kerr and even less so in perturbations of Kerr.

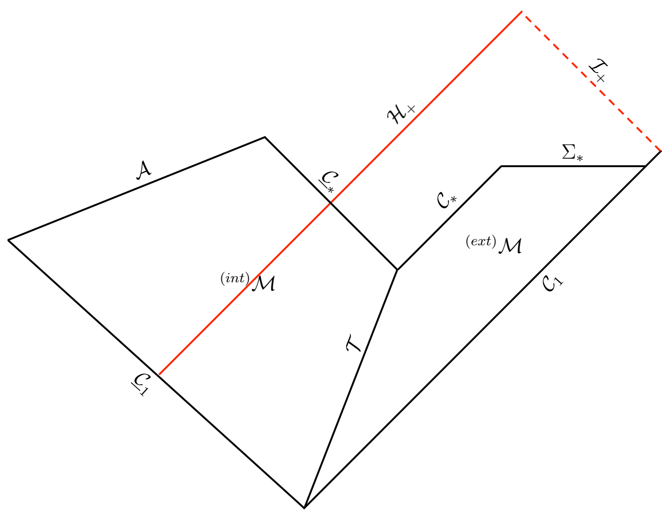


Figure 1.5: The GCM admissible space-time $\mathcal{M}$. By comparison to Figure ??, $\mathcal{M}$ does not have ${}^{(top)}\mathcal{M}$, the past boundaries $\mathcal{C}_0 \cup \underline{\mathcal{C}}_0$ and future boundary $\underline{\mathcal{C}} \cup \underline{\mathcal{C}}_*$ are null and the horizontal structures (induced by geodesic foliations) are integrable. As in Theorem 1.1.1, the crucial GCM sphere S_* is defined and constructed with no reference to the initial data.

Recently Dafermos-Holzegel-Rodnianski-Taylor [DHRT] have extended[47] the result of [K-S:Schw] by properly preparing a co-dimension 3 subset of the initial data such that the final state is still Schwarzschild. Like in [K-S:Schw], the starting point of [DHRT] is to anchor the entire construction on a far away[48] GCM type sphere S_*, in the sense

[47]The novelty of [DHRT], compared to [K-S:Schw], is the well preparation of the initial data, based on an additional three dimensional modulation. Note however that [DHRT] requires substantially stronger asymptotic conditions for the initial data compared to [K-S:Schw].

[48]That is $r \gg u$, similar to the dominant in r condition (3.3.4) of [K-S:Schw].

of [K-S:GCM1] [K-S:GCM2], with no direct reference to the initial data. It also uses the same definition of the angular momentum as in (7.19) of [K-S:GCM2]. Finally, the spacetime in [DHRT] is separated in an exterior region ${}^{(ext)}\mathcal{M}$ and an interior region ${}^{(int)}\mathcal{M}$, with the ingoing foliation of ${}^{(int)}\mathcal{M}$, initialized based on the information induced by ${}^{(ext)}\mathcal{M}$, as in [K-S:Schw]. We note, however, that [DHRT] does not use the geodesic foliation of [K-S:Schw], but instead both ${}^{(int)}\mathcal{M}$ and ${}^{(ext)}\mathcal{M}$ are foliated by double null foliations, and thus, the process of estimating the gauge dependent variables is somewhat different.

1.4 Main ideas in the proof of Theorem 1.1.1

1.4.1 The bootstrap region

As mentioned in section 1.1.3, the proof of Theorem 1.1.1 is centered around a continuity argument for a family of carefully constructed finite *generally covariant modulated (GCM) admissible spacetimes* $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$. As can be seen in Figure B.1 below, the future boundary of the spacetime is given by $\mathcal{A} \cup {}^{(top)}\Sigma \cup \Sigma_*$ where Σ_* is a spacelike, generally covariant modulated (GCM) hypersurface, that is a hypersurface verifying a set of crucial, well-specified, geometric conditions, essential to our proof of convergence to a final state.

The capstone as well as the most original part of the entire construction is the sphere S_*, the future boundary of Σ_*, which verifies a set of rigid, extrinsic and intrinsic, conditions. Once Σ_* is specified the whole GCM admissible spacetime $\mathcal{M}$ is determined by a more conventional construction, based on geometric transport type equations. More precisely ${}^{(ext)}\mathcal{M}$ can be determined from Σ_* by a specified outgoing foliation terminating in the timelike boundary $\mathcal{T}$, ${}^{(int)}\mathcal{M}$ is determined from $\mathcal{T}$ by a specified incoming one, and ${}^{(top)}\mathcal{M}$ is a complement of ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ which makes $\mathcal{M}$ a causal domain[49]. The past boundary $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$ of $\mathcal{M}$, which is itself to be constructed, is included in the initial layer $\mathcal{L}_0$ in which the spacetime is assumed to be known, i.e. a small perturbation of a Kerr solution. The passage from the initial data specified on Σ_0 to the initial layer spacetime $\mathcal{L}_0$ is justified by D. Shen in [Shen:Kerr-ext] by arguments similar to those of [Kl-Ni1]-[Kl-Ni2], based on the mathematical methods and techniques introduced in [Ch-Kl].

Each of the spacetime regions ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$, ${}^{(top)}\mathcal{M}$ come equipped with specific geomet-

[49]This is required because of the fact that, in our construction, the future boundary of ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ is not causal. By contrast, in [K-S:Schw], $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$.

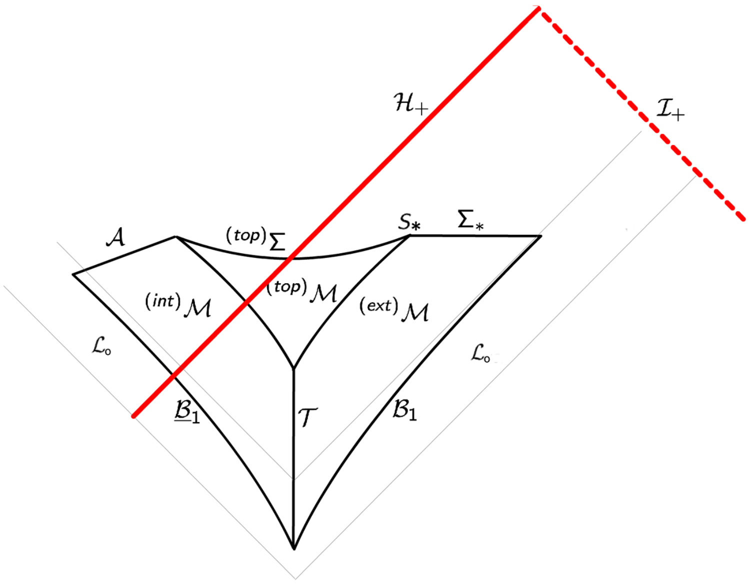


Figure 1.6: The Penrose diagram of a finite GCM admissible space-time $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$. The spacetime is prescribed in the initial layer $\mathcal{L}_0$ and has $\mathcal{A} \cup {}^{(top)}\Sigma \cup \Sigma_*$ as future boundary, with Σ_* a spacelike "generally covariant modulated (GCM)" hypersurface. Its past boundary, $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$, is itself part of the construction. ${}^{(ext)}\mathcal{M}$ is initialized by the GCM hypersurface Σ_* while ${}^{(int)}\mathcal{M}$ is initialized on $\mathcal{T}$ by the foliation induced by ${}^{(ext)}\mathcal{M}$. The main inovation is the GCM sphere S_*, defined and constructed with no reference to the initial data prescribed in the initial data layer $\mathcal{L}_0$.

ric structure including specific choices of null frames and functions such as $r, u, \underline{u}$. These are first defined on Σ_* and then transported to ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$, ${}^{(top)}\mathcal{M}$.

Another important insight in the proof is the separate treatment of the quasi-invariant[50] extreme curvature components $A, \underline{A}$ and all other Ricci and curvature components. In fact the entire hyperbolic character of the EV equations is carried over by $A, \underline{A}$, via the Teukolsky equations they verify, while all other quantities are controlled according to the following:

1. The control of $A, \underline{A}$ and the GCM conditions on Σ_*. This allows us to control all other quantities on Σ_*.

2. The control of all quantities on Σ_*, except A, from their control on Σ_* and the ∇_4 transport equations they verify. It is essential here that the corresponding equations have a triangular structure!

3. The control of all quantities in ${}^{(int)}\mathcal{M}$ using the control of $\underline{A}$ in ${}^{(int)}\mathcal{M}$, the control

[50] i.e. quadratic invariant

of all quantities on $\mathcal{T}$, induced by the control on ${}^{(ext)}\mathcal{M}$, and their ∇_3 transport equations. Once more the triangular structure of these equations important.

4. The control of all quantities in ${}^{(top)}\mathcal{M}$ usuing their control on ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ and 'the 'smallness" of ${}^{(top)}\mathcal{M}$.

1.4.2 Main intermediary results

The proof of Theorem 1.1.1 is divided in nine separate steps, Theorems M0–M8. These steps are briefly described below, see section 3.7 in [K-S:Kerr] for the precise statements:

1. *Theorem M0 (Control of the initial data in the bootstrap gauge).* The smallness of the initial perturbation is given in the frame of the initial data layer $\mathcal{L}_0$. Theorem M0 transfers this control to the bootstrap gauge in the initial data layer.

2. *Theorems M1–M2 (Decay estimates for α (Theorem M1) and $\underline{\alpha}$ (Theorem M2)).* This is achieved using Teukolsky equations and a Chandrasekhar type transform in perturbations of Kerr.

3. *Theorems M3–M5 (Decay estimates for all curvature, connection and metric components).* This is done making use of the GCM conditions on Σ_* as well as the control of α and $\underline{\alpha}$ established in Theorems M1 and M2. The proof proceeds in the following order:

   - Theorem M3 provides the crucial decay estimates on Σ_*,
   - Theorem M4 provides the decay estimates on ${}^{(ext)}\mathcal{M}$,
   - Theorem M5 provides the decay estimates on ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$.

4. *Theorems M6 (Existence of a bootstrap spacetime).* This theorem shows that there exists a GCM admissible spacetime satisfying the bootstrap assumptions, hence initializing the bootstrap procedure.

5. *Theorems M7 (Extension of the bootstrap region).* This theorem shows the existence of a slightly larger GCM admissible spacetime satisfying estimates improving the bootstrap assumptions on decay.

6. *Theorem M8 (Control of the top derivatives estimates).* This is based on an induction argument relative to the number of derivatives, energy-Morawetz estimates and the Maxwell like character of the Bianchi identities.

The paper [K-S:Kerr] provides the proof of Theorem M0, Theorems M3 to M7, and half of Theorem M8 (on the control of Ricci coefficients and metric components). The proof of Theorems M1 and M2, and of the other half of Theorem M8 (on the control of curvature components), based on nonlinear wave equations techniques, are provided in [GKS-2022]. The construction of GCM spheres in [K-S:GCM1] [K-S:GCM2], and of GCM hypersurfaces in [Shen] are used in the proof of Theorems M6 and M7 to construct respectively the terminal GCM sphere S_* and the last slice hypersurface Σ_* from S_*.

1.4.3 Main new ideas of the proof

Here is a short description of the main new ideas in the proof of Theorem 1.1.1 and how they compare with ideas used in other nonlinear results.

GCM admissible spacetimes

- As mentioned already the crucial concept in the proof of Theorem 1.1.1 is that of a GCM admissible spacetime, whose construction is anchored by the GCM sphere S_* in Figure B.1. GCM spheres[51], are codimension 2 compact surfaces, unrelated to the initial conditions, on which specific geometric quantities take Schwarzschildian values (made possible by taking into account the full general covariance of the Einstein vacuum equations). In addition to these extrinsic conditions the sphere S_* is endowed with a choice of "effective[52] isothermal coordinates", (θ, φ) verifying the following properties:

  - The metric on S_* takes the form $g = e^{2\phi} r^2 \big((d\theta)^2 + \sin^2\theta (d\varphi)^2\big)$.
  - The integrals on S_* of the $\ell = 1$ modes[53] $J^{(0)} := \cos\theta$, $J^{(-)} := \sin\theta \sin\varphi$ and $J^{(+)} := \sin\theta\cos\varphi$ vanish identically.

- Given the GCM sphere S_* and the effective isothermal coordinates (θ, φ) on it, our GCM procedure allows us, in particular, to define the mass m, the angular momentum a and a virtual axis of rotation which converge, in the limit, to the final parameters a_f, m_f and the axis of rotation of the final Kerr[54]. We refer the reader

[51] See the discussion in the introductions to [K-S:GCM1], [K-S:GCM2].
[52] This is meant to insure the rigidity of the uniformization map, see [K-S:GCM2].
[53] This is a natural generalization of $\ell = 1$ spherical harmonics.
[54] Previous definitions of the angular momentum in General Relativity were given in [Rizzi], [Chen], [Chen2], see also [Sz] for a comprehensive discussion of the subject.

to section 7.2 in [K-S:GCM2] for our intrinsic definition of a and of the virtual axis of symmetry on a GCM sphere.

- The boundary Σ_*, called a GCM hypersurface, is initialized at S_* and verifies additional conditions. In the polarized setting the first such construction appears in [K-S:Schw]. The general case needed for our theorem is treated in [Shen].
- The concepts of GCM spheres has appeared first in [K-S:Schw] in the context of polarized symmetry. The construction of GCM spheres, without any symmetries, in realistic perturbations of Kerr, is treated in [K-S:GCM1], [K-S:GCM2][55].
- The main novelty of the GCM approach is that it relies on gauge conditions initialized at a far away co-dimension 2 sphere S_*, with no direct reference to the initial conditions. Previously known geometric constructions, such as in [Ch-Kl], [Kl-Ni1] and [Kl-L-R], were based on codimension-1 foliations constructed on spacelike or null hypersurfaces and initialized on the initial hypersurface[56]. Gauge conditions initialized from the future with no direct reference to the initial conditions, which was initiated in [K-S:Schw], have since been used in other works, see [Giorgi] [Graf] [DHRT].
- The GCM construction introduces the following new important conceptual difficulty. The foliation on Σ_*, induced from the far away sphere S_*, needs to be connected, somehow, to the initial conditions (i.e. the initial layer $\mathcal{L}_0$ in Figure B.1). This is achieved in both [K-S:Schw] and [K-S:Kerr] by transporting[57] the sphere S_* to a sphere S_1 in the the initial layer and compare it, using the rigidity properties of the GCM conditions, to a sphere of the initial data layer. This induces a new foliation of the initial layer which differs substantially from the original one, due to a shift of the center of mass frame of the final black hole, known in the physics literature as a gravitational wave recoil[58].

Non integrability of the horizontal structure

As mentioned in section 1.1.1, the canonical horizontal structure induced by the principal null directions (e_3, e_4) in (1.1.4) of Kerr are non integrable. The lack of integrability is

[55]See also chapter 16 of [DHRT] in the particular case of perturbations of Schwarzschild, where the same concept appears instead under the name "teleological".

[56]The first such construction appears in the proof of the nonlinear stability of the Minkowski space [Ch-Kl] where the "inverse lapse foliation" was constructed on the "last slice", initialized at spacelike infinity i^0. Similar constructions, where the last slice is null rather than spacelike, appear in [Kl-Ni1] and [Kl-L-R].

[57]That is, we transport the $\ell = 1$ modes of some quantities from S_* to S_1, see section 8.3.1 in [K-S:Kerr].

[58]We refer the reader to section 8.3 in [K-S:Kerr] for the details.

dealt with by the Newman-Penrose (NP) formalism by general null frames (e_3, e_4, e_1, e_2), with e_1, e_2 a specified basis[59] of the horizontal structure induced by the null pair (e_3, e_4). It thus reduces all calculations to equations involving the Christoffel symbols of the frame, as scalar quantities. This un-geometric feature of the formalism makes it difficult to use it in the nonlinear setting of the Kerr stability problem. Indeed complex calculations depend on higher derivatives of all connection coefficients of the NP frame rather than only those which are geometrically significant. This seriously affects and complicates the structure of non-linear corrections and makes it difficult to avoid artificial gauge type singularities[60]. This difficulty is avoided in [Ch-Kl] by working with a tensorial approach adapted to S-foliations, i.e. $\{e_3, e_4\}^\perp$ coincides, at every point, with the tangent space to S.

In our work we extend, with minimal changes, the tensorial approach introduced in [Ch-Kl] to general non-integrable foliations. The idea is very simple: we define Ricci coefficients $\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \xi, \underline{\xi}, \omega, \underline{\omega}$ exactly as in [Ch-Kl], relative to an arbitrary basis of vectors $(e_a)_{a=1,2}$ of $\mathcal{H} := \{e_3, e_4\}^\perp$. In particular, the null fundamental forms χ and $\underline{\chi}$, are given by

$$\underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_a e_3, e_b), \qquad \chi_{ab} = \mathbf{g}(\mathbf{D}_a e_4, e_b).$$

Due to the lack of integrability of $\mathcal{H}$, the null fundamental forms χ and $\underline{\chi}$ are no longer symmetric. They can be both decomposed as follows

$$\chi_{ab} = \frac{1}{2}tr\,\chi\delta_{ab} + \frac{1}{2} \in_{ab} {}^{(a)}\mathrm{tr}\chi + \widehat{\chi}_{ab}, \qquad \underline{\chi}_{ab} = \frac{1}{2}\mathrm{tr}\,\underline{\chi}\delta_{ab} + \frac{1}{2} \in_{ab} {}^{(a)}\mathrm{tr}\underline{\chi} + \widehat{\underline{\chi}}_{ab},$$

where the new scalars ${}^{(a)}\mathrm{tr}\chi$, ${}^{(a)}\mathrm{tr}\underline{\chi}$ measure the lack of integrability of the horizontal structure. The null curvature components are also defined as in [Ch-Kl],

$$\alpha_{ab} = \mathbf{R}_{a4b4}, \quad \beta_a = \frac{1}{2}\mathbf{R}_{a434}, \quad \underline{\beta}_a = \frac{1}{2}\mathbf{R}_{a334}, \quad \underline{\alpha}_{ab} = \mathbf{R}_{a3b3}, \quad \rho = \frac{1}{4}\mathbf{R}_{3434}, \quad {}^*\rho = \frac{1}{4}\,{}^*\mathbf{R}_{3434}.$$

The null structure and null Bianchi equations can then be derived as in the integrable case, see chapter 7 in [Ch-Kl]. The only new features are the presence of the scalars ${}^{(a)}\mathrm{tr}\chi$, ${}^{(a)}\mathrm{tr}\underline{\chi}$ in the equations. Finally we note that the equations acquire additional simplicity if we pass to complex notations[61],

$$\begin{aligned} &A := \alpha + i\,{}^*\alpha, \quad B := \beta + i\,{}^*\beta, \quad P := \rho + i\,{}^*\rho, \quad \underline{B} := \underline{\beta} + i\,{}^*\underline{\beta}, \quad \underline{A} := \underline{\alpha} + i\,{}^*\underline{\alpha}, \\ &X := \chi + i\,{}^*\chi, \quad \underline{X} := \underline{\chi} + i\,{}^*\underline{\chi}, \quad H := \eta + i\,{}^*\eta, \quad \underline{H} := \underline{\eta} + i\,{}^*\underline{\eta}, \quad Z := \zeta + i\,{}^*\zeta. \end{aligned} \tag{1.4.1}$$

[59]Or rather the complexified vectors $m = e_1 + ie_2$ and $\overline{m} = e_1 - ie_2$.

[60]There are no smooth, global choices of a basis (e_1, e_2). The choice (1.1.5) in Kerr, for example, is singular at $\theta = 0, \pi$.

[61]The dual here is taken with respect to the antisymmetric horizontal 2-tensor $\in_{ab}$.

Frame transformations and choice of frames

Given an arbitrary perturbation of Kerr, there is no a-priori reason to prefer an horizontal structure to any other one obtained from the first by another perturbation of the same size. It is thus essential that we consider all possible frame transformations from one horizontal structure $(e_4, e_3, \mathcal{H})$ to another one $(e_4', e_3', \mathcal{H}')$ together with the transformation formulas $\Gamma \to \Gamma'$, $R \to R'$ they generate. The most general transformation formulas between two null frames is given in Lemma 3.1 of [K-S:GCM1]. It depends on two horizontal 1-forms $f, \underline{f}$ and a real scalar function λ and is given by

$$
\begin{aligned}
e_4' &= \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right),\\
e_a' &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2} f_a + \frac{1}{8}|f|^2 \underline{f}_a\right) e_3,\\
e_3' &= \lambda^{-1}\left(\left(1 + \frac{1}{2} f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b + \frac{1}{4}|\underline{f}|^2 e_4\right).
\end{aligned}
\tag{1.4.2}
$$

The very important transformation formulas $\Gamma \to \Gamma'$, $R \to R'$ are given in Proposition 3.3 of [K-S:GCM1].

Definition 1.4.1. *A spacetime $\mathcal{M}$, endowed with an horizontal structure $(e_3, e_4, \mathcal{H})$ is said to be an $O(\epsilon)$ perturbation of Kerr if all quantities which vanish in Kerr are $O(\epsilon)$, and if all other quantities stay bounded in an $O(\epsilon)$ neighborhood of their corresponding*[62] *Kerr values.*

The definition is, of course, ambiguous in the sense that any other horizontal structure $(e_3', e_4', \mathcal{H}')$ connected to $(e_3, e_4, \mathcal{H})$ by the frame transformation (1.4.2) with $f, \underline{f} = O(\epsilon)$ and $\lambda = 1 + O(\epsilon)$ is also an $O(\epsilon)$-perturbation of Kerr. Nevertheless the definition is useful in that it brings to light the remarkable fact that the extreme curvature components are in fact $O(\epsilon^2)$ invariant. This can be easily seen from the transformation formulas

$$
\begin{aligned}
\lambda^{-2}\alpha' &= \alpha + \left(f\widehat{\otimes}\beta - {}^*f\widehat{\otimes}\,{}^*\beta\right) + \left(f\widehat{\otimes} f - \frac{1}{2}\,{}^*f\widehat{\otimes}\,{}^*f\right)\rho + \frac{3}{2}\left(f\widehat{\otimes}\,{}^*f\right)\,{}^*\rho + O(\epsilon^3),\\
\lambda^{2}\underline{\alpha}' &= \underline{\alpha} + \left(\underline{f}\widehat{\otimes}\underline{\beta} - {}^*\underline{f}\widehat{\otimes}\,{}^*\underline{\beta}\right) + \left(\underline{f}\widehat{\otimes}\underline{f} - \frac{1}{2}\,{}^*\underline{f}\widehat{\otimes}\,{}^*\underline{f}\right)\rho + \frac{3}{2}\left(\underline{f}\widehat{\otimes}\,{}^*\underline{f}\right)\,{}^*\rho + O(\epsilon^3),
\end{aligned}
$$

see Proposition 2.2.3 of [K-S:Kerr].

[62] To make this precise, we also need a definition of functions (r, θ) and of a complex 1-form $\mathfrak{J}$, see section **??**.

Remark 1.4.2. *It is this fact that allows us to treat $\alpha, \underline{\alpha}$ differently from all other quantities. In addition to being less sensitive to frame transformations they do also verify wave equations, the Teukolsky equations, which decouple, in linear theory, from all other curvature components. See further discussion below.*

The case of $\mathcal{K}(a,m)$, $a \neq 0$ presents an interesting new feature which can be described as follows:

- To capture the simplicity induced by the principle null directions in Kerr it is natural to work with non-integrable frames. We do in fact define all our main quantities relative to frames for which all quantities which vanish in Kerr are of the size of the perturbation.
- A crucial aspect of all important results in GR, based on integrable S- foliations, is that one can rely on elliptic Hodge theory on each 2-surface S. This is no longer possible in context where our main quantities and the basic equations they verify are defined relative to non integrable frames. In our work we deal with this problem by passing back and forth, whenever needed, from the main non-integrable frame to a well chosen adapted integrable frame, according to the transformation formulas mentioned above.

Renormalization procedure and the canonical complex 1-form $\mathfrak{J}$

We first notice that our main complex quantities introduced in (1.4.1) take a particularly simple form in the principal null frame (1.1.4) of Kerr:

$$\begin{aligned} &A = \underline{A} = B = \underline{B} = 0, \qquad P = -\frac{2m}{q^3},\\ &\widehat{X} = \widehat{\underline{X}} = 0, \qquad \mathrm{tr}X = \frac{2}{q}\frac{\Delta}{|q|^2}, \qquad \mathrm{tr}\underline{X} = -\frac{2}{\overline{q}},\\ &Z = \frac{aq}{|q|^2}\mathfrak{J}, \qquad H = \frac{aq}{|q|^2}\mathfrak{J}, \qquad \underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}, \end{aligned} \tag{1.4.3}$$

where $q = r + ia\cos\theta$, and where the regular[63] complex 1-form $\mathfrak{J}$ is given by

$$\mathfrak{J}_1 = \frac{i\sin\theta}{|q|}, \qquad \mathfrak{J}_2 = \frac{\sin\theta}{|q|}, \tag{1.4.4}$$

see sections 2.4.2 and 2.4.3 in [K-S:Kerr]. In particular, the following holds for the complexified horizontal tensors of (1.4.1) in the principal null frame (1.1.4) of Kerr:

[63]Note that $\mathfrak{J}$ is regular including at $\theta = 0, \pi$.

- the complex scalars P, $\mathrm{tr}X$ and $\mathrm{tr}\underline{X}$ are functions of r and $\cos\theta$,
- the non vanishing complex 1-forms H, $\underline{H}$ and Z consist of functions of r and $\cos\theta$ multiplied by $\mathfrak{J}$,
- the traceless symmetric complex 2-tensors A, $\underline{A}$, $\widehat{X}$ and $\widehat{\underline{X}}$ vanish identically.

Based on that observation, for a given horizontal structure perturbing the one of Kerr, we can define a renormalization procedure by which, once we have[64] suitable constants (a, m), suitable scalar functions (r, θ), and a suitable complex 1-form $\mathfrak{J}$, and after subtracting the corresponding values in Kerr computed from $(a, m, r, \theta, \mathfrak{J})$ for all the Ricci and curvature coefficients, we obtain quantities which are first order in the perturbation.

More precisely, once (a, m), (r, θ) and $\mathfrak{J}$ have been chosen, we renormalize the quantities in (1.4.1) that do not vanish in Kerr as follows[65]:

$$\begin{aligned}
\check{P} &:= P + \frac{2m}{q^3}, \qquad \widecheck{\mathrm{tr}X} := \mathrm{tr}X - \frac{2}{q}\frac{\Delta}{|q|^2}, \qquad \widecheck{\mathrm{tr}\underline{X}} := \mathrm{tr}\underline{X} + \frac{2}{\overline{q}},\\
\check{Z} &:= Z - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \check{H} := H - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \widecheck{\underline{H}} := \underline{H} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}.
\end{aligned} \tag{1.4.5}$$

Principal Geodesic and Principal Temporal structures

In addition to the GCM gauge conditions on Σ_*, we need to construct a gauge on $\mathcal{M}$ which relates the non integrable horizontal structure to the scalars (r, θ) and the complex 1-form $\mathfrak{J}$. Two such gauges were introduced in [K-S:Kerr]:

- Principal Geodesic (PG) structure, which is a generalization of the geodesic foliation to non-integrable horizontal structures,
- Principal Temporal (PT) structure, which favors transport equations along a null direction.

The PG structure[66] is well suited for decay estimates, but fails to be well posed. Indeed, due to the lack of integrability of the horizontal structure, we cannot control the null

[64] The constants m and a are computed on our GCM sphere S_*, see section 1.4.3. r, θ and $\mathfrak{J}$ are chosen on S_*, transported to Σ_* and then to $\mathcal{M}$. The horizontal structure is also defined first on Σ_* and then transported to $\mathcal{M}$.

[65] The renormalization is written here in the case of a null pair (e_3, e_4) with an ingoing normalization.

[66] Note that, in the integrable context of [K-S:Schw], the PG structure coincides with the standard (integrable) geodesic foliation used there. Thus the PG structure, defined in [K-S:Kerr], is a suitable generalization to the non-integrable case of perturbations of Kerr.

structure equations[67] without a loss of derivative. The PT structure, on the other hand, is designed so that the loss of derivatives in the null structure equations, in the incoming or outgoing direction, is completely avoided. Note however that the PT structure is not well suited to the derivation of decay estimates on ${}^{(ext)}\mathcal{M}$ where r can take arbitrary large values. In [K-S:Kerr] we work with both gauge conditions, depending on the goal we want to achieve, and rely on the transformation formulas (1.4.2) to pass from one to the other.

In the outgoing normalization both the outgoing PG and PT structures consist of a choice $(e_3, e_4, \mathcal{H})$, with e_4 null geodesic, together with a scalar functions r, θ and a complex 1-form $\mathfrak{J}$ such that $e_4(r) = 1$, $e_4(\theta) = 0$, $\nabla_4(q\mathfrak{J}) = 0$ where $q = r + ia\cos\theta$. In addition:

1. In a PG structure the gradient of r, given by $N = \mathbf{g}^{\alpha\beta}\partial_\beta r\partial_\alpha$, is perpendicular to $\mathcal{H}$,
2. In a PT structure $\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}$, i.e. $\check{\underline{H}} = 0$ in view of (2.1.1).

A similar definition of incoming PG and PT structures is obtained by interchanging the roles of e_3, e_4. Note that both structures still need to be initialized. The outgoing PG and PT structures of ${}^{(ext)}\mathcal{M}$ are both initialized on Σ_* from the GCM frame of Σ_*, while the ingoing PT structures of ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$ are initialized on the the timelike hypersurface $\mathcal{T}$, see Figure B.1, using the data induced by the outgoing structures.

Control of the extreme curvature components $A, \underline{A}$

It was already observed by Teukolsky that, in linear theory, the extreme components of the curvature are both gauge invariant and verify decoupled wave equations[68]. In our nonlinear context this translates to the statement that the horizontal 2-tensors $A, \underline{A}$, defined relative to an $O(\epsilon)$ perturbation of the principal frame of Kerr, are $O(\epsilon^2)$-invariant, relative to $O(\epsilon)$ frame transformations[69], and verify tensorial wave equations of the form

$$\dot{\square}_2 A + L[A] \;=\; \mathrm{Err}(\check{\Gamma}, \check{R}), \qquad \dot{\square}_2 \underline{A} + \underline{L}\,[\underline{A}] = \underline{\mathrm{Err}}(\check{\Gamma}, \check{R}). \tag{1.4.6}$$

Here $\dot{\square}_2$ denotes the wave operator on horizontal symmetric traceless 2-tensors, L and $\underline{L}$ are linear first order operators and $\check{\Gamma}, \check{R}$ denote the linearized Ricci and curvature coefficients. The error terms $\mathrm{Err}(\check{\Gamma}, \check{R})$, $\underline{\mathrm{Err}}(\check{\Gamma}, \check{R})$ are nonlinear expressions in $\check{\Gamma}, \check{R}$.

[67]In integrable situation, like in the case of S-foliations, the Hodge systems on the leaves of the S-foliation allows us to avoid the loss.

[68]See discussion in section 1.2.5.

[69]This means that $f, \underline{f}$, $\lambda - 1$ are $O(\epsilon)$ in the transformation formulas (1.4.2).

In linear theory, i.e. when $\mathbf{g}$ is the Kerr metric and the error terms are not present, these equations have been treated by [DHR] in Schwarzschild[70] and by [Ma] and [DHR-Kerr] in slowly rotating[71] Kerr, i.e. $|a|/m \ll 1$. More precisely both results derive realistic decay estimates for $A, \underline{A}$. The methods are however not robust. Indeed, a crucial ingredient in the proof, the Energy-Morawetz estimates, is based on separation of variables. The control of A and $\underline{A}$ in perturbations of Kerr in [GKS-2022] contains the following new features:

- *Derivation of the gRW equation.* The derivation of the generalized Regge-Wheeler equations in Kerr, in [Ma] and [DHR-Kerr], is done starting with the complex, scalar, Teukolsky equations, derived via the NP, or GHP formalism, by applying a Chandrasekhar type transformation. In part I of [GKS-2022] we extend their derivation, using our non-integrable horizontal formalism, to perturbations of Kerr. By contrast with [Ma], [DHR-Kerr], we derive gRW equations for the horizontal symmetric traceless 2-tensors[72] $\mathfrak{q}, \underline{\mathfrak{q}}$, rather than for complex scalars. The main difficulty here is to make sure that the non-linear error terms verify a favorable structure.
- *Nonlinear error terms.* The control of the nonlinear terms and their associated null structure was already understood in perturbations of Schwarzschild in [K-S:Schw] and is extended to perturbations of Kerr in [GKS-2022].
- *Energy-Morawetz.* To derive energy-morawetz estimates for $A, \underline{A}$ in Part II of [GKS-2022] we vastly extend the pioneering idea of Andersson and Blue [A-B], based on commutations with $\mathbf{T}, \mathbf{Z}$ and the second order Carter operator $\mathcal{C}$, developed in the context of the scalar wave equation in slowly rotating Kerr, to treat our tensorial Teukolsky and gRW equations in perturbations of Kerr.

Comments on the full sub-extremal range

Though the full sub-extremal range $|a| < m$ remains open we remark that a large part of our work does not require the smallness of $|a|/m$. This is the case for [K-S:GCM1] [K-S:GCM2] [Shen] and [K-S:Kerr]. In fact the smallness assumption is only needed in [GKS-2022], mostly in the derivation of the main Energy-Morawetz estimates in parts II and III.

[70] See discussion in section 1.3.3.
[71] See discussion in section 1.3.4.
[72] Derived from $A, \underline{A}$, see Definition 5.2.2 and 5.3.3 in [GKS-2022].

Chapter 2

Introduction III.

In these lectures I will concentrate on the results proved in [GKS-2022] more precisely on the proof of Theorems M1 and M2 as well the curvature estimates of Theorem M8, which were stated without proof in sections 3.7.1 and 9.4.7 of [K-S:Kerr].

2.1 Geometric set-up

2.1.1 Spacetime $\mathcal{M}$

The geometric setting of our work consists of an Einstein vacuum Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ with boundaries equipped with the following:

1. A regular horizontal structure defined by a null pair (e_3, e_4), and the space $\mathcal{H}$ orthogonal to it. Note that the horizontal structure considered here is not integrable[1]. The formalism of non-integrable horizontal structures, on which of our entire work is based, is developed in full in Chapter 2 of [GKS-2022].

2. Two constants (a, m) with $|a| < m$, two scalar functions (r, θ) and a time function τ on $\mathcal{M}$. In addition, $\mathcal{M}$ possesses a horizontal complex 1-form[2] $\mathfrak{J}$, used to linearize all horizontal 1-forms in perturbations of Kerr.

[1] In other words, the space $\mathcal{H}$ forms a non integrable distribution. The formalism was originally mentioned in [I-Kl] and developed in [GKS-2020].

[2] By this, we mean $\mathfrak{J} = j + i\,{}^*j$ where j is a real horizontal 1-form. In Kerr this quantity is specifically introduced in Definition 5.2.4.

3. Boundaries given by $\partial\mathcal{M} = \mathcal{A} \cup \Sigma(\tau_*) \cup \Sigma_* \cup \Sigma(1)$ where
   - $\mathcal{A}$ is the spacelike hypersurface given by
   $$\mathcal{A} \;\; := \;\; \mathcal{M} \cap \{r = r_+(1-\delta_{\mathcal{H}})\}, \qquad r_+ := m + \sqrt{m^2 - a^2},$$
   where $\delta_{\mathcal{H}} > 0$ a sufficiently small constant.
   - $\Sigma(1)$ and $\Sigma(\tau_*)$ denote the spacelike level hypersurfaces $\tau = 1$ and $\tau = \tau_*$, with $\tau_* > 1$ and $1 \leq \tau \leq \tau_*$ on $\mathcal{M}$.
   - Σ_* is a uniformly spacelike hypersurface connecting $\Sigma(1)$ to $\Sigma(\tau_*)$.
4. Two spacetime regions ${}^{(int)}\mathcal{M}$ and ${}^{(ext)}\mathcal{M}$ such that
$$\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}, \qquad {}^{(ext)}\mathcal{M} = \mathcal{M}_{r \geq r_0}, \qquad {}^{(int)}\mathcal{M} = \mathcal{M}_{r \leq r_0},$$
where $r_0 \gg m$ is a sufficiently large constant.

Remark 2.1.1. *Note that the spacetime $\mathcal{M}$ considered above does not require any specific gauge conditions. Indeed, in this paper, we only provide gauge independent curvature estimates. The control of Ricci coefficients is provided in [K-S:Kerr] where specific gauge choices are made, see section 2.3 and 2.8 for the definitions of PG and PT structures in [K-S:Kerr]. We also note that the scalar functions r, θ and τ are not aligned with the frame, i.e. unlike in the stability of Minkowski space, in [Ch-Kl], and all other subsequent works*[3]*, our frames are in no way adapted to foliations.*

The function τ is used to define the regions of integrations $\mathcal{M}(\tau_1, \tau_2)$ where $\tau_1 \leq \tau \leq \tau_2$. We also define the following significant regions of $\mathcal{M}$, see Definition **??**.

Definition 2.1.2. *We define the following regions of $\mathcal{M}$:*

1. *We define the trapping region of $\mathcal{M}$ to be the set*
$$\mathcal{M}_{trap} := \mathcal{M} \cap \left\{ \frac{|\mathcal{T}|}{r^3} \leq \delta_{trap} \right\}, \qquad \delta_{trap} = \frac{1}{10},$$
where $\mathcal{T} = \mathcal{T} = r^3 - 3mr^2 + a^2 r + ma^2$. This is the region that contains all trapped null geodesics, for sufficiently small a/m.
2. *We denote $\mathcal{M}_{tr\not{a}p}$ the complement to the trapping region $\mathcal{M}_{trap}$.*
3. *We denote $\mathcal{M}_{red} := \mathcal{M} \cap \big\{ r \leq r_+(1 + 2\delta_{red}) \big\}$, for a sufficiently small constant $\delta_{red} > 0$, the region where the red shift effect of the horizon is manifest.*

[3]We note however that in the treatment of the Regge Wheeler equation in Chapter 10 of [K-S:Schw] the foliations used are also not aligned with the frame.

2.1.2 Ricci and curvature coefficients

Definition of the Ricci and curvature coefficients

We can define, with respect to the horizontal structure associated to (e_3, e_4), connection and curvature coefficients similar to those in the integrable case, as in [Ch-Kl],

$$\begin{aligned}
&\underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_a e_3, e_b), \qquad \chi_{ab} = \mathbf{g}(\mathbf{D}_a e_4, e_b), \qquad \underline{\xi}_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_a), \qquad \xi_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_a),\\
&\underline{\omega} = \frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_3, e_4), \qquad \omega = \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3), \qquad \underline{\eta}_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_a), \qquad \eta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_a),\\
&\zeta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_a e_4, e_3),
\end{aligned}$$

$$\alpha_{ab} = \mathbf{R}_{a4b4}, \quad \beta_a = \frac{1}{2}\mathbf{R}_{a434}, \quad \underline{\beta}_a = \frac{1}{2}\mathbf{R}_{a334}, \quad \underline{\alpha}_{ab} = \mathbf{R}_{a3b3}, \quad \rho = \frac{1}{4}\mathbf{R}_{3434}, \qquad {}^*\rho = \frac{1}{4}\,{}^*\mathbf{R}_{3434},$$

and derive the corresponding null structure and null Bianchi equations. The non-symmetric 2 tensors $\chi, \underline{\chi}$ are decomposed as follows.

$$\chi_{ab} \;=\; \widehat{\chi}_{ab} + \frac{1}{2}\delta_{ab}\mathrm{tr}\,\chi + \frac{1}{2}\in_{ab}\ {}^{(a)}\mathrm{tr}\chi, \qquad \underline{\chi}_{ab} = \widehat{\underline{\chi}}_{ab} + \frac{1}{2}\delta_{ab}\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\in_{ab}\ {}^{(a)}\mathrm{tr}\underline{\chi},$$

where the scalars $\mathrm{tr}\,\chi$, $\mathrm{tr}\,\underline{\chi}$ and ${}^{(a)}\mathrm{tr}\chi$, ${}^{(a)}\mathrm{tr}\underline{\chi}$ are given by

$$\mathrm{tr}\,\chi := \delta^{ab}\chi_{ab}, \qquad \mathrm{tr}\,\underline{\chi} := \delta^{ab}\underline{\chi}_{ab}, \qquad {}^{(a)}\mathrm{tr}\chi := \in^{ab}\chi_{ab}, \qquad {}^{(a)}\mathrm{tr}\underline{\chi} := \in^{ab}\underline{\chi}_{ab}.$$

Remark 2.1.3. *The non integrability of (e_3, e_4) corresponds to the non vanishing ${}^{(a)}tr\chi$ and ${}^{(a)}tr\underline{\chi}$. A well known example of a non integrable null frame, is the principal null frame of Kerr for which ${}^{(a)}tr\chi$ and ${}^{(a)}tr\underline{\chi}$ are indeed non trivial, see section 5.2.*

2.1.3 Basic equations and complexification

The null structure and null Bianchi equations verified by the Ricci and curvature coefficients are derived in sections 2.2. These equations simplify considerably, see section 2.4, by introducing complex notations:

$$\begin{aligned}
&A := \alpha + i\,{}^*\alpha, \quad B := \beta + i\,{}^*\beta, \quad P := \rho + i\,{}^*\rho, \quad \underline{B} := \underline{\beta} + i\,{}^*\underline{\beta}, \quad \underline{A} := \underline{\alpha} + i\,{}^*\underline{\alpha},\\
&X := \chi + i\,{}^*\chi, \quad \underline{X} := \underline{\chi} + i\,{}^*\underline{\chi}, \quad H := \eta + i\,{}^*\eta, \quad \underline{H} := \underline{\eta} + i\,{}^*\underline{\eta}, \quad Z := \zeta + i\,{}^*\zeta,\\
&\Xi := \xi + i\,{}^*\xi, \quad \underline{\Xi} := \underline{\xi} + i\,{}^*\underline{\xi},
\end{aligned}$$

where * denotes the Hodge dual. In particular, note that $\mathrm{tr}X = \mathrm{tr}\,\chi - i\,{}^{(a)}\mathrm{tr}\chi$, $\mathrm{tr}\underline{X} = \mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi}$, while $\widehat{X}$ and $\widehat{\underline{X}}$ denote the symmetric traceless part of X and $\underline{X}$ respectively. Further useful simplifications of the equations can be obtained with the help of conformally invariant derivative operators introduced in section 2.2.9.

$$
\begin{aligned}
\check{P} &:= P + \frac{2m}{q^3}, \qquad \widecheck{\mathrm{tr}X} := \mathrm{tr}X - \frac{2}{q}\frac{\Delta}{|q|^2}, \qquad \widecheck{\mathrm{tr}\underline{X}} := \mathrm{tr}\underline{X} + \frac{2}{\overline{q}}, \\
\check{Z} &:= Z - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \check{H} := H - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \check{\underline{H}} := \underline{H} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}.
\end{aligned}
\tag{2.1.1}
$$

Notation (Γ_g, Γ_b) for Ricci coefficients

We group the linearized Ricci coefficients in two subsets reflecting their expected decay properties, see section 4.1.2 [GKS-2022]:

$$
\begin{aligned}
\Gamma_g &:= \Big\{ \widecheck{\mathrm{tr}X}, \quad \widehat{X}, \quad \widecheck{\mathrm{tr}\underline{X}}, \quad \check{\underline{H}}, \quad \check{Z}, \quad \check{\omega}, \quad \Xi \Big\}, \\
\Gamma_b &:= \Big\{ \widehat{\underline{X}}, \quad \check{H}, \quad \underline{\omega}, \quad \underline{\Xi} \Big\}.
\end{aligned}
$$

Remark 2.1.4. *In fact, (Γ_g, Γ_b) also include the linearization of the derivatives of the scalar functions $(r, \cos\theta)$, and of the complex horizontal 1-form $\mathfrak{J}$, see section 4.1.2.*

The justification for the above decompositions has to do with the expected decay properties of the linearized components in perturbations of Kerr, with respect to τ and r. See discussion in section 2.2.3 below.

More precisely,

$$
\begin{aligned}
\big|\mathfrak{d}^{\le s}\Gamma_g\big| &\lesssim \epsilon \min\Big\{ r^{-2}\tau^{-1/2-\delta_{dec}},\ r^{-1}\tau^{-1-\delta_{dec}} \Big\}, \\
\big|\mathfrak{d}^{\le s}\Gamma_b\big| &\lesssim \epsilon r^{-1}\tau^{-1-\delta_{dec}},
\end{aligned}
\tag{2.1.2}
$$

for a small constant $\delta_{dec} > 0$, where $\mathfrak{d} = \{\nabla_3, r\nabla_4, r\nabla\}$ denotes weighted derivatives, and $\epsilon > 0$ is a sufficiently small bootstrap constant. We note also that the curvature components $\underline{A}, r\underline{B}$ behave in the same way as Γ_b, while $r(\check{P}, B, A)$ behave like Γ_g. Moreover A, B get the optimal decay in powers of r, i.e.

$$
|A|, |B| \lesssim \epsilon r^{-7/2-\delta_{dec}}.
$$

2.2 Main theorems

We refer to section 3.4 of [K-S:Kerr] for a precise statement of our Main Theorem concerning the stability of Kerr and to section 3.7 of [K-S:Kerr] the main steps in the proof. Here we concentrate on a simplified set of assumptions needed for the proof of Theorems M1, M2 and the curvature estimates for Theorem M8.

2.2.1 Smallness constants

The following constants are involved in the statement of Theorems M0-M8, see section 3.4. in [K-S:Kerr]:

- The constants $m_0 > 0$ and $|a_0| \ll m_0$ are the mass and the angular momentum of the Kerr solution relative to which our initial perturbation is measured.
- The integer k_{large} which corresponds to the maximum number of derivatives of the solution.
- The size of the initial data perturbation is measured by $\epsilon_0 > 0$.
- The size of the bootstrap assumption norms are measured by $\epsilon > 0$.
- $r_0 > 0$ is tied to ${}^{(int)}\mathcal{M} \cap {}^{(ext)}\mathcal{M} = \{r = r_0\}$.
- The constant $\delta_{\mathcal{H}}$ tied to the definition of $\mathcal{A} = \{r = r_+(1-\delta_{\mathcal{H}})\}$.
- δ_{dec} is tied to decay estimates in τ for the linearized quantities of section **??**.

These constants are chosen such that

$$\begin{aligned} & 0 < \delta_{\mathcal{H}},\ \delta_{dec} \ \ll \min\{m_0 - |a_0|, 1\}, \\ & r_0 \gg \max\{m_0, 1\}, \qquad k_{large} \gg \frac{1}{\delta_{dec}}. \end{aligned} \tag{2.2.1}$$

Then, ϵ and ϵ_0 are chosen such that

$$0 < \epsilon_0, \epsilon \ll \min\left\{\delta_{dec}, \frac{1}{r_0}, \frac{1}{k_{large}}, m_0 - |a_0|, 1\right\}, \tag{2.2.2}$$

$$\epsilon_0, \epsilon \ll |a_0| \quad \text{in the case } a_0 \neq 0, \tag{2.2.3}$$

and

$$\epsilon = \epsilon_0^{\frac{2}{3}}. \tag{2.2.4}$$

Also, we introduce the integer k_{small} which corresponds to the number of derivatives for which the solution satisfies decay estimates. It is related to k_{large} by

$$k_{small} = \left\lfloor \frac{1}{2} k_{large} \right\rfloor + 1. \tag{2.2.5}$$

From now on, in the rest of the paper, $\lesssim$ means bounded by a constant depending only on geometric universal constants (such as Sobolev embeddings, elliptic estimates,...) as well as the constants

$$m_0,\ a_0,\ \delta_{\mathcal{H}},\ \delta_{dec},\ r_0,\ k_{large},$$

but not on ϵ and ϵ_0.

2.2.2 Initial data assumptions

The initial data norm denoted by $\mathfrak{I}_k$, measures the size of the perturbation from Kerr at $\tau = 1$, for the top k derivatives of the curvature tensor[4].

Definition 2.2.1. *We define the following initial data norms on* Σ_1

$$\begin{aligned} \mathfrak{I}_k := \sup_{S \subset \Sigma_1} r^{\frac{5}{2}+\delta_B} \Big(\big\| \mathfrak{d}^k (A, B) \big\|_{L^2(S)} + \big\| \mathfrak{d}^k B \big\|_{L^2(S)} \Big) \\ + \sup_{S \subset \Sigma_1} \Big(r^2 \big\| \mathfrak{d}^k \check{P} \big\|_{L^2(S)} + r \big\| \mathfrak{d}^k \underline{B} \big\|_{L^2(S)} + \big\| \mathfrak{d}^k \underline{A} \big\|_{L^2(S)} \Big). \end{aligned} \tag{2.2.6}$$

In [GKS-2022] we make the following assumption on the control of the initial data norm[5]

$$\mathfrak{I}_{k_{large}+7} \leq \epsilon_0. \tag{2.2.7}$$

The bound (2.2.7) will be used both in Part II and Part III as assumptions on the initial data.

[4]The definition used here differs slightly from the one in Definition 9.4.9 in [K-S:Kerr], but easily follows from it by a local existence argument.

[5]The original assumption on initial data in [K-S:Kerr] is stated for $k_{large} + 10$ derivatives, see (3.4.7) in that paper, in a given frame of an initial data layer $\mathcal{L}(a_0, m_0)$. The control in the frames used in this paper are obtained in Theorem M0 of section 3.7.1 in [K-S:Kerr], and in Theorem 9.4.12 in [K-S:Kerr] for $k_{large} + 7$ derivatives.

2.2.3 Quantitative assumptions on the spacetime $\mathcal{M}$

The quantitative assumptions made in this article depend on a large positive integer k_L, representing the maximal number of derivatives for the linearized Ricci and curvature coefficients $(\check{\Gamma}, \check{R})$ which are required in the proof. There are in fact two types of assumptions:

1. For the proof of Theorem M1 and M2 of [K-S:Kerr], we rely on the following pointwise quantitative assumptions on Γ_b and Γ_g, for $k \leq k_L$,
$$\begin{aligned}\Big(r^2\tau^{\frac{1}{2}+\delta_{dec}} + r\tau^{1+\delta_{dec}}\Big)|\mathfrak{d}^{\leq k}\Gamma_g| &\leq \epsilon,\\ r\tau^{1+\delta_{dec}}|\mathfrak{d}^{\leq k}\Gamma_b| &\leq \epsilon.\end{aligned} \tag{2.2.8}$$
for a small constant $\delta_{dec} > 0$, where $\mathfrak{d} = \{\nabla_3, r\nabla_4, r\nabla\}$ denotes weighted derivatives, and $\epsilon > 0$ is a sufficiently small bootstrap constant.

2. For the proof of the curvature estimates of Theorem M8 of [K-S:Kerr], we introduce weighted energy-Morawetz type norms for curvature and Ricci coefficients, denoted respectively by $\mathfrak{R}_k$ and $\mathfrak{G}_k$, see section 13.5 for the precise definition. We then rely on the following quantitative assumptions on $\mathfrak{R}_k$ and $\mathfrak{G}_k$
$$\mathfrak{R}_k + \mathfrak{G}_k \quad \leq \epsilon, \qquad 0 \leq k \leq k_L, \tag{2.2.9}$$
as well as the following pointwise quantitative assumptions on Γ_b and Γ_g
$$r^2|\mathfrak{d}^k\Gamma_g| + r|\mathfrak{d}^k\Gamma_b| \quad \leq \frac{\epsilon}{\tau_{trap}^{1+\delta_{dec}}}, \qquad 0 \leq k \leq \frac{k_L}{2}, \tag{2.2.10}$$
where the scalar function τ_{trap} defined by
$$\tau_{trap} := \begin{cases} 1+\tau & \text{on} \quad \mathcal{M}_{trap},\\ 1 & \text{on} \quad \mathcal{M}_{\not{trap}}.\end{cases}$$

The integer k_L is chosen as follows:

- For the proof of Theorem M1 and M2 of [K-S:Kerr] (restated in Theorem 2.2.2 and 2.2.3 below), we choose $k_L = k_{small}+120$. Then, (2.2.8) follows by interpolation from the bootstrap assumptions (3.5.1) (3.5.2) in [K-S:Kerr] together with the construction of the global frame in section 3.6.3 of [K-S:Kerr], where (3.5.1) in [K-S:Kerr] are bootstrap assumptions on boundedness for $k \leq k_{large}$ derivatives, and (3.5.2) in [K-S:Kerr] are bootstrap assumptions on decay for $k \leq k_{small}$ derivatives.

- For the proof of the curvature estimates of Theorem M8 (see Theorem 2.2.4 below), we choose $k_L = k_{large} + 7$. Then, (2.2.9) follows from the bootstrap assumptions (9.4.20) of [K-S:Kerr] together with the construction of the global frame in section 9.4 of [K-S:Kerr]. Also, (2.2.10) is a non sharp consequence of the bootstrap assumptions (9.4.22) in [K-S:Kerr] together with the construction of the global frame in section 9.4 of [K-S:Kerr].

2.2.4 Statement of the main theorems

Recall that the nonlinear stability of the Kerr family for small angular momentum, i.e $|a|/m \ll 1$, is stated in the Main Theorem in section 3.4 of [K-S:Kerr]. The proof is divided in a sequence of nine intermediary steps, called Theorem M0–M8, see section 3.7 in [K-S:Kerr]. The goal of the present paper is to provide the proof of Theorems M1 and M2 as well the curvature estimates of Theorem M8, which were stated without proof in Theorem 9.4.15 of [K-S:Kerr] and all involve curvature estimates of hyperbolic type.

Theorems M1 and M2

In what follows, we restate[6] Theorem M1 and M2, see section 3.7.1 in [K-S:Kerr].

Theorem 2.2.2 (Theorem M1 in [K-S:Kerr])**.** *Assume that the spacetime $\mathcal{M}$ as defined in section 2.1.1 verifies the quantitative assumptions* (2.2.8)*, and the assumption* (2.2.7) *on initial data. Then, if $\epsilon_0 > 0$ is sufficiently small, there exists $\delta_{extra} > \delta_{dec}$ such that we have the following estimates in $\mathcal{M}$, for all $k \le k_L - 20$,*

$$\sup_{\mathcal{M}} \Big(r^2\tau^{1+\delta_{extra}} + r^3(2r+\tau)^{\frac{1}{2}+\delta_{extra}}\Big)\Big(|\mathfrak{d}^k A| + r|\mathfrak{d}^{k-1}\nabla_3 A|\Big) \lesssim \epsilon_0.$$

Also, the quantity $\mathfrak{q}$ introduced below, see section 2.3.1, satisfies, for all $k \le k_L - 20$,

$$\int_{\Sigma_*(\ge\tau)} |\nabla_3\mathfrak{d}^{k-1}\mathfrak{q}|^2 \quad\lesssim\quad \epsilon_0^2\tau^{-2-2\delta_{extra}}.$$

Theorem 2.2.3 (Theorem M2 in [K-S:Kerr])**.** *In addition to the assumptions of Theorem 2.2.2, we make the following assumption[7] on Σ_**

$$\min_{\Sigma_*} r \ge \delta_*\epsilon_0^{-1}\tau_*^{1+\delta_{dec}} \tag{2.2.11}$$

[6] A more precise statement is given in Theorems **??** and **??**.

[7] This is the dominant condition of r on Σ_*, see (3.4.5) in [K-S:Kerr].

for some small universal constant $\delta_* > 0$. Then, we have the following decay estimates for $\underline{A}$ along Σ_*

$$\max_{0\leq k\leq k_L-40}\int_{\Sigma_*}\tau^{2+2\delta_{dec}}|\mathfrak{d}^k\underline{A}|^2\lesssim\epsilon_0^2.$$

Both results are proved in Part II of [GKS-2022].

Curvature estimates in Theorem M8

Theorem M8 in [K-S:Kerr] is proved through an iteration procedure described in section 9.4.7 of [K-S:Kerr]. The control of the Ricci coefficients have been derived in Chapter 9 of [K-S:Kerr]. In the present paper, we derive the remaining estimates for the proof of Theorem M8, i.e the estimates for curvature stated in Theorem 9.4.15 of [K-S:Kerr]. To this end, we introduce weighed L^2 type norms $\mathfrak{R}_k$ and $\mathfrak{G}_k$ respectively for curvature and Ricci coefficients[8], and decompose $\mathfrak{R}_k$ and $\mathfrak{G}_k$ in their restrictions ${}^{(int)}\mathfrak{R}$, ${}^{(int)}\mathfrak{G}$ to ${}^{(int)}\mathcal{M}$ and ${}^{(ext)}\mathfrak{R}$, ${}^{(ext)}\mathfrak{G}$ to ${}^{(ext)}\mathcal{M}$, see section 13.5 in [GKS-2022] for the precise definition of these norms. In view of the results in Chapter 9 of [K-S:Kerr], the proof of Theorem 8 reduces to the following result on the control of the curvature norm $\mathfrak{R}_k$.

Theorem 2.2.4 (Theorem 9.4.15 of [K-S:Kerr])**.** *Assume that the spacetime $\mathcal{M}$ as defined in section 2.1.1 verifies the quantitative assumptions* (2.2.9) (2.2.10) *for $k_L = k_{large} + 7$, and the assumption* (2.2.7) *on initial data. Let $k_{small} - 1 \leq J \leq k_{large} + 6$. Then, we have the following boundedness estimates for all components of curvature*

$$\begin{aligned}
{}^{(int)}\mathfrak{R}^2_{J+1}\lesssim\,& r_0^{18}\Big(\epsilon_J(\mathfrak{G}_{J+1}+\mathfrak{R}_{J+1})+\epsilon_J^2+\epsilon_0^2\Big)+|a|r_0^3\mathfrak{G}^2_{J+1}\\
&+r_0^{\frac{27}{4}}\mathfrak{G}^{\frac{3}{2}}_{J+1}\Big(\epsilon_0+\sqrt{\epsilon_J}\sqrt{\mathfrak{G}_{J+1}+\mathfrak{R}_{J+1}}\Big)^{\frac{1}{2}},\\
{}^{(ext)}\mathfrak{R}^2_{J+1}\lesssim\,& r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}^2_{J+1}+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}^2_{J+1}+\epsilon_0^2,
\end{aligned}$$

where the constant in $\lesssim$ is independent of r_0 and ϵ_J is such that $\mathfrak{G}_J + \mathfrak{R}_J \leq \epsilon_J$.

Part III of [GKS-2022] is entirely dedicated to the proof of Theorem 2.2.4.

[8]As well as derivatives of $(r, \cos\theta)$ and $\mathfrak{J}$.

2.3 Derivation and estimates for the gRW equations

2.3.1 Teukolsky and gRW equations in our approach

In section 2.1.1 we derive, using the formalism developed in the previous sections[9], the nonlinear version of the Teukolsky equations for A and $\underline{A}$ of the form

$$\mathcal{L}[A] = \mathrm{Err}[\mathcal{L}[A]], \qquad \underline{\mathcal{L}}[\underline{A}] = \mathrm{Err}[[\underline{\mathcal{L}}[\underline{A}]], \tag{2.3.1}$$

where $\mathcal{L}, \underline{\mathcal{L}}$ are second order tensorial wave operators on our spacetime $\mathcal{M}$, and where $\mathrm{Err}[\mathcal{L}[A]]$, $\mathrm{Err}[\,\underline{\not{\mathcal{L}}}[\underline{A}]]$ are nonlinear errors depending on all linearized Ricci and curvature coefficients.

Just as in linear theory, to be able to control $A, \underline{A}$ we need to perform transformations $\mathfrak{q} = \mathfrak{q}[A]$, $\underline{\mathfrak{q}} = \underline{\mathfrak{q}}[\underline{A}]$, which take solutions $A, \underline{A}$ of the Teukolsky equation (2.3.1) into solutions of nonlinear, tensorial, versions of Regge-Wheeler equations, which we call gRW equations.

In the setting of polarized perturbations of Schwarzschild [K-S:Schw], the derivation of the RW equation for[10] $\mathfrak{q}$ was performed using null frames, which had the feature to be both adapted to an integrable foliation and diagonalize the curvature tensor up to error terms. One could thus rely on the geometric formalism developed in the context of the proof of the nonlinear stability of Minkowski space [Ch-Kl]. In Chapter 2 of [GKS-2022] we rely on an extension of the formalism of [Ch-Kl] which allows for non integrable null frames. Our results on the derivation of gRW in perturbations of Kerr are obtained in Chapter 5 of [GKS-2022] and can be summarized as follows.

Theorem 2.3.1. *There exist complex 2 tensors $\mathfrak{q}, \underline{\mathfrak{q}} \in \mathfrak{s}_2(\mathbb{C})$ derived from $A, \underline{A}$ as follows,*

$$\begin{aligned} \mathfrak{q} = &\quad q\overline{q}^3 \left({}^{(c)}\nabla_3 \, {}^{(c)}\nabla_3 A + C_1 \, {}^{(c)}\nabla_3 A + C_2 A \right), \\ \underline{\mathfrak{q}} = &\quad \overline{q} q^3 \left({}^{(c)}\nabla_4 \, {}^{(c)}\nabla_4 A + \underline{C}_1 \, {}^{(c)}\nabla_3 A + \underline{C}_2 A \right), \end{aligned} \tag{2.3.2}$$

[9]This follows from the complex form of the null Bianchi identities, see Proposition 3.4.17.
[10]Note that [K-S:Schw] did not rely on $\underline{\mathfrak{q}}$.

where $q = r + ia\cos\theta$ ${}^{(c)}\nabla_3$, ${}^{(c)}\nabla_4$ are conformal derivatives, see section **??**, *and*

$$
\begin{aligned}
C_1 &= 2tr\,\underline{\chi} - 2\frac{{}^{(a)}tr\underline{\chi}^2}{tr\,\underline{\chi}} - 4i\,{}^{(a)}tr\underline{\chi},\\
C_2 &= \frac{1}{2}tr\,\underline{\chi}^2 - 4\,{}^{(a)}tr\underline{\chi}^2 + \frac{3}{2}\frac{{}^{(a)}tr\underline{\chi}^4}{tr\,\underline{\chi}^2} + i\left(-2tr\,\underline{\chi}\,{}^{(a)}tr\underline{\chi} + 4\frac{{}^{(a)}tr\underline{\chi}^3}{tr\,\underline{\chi}}\right),\\
\underline{C}_1 &= 2tr\,\chi - 2\frac{{}^{(a)}tr\chi^2}{tr\,\chi} - 4i\,{}^{(a)}tr\chi,\\
\underline{C}_2 &= \frac{1}{2}tr\,\chi^2 - 4\,{}^{(a)}tr\chi^2 + \frac{3}{2}\frac{{}^{(a)}tr\chi^4}{tr\,\chi^2} + i\left(-2tr\,\chi\,{}^{(a)}tr\chi + 4\frac{{}^{(a)}tr\chi^3}{tr\,\chi}\right),
\end{aligned}
\tag{2.3.3}
$$

which verify gRW equations of the form[11]

$$
\begin{aligned}
\dot{\square}_2\mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V\mathfrak{q} &= L_{\mathfrak{q}}[A] + Err[\dot{\square}_2\mathfrak{q}],\\
\dot{\square}_2\underline{\mathfrak{q}} + i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\underline{\mathfrak{q}} - \underline{V}\underline{\mathfrak{q}} &= L_{\underline{\mathfrak{q}}}[\underline{A}] + Err[\dot{\square}_2\underline{\mathfrak{q}}],
\end{aligned}
\tag{2.3.4}
$$

with $\mathbf{T}$ *an appropriately defined deformation of the stationary Killing v-field in Kerr. The potentials $V, \underline{V}$ are real and positive and the terms $L_{\mathfrak{q}}[A], L_{\underline{\mathfrak{q}}}[\underline{A}]$ are linear in A, resp $\underline{A}$ and have have important specific properties described in detail in sections 5.2.3 and 5.3.4 of [GKS-2022]. Finally the error terms $Err[\dot{\square}_2\mathfrak{q}]$, $Err[\dot{\square}_2\underline{\mathfrak{q}}]$ depending on all linearized Ricci and curvature coefficients are acceptable error terms, i.e. they verify important structural properties, reminiscent to the null condition.*

Remark 2.3.2. *Due to the presence of the linear terms in A, resp. $\underline{A}$, on the right hand side of* (2.3.4)*, one has to view the wave equations in* (2.3.4) *as coupled with the defining equations for $\mathfrak{q}$, $\underline{\mathfrak{q}}$ given by* (2.3.2)*, that is coupled*[12] *with second order transport type equations in A, resp. $\underline{A}$.*

Remark 2.3.3. *Note that, in the case of Kerr, the corresponding gRW type equations in [Ma] are complex scalars $\psi^{[\pm]}$ verifying the equations*[13]

$$
\square_{a,m}\psi^{[\pm]} + ia\ c(r,\theta)\partial_t\psi^{[\pm]} + V(r,\theta)\psi^{[\pm]} = aL_{\pm}(\alpha^{[\pm 2]}). \tag{2.3.5}
$$

These scalars are connected to our tensorial quantities $\mathfrak{q}, \underline{\mathfrak{q}}$ via the relations $\psi^{[+]} = \mathfrak{q}(e_1, e_1)$, $\psi^{[-]} = \underline{\mathfrak{q}}(e_1, e_1)$. The equations (2.3.5) *can be obtained by projecting our tensorial equations* (2.3.4)*. Note however that the projection modifies the equations by the appearance of Christoffel symbols*[14] *of the horizontal frame*[15].

[11] Here $\dot{\square}_2$ is the covariant wave operator for horizontal 2-tensors, see section 2.3. in the paper.
[12] This is different from the case of Schwarzschild, see [K-S:Schw], where these equations decouple.
[13] With $\square_{a,m}$ the Kerr D'Alembertian, c, V are real function of r, θ and $L_{\pm}(\alpha^{[\pm 2]})$ lower order terms.
[14] Singular on the axis, i.e. at $\theta = 0, \pi$.
[15] See Section 5.2.2 of the paper for a discussion of the projection and the relation with equation (2.3.5).

2.3.2 RW model equations

The most demanding part in the analysis of the gRW equations (2.3.4) is to derive global Energy-Morawetz type estimates for $(\mathfrak{q}, A)$ and respectively $(\underline{\mathfrak{q}}, \underline{A})$. To do this, it helps to analyze first the reduced equations in which the right hand side of both equations are treated as sources. Taking also $\psi = \Re(\mathfrak{q})$, $\underline{\psi} = \Re(\underline{\mathfrak{q}})$ we are led to the real RW model equations

$$
\begin{aligned}
\square_2\psi - V\psi &= -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N, \qquad V = \frac{4\Delta}{(r^2+a^2)|q|^2},\\
\dot{\square}_2\underline{\psi} - V\underline{\psi} &= \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\underline{\psi} + \underline{N}, \qquad V = \frac{4\Delta}{(r^2+a^2)|q|^2}.
\end{aligned}
\tag{2.3.6}
$$

A significant part in the proof of Theorems 2.2.2-2.2.3 is to derive the following result for $\psi, \underline{\psi}$.

Theorem 2.3.4. *The following estimates hold true for solutions $\psi, \underline{\psi} \in \mathfrak{s}_2$ of the wave equations* (2.3.6) *on spacetime region $\mathcal{M}(\tau_1, \tau_2)$, for all $\delta \le p \le 2-\delta$ and $2 \le s \le k_L$,*

$$
BEF^s_p[\psi](\tau_1,\tau_2) \lesssim E^s_p[\psi](\tau_1) + \mathcal{N}^s_p[\psi, N](\tau_1,\tau_2), \tag{2.3.7}
$$

$$
BEF^s_p[\underline{\psi}](\tau_1,\tau_2) \lesssim E^s_p[\underline{\psi}](\tau_1) + \mathcal{N}^s_p[\underline{\psi}, \underline{N}](\tau_1,\tau_2), \tag{2.3.8}
$$

where

$$
BEF^s_p[\psi](\tau_1,\tau_2) := \sup_{\tau\in[\tau_1,\tau_2]} E^s_p[\psi](\tau) + B^s_p[\psi](\tau_1,\tau_2) + F^s_p[\psi](\tau_1,\tau_2). \tag{2.3.9}
$$

The energy flux norms $E^s_p[\psi], F^s_p[\psi]$, bulk norms $B^s_p[\psi]$ and source norms $\mathcal{N}^s_p$, with p referring to r^p weights and s to the number of derivatives, are defined in section **??** of these notes. For the sake of this introduction it suffices to take a closer look at the crucial bulk terms B^s_p, which degenerate at the trapped set $\mathcal{M}_{trap}$, see Definition 2.1.2.

Definition 2.3.5. *For $0 < p < 2$ we define, with $\mathfrak{d} = (r\nabla_4, r\nabla, \nabla_3)$, the bulk norms $B^s_p[\psi](\tau_1,\tau_2) := \sum_{k\le s} B_p[\mathfrak{d}^k\psi]$*

$$
\begin{aligned}
B_p[\psi](\tau_1,\tau_2) &:= Mor[\psi](\tau_1,\tau_2) + \int_{\mathcal{M}_{r\ge 4m}(\tau_1,\tau_2)} r^{-1-\delta}|\nabla_3\psi| + r^{p-3}\Big(|\mathfrak{d}\psi|^2 + |\psi|^2\Big),\\
Mor[\psi](\tau_1,\tau_2) &:= \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\nabla_{\widehat{R}}\psi|^2 + r^{-3}|\psi|^2 + \int_{\mathcal{M}_{\cancel{trap}}(\tau_1,\tau_2)} \left(r^{-2}|\nabla_3\psi|^2 + r^{-1}|\nabla\psi|^2\right).
\end{aligned}
$$

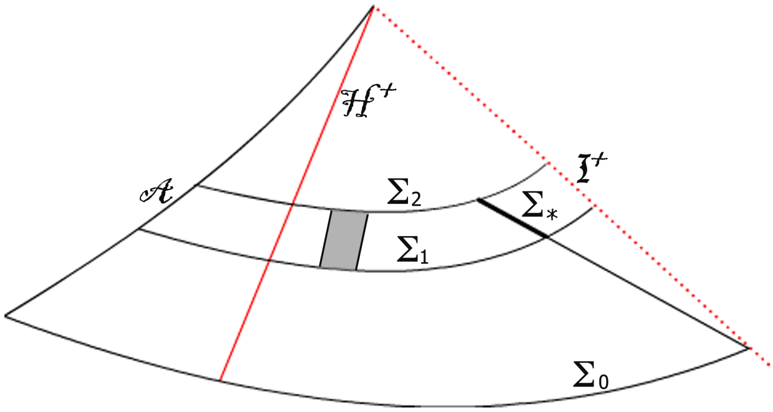


Figure 2.1: The spacetime region $\mathcal{M}(\tau_1, \tau_2) = \mathcal{M} \cap \{\tau_1 \leq \tau \leq \tau_2\}$ between the spacelike hypersurfaces $\Sigma_1 = \Sigma(\tau_1)$ and $\Sigma_2 = \Sigma(\tau_2)$, with the grey region denoting the trapped set.

The important thing in this definition is that $B_p[\psi]$ controls the spacetime integrals of $|\nabla_{\widehat{R}}\psi|^2$ and $|\psi|^2$ everywhere and all other derivatives away from the trapped set.

In addition, we also derive estimates for the quantity $\check{\psi} := r^2(e_4\psi + \frac{r}{|q|^2}\psi)$ for which one can prove stronger r^p estimates[16], see Theorem **??**.

The content of the section below are to be found in the Introduction to [GKS-2022]

2.4 Main steps in the proof of Theorems M1 and M2

2.5 Main ideas in the proof of Theorem 2.3.4

2.6 Main ideas in the proof of Theorem 2.2.4

[16]These results are the analog in perturbations of Kerr, to Theorem 5.17 and Theorem 5.18 of [K-S:Schw] for perturbations of Schwarzschild. They are based on improved r^p weighted hierarchy first introduced in [AArGa].

Part II

Formalism and derivation of the main equations

Chapter 3

The geometric formalism of null horizontal structures

We summarize the content of Chapters 2 in [GKS-2022] which provides the general formalism used in our stability of Kerr papers. The formalism extends the one used for perturbations of Minkowski space [Ch-Kl] to perturbations of Kerr spacetimes. Such formalism can be adapted to any Lorentzian spacetime possessing a null pair and not necessarily foliated by surfaces. The formalism in this section is very general and does not rely on the Einstein equation.

3.0.1 Null pairs and horizontal structures

Let $(\mathcal{M}, \mathbf{g})$ be a Lorentzian spacetime. Consider an arbitrary null pair $e_3 = \underline{L}$, $e_4 = L$, i.e.

$$\mathbf{g}(e_3, e_3) = \mathbf{g}(e_4, e_4) = 0, \qquad \mathbf{g}(e_3, e_4) = -2.$$

Definition 3.0.1. *We say that a vectorfield X is $(L, \underline{L})$-horizontal, or simply horizontal, if*

$$\mathbf{g}(L, X) = \mathbf{g}(\underline{L}, X) = 0.$$

We denote by $\mathbf{O}(\mathcal{M})$ the set of horizontal vectorfields on $\mathcal{M}$. Given a fixed orientation on $\mathcal{M}$, with corresponding volume form $\in$, we define the induced volume form on $\mathbf{O}(\mathcal{M})$ by,

$$\in (X, Y) := \frac{1}{2} \in (X, Y, \underline{L}, L). \tag{3.0.1}$$

Given a null pair $(L, \underline{L})$, the horizontal vectorfields $\mathbf{O}(\mathcal{M})$ define a horizontal distribution, i.e. a sub-bundle of the tangent bundle $\mathbf{T}(\mathcal{M})$ of the manifold. In the standard terminology used in differential topology, a subbundle $E \subset \mathbf{T}(\mathcal{M})$ of the tangent bundle is said to be integrable if for any vectorfields X and Y taking values in E, the Lie bracket $[X, Y]$ takes values in E as well. We recall that Frobenius' theorem states that a subbundle E is integrable (or involutive) if and only if the subbundle E arises from a regular foliation of $\mathcal{M}$, i.e. if locally the subbundle E can be realized as the tangent space of a submanifold of $\mathcal{M}$.

In the context of Lorentzian spacetimes, we are often interested in foliations of the manifold given by compact surfaces S, called S-folaitions in [Ch-Kl]. We therefore formulate the following definition.

Definition 3.0.2. *We say that the horizontal structure $\mathbf{O}(\mathcal{M})$ is integrable if there exists a foliation by compact surfaces S, i.e. an S-foliation of $\mathcal{M}$, such that the horizontal vectors in $\mathbf{O}(\mathcal{M})$ at every point coincide with the tangent space of S, i.e. $\mathbf{O}(\mathcal{M}) = \mathbf{T}(S)$.*

Here we will work with general, not necessarily integrable, horizontal structures.

Clearly, any linear combination of horizontal vectorfields is again horizontal. However, the commutator $[X, Y]$ of two horizontal vectorfields may fail to be horizontal. Such failure is precisely related to the existence[1] of an S-foliation. More precisely, if $\mathbf{O}(\mathcal{M})$ is integrable according to Definition 3.0.2, i.e. admits an S-foliation, then $X, Y \in \mathbf{O}(\mathcal{M})$ implies that $[X, Y] \in \mathbf{O}(\mathcal{M})$. Conversely, if $\mathbf{O}(\mathcal{M})$ is not close under the Lie bracket, then it can not be foliated by compact surfaces.

Given an arbitrary vectorfield X we denote by ${}^{(h)}X$ its horizontal projection,

$$ {}^{(h)}X \quad = \quad X + \frac{1}{2}\mathbf{g}(X, \underline{L})L + \frac{1}{2}\mathbf{g}(X, L)\,\underline{L}. $$

Definition 3.0.3. *A k-covariant tensor-field U is said to be horizontal, and denoted $U \in \mathbf{O}_k(\mathcal{M})$, if for any vectorfields $X_1, \ldots X_k$ we have,*

$$ U(X_1, \ldots X_k) = U({}^{(h)}X_1, \ldots {}^{(h)}X_k). $$

We can define the projection operator,

$$ \Pi^{\mu\nu} \quad = \quad \mathbf{g}^{\mu\nu} + \frac{1}{2}(\,\underline{L}^\mu L^\nu + L^\mu\,\underline{L}^\nu). $$

Clearly $\Pi^\mu_\alpha \Pi^\beta_\mu = \Pi^\beta_\alpha$. An arbitrary tensor $U_{\alpha_1 \ldots \alpha_m}$ is horizontal, if

$$ \Pi^{\beta_1}_{\alpha_1} \ldots \Pi^{\beta_m}_{\alpha_m} U_{\beta_1 \ldots \beta_m} = U_{\alpha_1 \ldots \alpha_m}. $$

[1]Consistent to Frobenius' theorem.

Definition 3.0.4. *For any horizontal X, Y we define*[2]

$$\gamma(X,Y) \;=\; \mathbf{g}(X,Y) \tag{3.0.2}$$

and

$$\begin{cases} \chi(X,Y) = \mathbf{g}(\mathbf{D}_X L, Y), \\ \underline{\chi}(X,Y) = \mathbf{g}(\mathbf{D}_X \underline{L}, Y). \end{cases} \tag{3.0.3}$$

where $\mathbf{D}$ *denotes the covariant derivative of* $\mathbf{g}$.

Observe that χ and $\underline{\chi}$ are symmetric if and only if the horizontal structure is integrable. Indeed this follows easily from the formulas,

$$\begin{aligned} \chi(X,Y) - \chi(Y,X) &= \mathbf{g}(\mathbf{D}_X L, Y) - \mathbf{g}(\mathbf{D}_Y L, X) = -\mathbf{g}(L, [X,Y]), \\ \underline{\chi}(X,Y) - \underline{\chi}(Y,X) &= \mathbf{g}(\mathbf{D}_X \underline{L}, Y) - \mathbf{g}(\mathbf{D}_Y \underline{L}, X) = -\mathbf{g}(\underline{L}, [X,Y]). \end{aligned}$$

We can view γ, χ and $\underline{\chi}$ as horizontal 2-covariant tensor-fields by extending their definition to arbitrary vectorfields X, Y according to,

$$\gamma(X,Y) \;=\; \gamma({}^{(h)}X, {}^{(h)}Y)$$

and

$$\chi(X,Y) \;=\; \chi({}^{(h)}X, {}^{(h)}Y), \qquad \underline{\chi}(X,Y) = \underline{\chi}({}^{(h)}X, {}^{(h)}Y).$$

Given a general 2-covariant horizontal tensor U we decompose it in its symmetric and antisymmetric part as follows,

$$\begin{aligned} {}^{(s)}U(X,Y) &= \frac{1}{2}\big(U(X,Y) + U(Y,X)\big), \\ {}^{(a)}U(X,Y) &= \frac{1}{2}\big(U(X,Y) - U(Y,X)\big). \end{aligned}$$

Given a horizontal structure defined by $e_3 = \underline{L}$, $e_4 = L$ we associate a null frame by choosing orthonormal horizontal vectorfields e_1, e_2 such that $\gamma(e_a, e_b) = \delta_{ab}$. By convention, we say that (e_1, e_2) is positively oriented on $\mathbf{O}(\mathcal{M})$ if,

$$\in (e_1, e_2) = \in (e_1, e_2, e_3, e_4) = 1. \tag{3.0.4}$$

Remark 3.0.5. *We note that the particular choice of an orthonormal basis for $\mathcal{H}$ is immaterial. All the quantities we work with are tensorial with respect to the horizontal structure.*

[2] In the particular case where the horizontal structure is integrable, γ is the induced metric, and χ and $\underline{\chi}$ are the null second fundamental forms.

Given a covariant horizontal 2-tensor U and an arbitrary orthonormal horizontal frame $(e_a)_{a=1,2}$ we have,

$$^{(s)}U_{ab} = \frac{1}{2}(U_{ab} + U_{ba}), \qquad ^{(a)}U_{ab} = \frac{1}{2}(U_{ab} - U_{ba}).$$

Definition 3.0.6. *The trace of a horizontal 2-tensor U is defined by*

$$tr(U) := \delta^{ab} U_{ab} = \delta^{ab\,(s)} U_{ab}. \tag{3.0.5}$$

We define the anti-trace of U by,

$$^{(a)}tr(U) := \in^{ab} U_{ab} = \in^{ab} \ ^{(a)}U_{ab}. \tag{3.0.6}$$

Observe that the first trace is independent of the particular choice of the frame e_1, e_2. On the other hand, for fixed e_3, e_4, $^{(a)}tr$ depends on the orientation of e_1, e_2. Also, by interchanging e_3, e_4, $^{(a)}tr$ changes sign.

A general horizontal 2-tensor U can be decomposed according to,

$$U_{ab} \quad = \quad ^{(s)}U_{ab} + \ ^{(a)}U_{ab} = \widehat{U}_{ab} + \frac{1}{2}\delta_{ab}\,\mathrm{tr}(U) + \frac{1}{2} \in_{ab} \ ^{(a)}\mathrm{tr}(U). \tag{3.0.7}$$

where $\widehat{U}$ denotes the symmetric traceless part of U.

Definition 3.0.7. *We introduce the notation*

$$tr\,\chi := tr(\chi), \quad ^{(a)}tr\chi := \ ^{(a)}tr(\chi), \quad tr\underline{\chi} := tr(\underline{\chi}), \quad ^{(a)}tr\underline{\chi} := \ ^{(a)}tr(\underline{\chi}). \tag{3.0.8}$$

The quantities $\widehat{\chi}, tr\,\chi$ and $\widehat{\underline{\chi}}, tr\,\underline{\chi}$ are called, respectively, the shear and expansion of the horizontal distribution $\mathbf{O}(\mathcal{M})$. The scalars $^{(a)}tr\chi$ and $^{(a)}tr\underline{\chi}$ measure the integrability defects of the distribution.

Accordingly, we decompose $\chi, \underline{\chi}$ as follows

$$\begin{aligned}
\chi_{ab} &= \widehat{\chi}_{ab} + \frac{1}{2}\delta_{ab} tr\,\chi + \frac{1}{2} \in_{ab} \ ^{(a)}tr\chi, \\
\underline{\chi}_{ab} &= \widehat{\underline{\chi}}_{ab} + \frac{1}{2}\delta_{ab} tr\underline{\chi} + \frac{1}{2} \in_{ab} \ ^{(a)}tr\underline{\chi}.
\end{aligned}$$

The scalars $tr\,\chi, tr\underline{\chi}$ are called expansions and $\chi, \underline{\chi}$ are called the shears of the horizontal structure.

In what follows we fix a null pair e_3, e_4 and an orientation on $\mathbf{O}(\mathcal{M})$. Consider the set of all smooth k-horizontal tensorfields $\xi = \xi_{a_1 \ldots a_k}$ which are fully symmetric and traceless, i.e.

$$\xi = \xi_{(a_1 \ldots a_k)}, \qquad \gamma^{a_i a_j} \xi_{a_1 \ldots a_i \ldots a_j \ldots a_k} = 0.$$

Definition 3.0.8. *We denote*[3] *by* $\mathbf{O}_k(\mathcal{M})$ *the set of all horizontal tensor-fields of rank* k *on* $\mathcal{M}$*. We denote by* $\mathfrak{s}_0 = \mathfrak{s}_0(\mathcal{M})$ *the set of pairs of scalar functions on* $\mathcal{M}$*,* $\mathfrak{s}_1 = \mathfrak{s}_1(\mathcal{M})$ *the set of horizontal* 1*-forms on* $\mathcal{M}$ *and for,* $k \geq 2$*,* $\mathfrak{s}_k(\mathcal{M})$ *the set of fully symmetric traceless horizontal tensors of rank* k*. In particular* $\mathfrak{s}_2 = \mathfrak{s}_2(\mathcal{M})$ *denotes the set of symmetric traceless horizontal* 2*-tensors on* $\mathcal{M}$*.*

In particular, $\mathrm{t}r\,\chi, \mathrm{tr}\,\underline{\chi}$, ${}^{(a)}\mathrm{tr}\chi$, ${}^{(a)}\mathrm{tr}\underline{\chi} \in \mathfrak{s}_0$ and $\widehat{\chi}, \widehat{\underline{\chi}} \in \mathfrak{s}_2$. Any horizontal 1-form belongs to $\mathfrak{s}_1$.

Definition 3.0.9. *We define the left and right duals of a horizontal of tensors* $\xi \in \mathfrak{s}_k$*,* $k = 1, 2$

$$\begin{aligned} {}^*\xi_a &= \in_{ab} \xi_b, \qquad & \xi^*{}_a &= \xi_b \in_{ba}, \\ ({}^*\xi)_{ab} &= \in_{ac} \xi_{cb}, \qquad & (\xi^*)_{ab} &= \xi_{ac} \in_{cb} . \end{aligned}$$

Lemma 3.0.10. *Given* $\xi \in \mathfrak{s}_{1,2}$*, we have*

$$^*({}^*\xi) = -\xi, \qquad {}^*\xi = -\xi^*.$$

Proof. Straightforward verification. $\square$

Given $\xi, \eta \in \mathfrak{s}_{1,2}$ we define all the possible dot products between then

$$\xi \cdot \eta = \begin{cases} \xi^a \eta_a & \text{if} \quad \xi, \eta \in \mathfrak{s}_1 \\ \xi^a \eta_{ab}, & \text{if} \quad \xi \in \mathfrak{s}_1, \quad \eta \in \mathfrak{s}_2 \\ \xi_{ab} \eta^b, & \text{if} \quad \xi \in \mathfrak{s}_2, \quad \eta \in \mathfrak{s}_1 \\ \xi_{ab} \eta^{ab}, & \text{if} \quad \xi, \eta \in \mathfrak{s}_2. \\ \xi_{ac} \eta^c{}_b, & \text{if} \quad \xi, \eta \in \mathfrak{s}_2. \end{cases}$$

Lemma 3.0.11. *Given* $\xi, \eta \in \mathfrak{s}_{1,2}$ *we have,*

$$^*\xi \cdot \eta = -\xi^{\,*} \cdot \eta$$

Proof. Straightforward verification. $\square$

Lemma 3.0.12. *Given* $\xi, \eta \in \mathfrak{s}_2$ *we have, with respect to an arbitrary orthonormal basis,*

$$\xi_{ac}\eta_{cb} + \eta_{ac}\xi_{cb} = \delta_{ab}\, \xi \cdot \eta$$

[3] Using the convention of raising and lowering indices we make no distinction here between covariant and contravariant tensors.

Proof. Straightforward verification using an orthonormal basis e_1, e_2. $\square$

Definition 3.0.13. *Given $\xi, \eta \in \mathfrak{s}_1$ we denote*

$$
\begin{aligned}
\xi \cdot \eta &:= \delta^{ab}\xi_a\eta_b, \\
\xi \wedge \eta &:= \in^{ab} \xi_a\eta_b, \\
(\xi\widehat{\otimes}\eta)_{ab} &:= \xi_a\eta_b + \xi_b\eta_a - \delta_{ab}\xi \cdot \eta.
\end{aligned}
$$

Given $\xi \in \mathfrak{s}_1$, $\eta \in \mathfrak{s}_2$ we denote

$$
(\xi \cdot \eta)_a := \delta^{bc}\xi_b\eta_{ac}.
$$

Given $\xi, \eta \in \mathfrak{s}_2$ we denote

$$
(\xi \wedge \eta)_{ab} := \in^{ab} \xi_{ac}\eta_{cb}.
$$

Lemma 3.0.14. *Given $\xi, \eta \in \mathfrak{s}_1$,*

$$
{}^*\xi\widehat{\otimes}\eta = \xi\widehat{\otimes}\, {}^*\eta, \qquad {}^*(\xi\widehat{\otimes}\eta) = {}^*\xi\widehat{\otimes}\eta, \qquad {}^*\xi\widehat{\otimes}\, {}^*\eta = -\xi\widehat{\otimes}\eta.
$$

Proof. Write

$$
\begin{aligned}
{}^*(\xi\widehat{\otimes}\eta)_{11} &= (\xi\widehat{\otimes}\eta)_{21} = \xi_2\eta_1 + \xi_1\eta_2, \\
(\xi\widehat{\otimes}\, {}^*\eta)_{11} &= \xi_1(\, {}^*\eta)_1 - \xi_2(\, {}^*\eta)_2 = \xi_1\eta_2 + \xi_2\eta_1, \\
(\, {}^*\xi\widehat{\otimes}\eta)_{11} &= (\, {}^*\xi)_1\eta_1 - (\, {}^*\xi)_2\eta_2 = \xi_2\eta_1 + \xi_1\eta_2.
\end{aligned}
$$

$$
\begin{aligned}
{}^*(\xi\widehat{\otimes}\eta)_{12} &= \xi_2\eta_2 - \xi_1\eta_1, \\
(\xi\widehat{\otimes}\, {}^*\eta)_{12} &= \xi_1\, {}^*\eta_2 + \xi_2\, {}^*\eta_1 = -\xi_1\eta_1 + \xi_2\eta_2, \\
(\, {}^*\xi\widehat{\otimes}\eta)_{12} &= {}^*\xi_1\eta_2 + {}^*\xi_2\eta_1 = -\xi_1\eta_1 + \xi_2\eta_2.
\end{aligned}
$$

Hence,

$$
{}^*(\xi\widehat{\otimes}\eta) = {}^*\xi\widehat{\otimes}\eta = \xi\widehat{\otimes}\, {}^*\eta.
$$

$\square$

Lemma 3.0.15. *Given $\xi, \eta \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$ we have*

$$
\xi\widehat{\otimes}(\eta \cdot u) + \eta\widehat{\otimes}(\xi \cdot u) = 2(\xi \cdot \eta)u.
$$

Proof. Straightforward verification using direct verification as above. $\square$

3.0.2 Horizontal covariant derivative

Given $X, Y \in \mathbf{O}(\mathcal{M})$ the covariant derivative $\mathbf{D}_X Y$ fails in general to be horizontal. We thus define the horizontal covariant operator ∇ as follows,

$$\nabla_X Y \ := \ {}^{(h)}(\mathbf{D}_X Y) = \mathbf{D}_X Y - \frac{1}{2}\underline{\chi}(X,Y)L - \frac{1}{2}\chi(X,Y)\,\underline{L}. \tag{3.0.9}$$

Proposition 3.0.16. *For all $X, Y \in \mathbf{O}(\mathcal{M})$,*

$$\begin{aligned}\nabla_X Y - \nabla_Y X &= [X,Y] - {}^{(a)}\underline{\chi}(X,Y)L - {}^{(a)}\chi(X,Y)\,\underline{L}\\ &= [X,Y] - \frac{1}{2}\left({}^{(a)}tr\underline{\chi}\,L + {}^{(a)}tr\chi\ \underline{L}\right) \in (X,Y).\end{aligned}$$

In particular,

$$^{(h)}[X,Y] \ = \ \frac{1}{2}\left({}^{(a)}tr\underline{\chi}\,L + {}^{(a)}tr\chi\ \underline{L}\right) \in (X,Y). \tag{3.0.10}$$

For all $X, Y, Z \in \mathbf{O}(\mathcal{M})$,

$$Z\gamma(X,Y) = \gamma(\nabla_Z X, Y) + \gamma(X, \nabla_Z Y).$$

Remark 3.0.17. *In the integrable case, ∇ coincides with the Levi-Civita connection of the metric induced on the integral surfaces of $\mathbf{O}(\mathcal{M})$.*

Given a general covariant, horizontal tensor-field U we define its horizontal covariant derivative according to the formula,

$$\begin{aligned}\nabla_Z U(X_1, \ldots X_k) = Z(U(X_1, \ldots X_k)) \ &- \ U(\nabla_Z X_1, \ldots X_k) - \\ \ldots \ &- \ U(X_1, \ldots \nabla_Z X_k).\end{aligned}$$

Given X horizontal, $\mathbf{D}_L X$ and $\mathbf{D}_{\underline{L}} X$ are in general not horizontal. We define $\nabla_L X$ and $\nabla_{\underline{L}} X$ to be the horizontal projections of the former. More precisely,

$$\begin{aligned}\nabla_L X \ &:= \ {}^{(h)}(\mathbf{D}_L X) = \mathbf{D}_L X - \mathbf{g}(X, \mathbf{D}_L \underline{L})L - \mathbf{g}(X, \mathbf{D}_L L)\,\underline{L},\\ \nabla_{\underline{L}} X \ &:= \ {}^{(h)}(\mathbf{D}_{\underline{L}} X) = \mathbf{D}_{\underline{L}} X - \mathbf{g}(X, \mathbf{D}_{\underline{L}} \underline{L})L - \mathbf{g}(X, \mathbf{D}_{\underline{L}} L)\,\underline{L}.\end{aligned}$$

We can extend the operators ∇_L and $\nabla_{\underline{L}}$ to arbitrary k-covariant, horizontal tensor-fields U as follows,

$$\begin{aligned}\nabla_L U(X_1, \ldots, X_k) = L(U(X_1, \ldots, X_k)) \ &- \ U(\nabla_L X_1, \ldots, X_k) - \\ \ldots \ &- \ U(X_1, \ldots, \nabla_L X_k),\\ \nabla_{\underline{L}} U(X_1, \ldots, X_k) = \underline{L}(U(X_1, \ldots, X_k)) \ &- \ U(\nabla_{\underline{L}} X_1, \ldots, X_k) - \\ \ldots \ &- \ U(X_1, \ldots, \nabla_{\underline{L}} X_k).\end{aligned}$$

The following proposition follows easily from the definition.

Proposition 3.0.18. *The operators ∇, ∇_L and $\nabla_{\underline{L}}$ take horizontal tensor-fields into horizontal tensor-fields. We have,*

$$\nabla\gamma = \nabla_L\gamma = \nabla_{\underline{L}}\gamma = 0. \tag{3.0.11}$$

We now extend the definition of horizontal covariant derivative to any $X \in \mathbf{T}(\mathcal{M})$ in the tangent space of $\mathcal{M}$ and $Y \in \mathbf{O}(\mathcal{M})$.

Definition 3.0.19. *Given $X \in \mathbf{T}(\mathcal{M})$ and $Y \in \mathbf{O}(\mathcal{M})$ we define,*

$$\dot{\mathbf{D}}_X Y \ := \ {}^{(h)}(\mathbf{D}_X Y).$$

Given an orthonormal frame $e_1, e_2 \in \mathbf{O}(\mathcal{M})$ we write

$$\dot{\mathbf{D}}_\mu e_a \ = \ \sum_{b=1,2} (\Lambda_\mu)_{ba}\, e_b, \qquad (\Lambda_\mu)_{\alpha\beta} := \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha).$$

Definition 3.0.20. *Given a general, covariant, S-horizontal tensor-field U we define its horizontal covariant derivative according to the formula,*

$$\dot{\mathbf{D}}_X U(Y_1, \dots Y_k) = X(U(Y_1, \dots Y_k)) \ - \ U(\dot{\mathbf{D}}_X Y_1, \dots Y_k) - \dots - U(Y_1, \dots \dot{\mathbf{D}}_X Y_k),$$

where $X \in \mathbf{T}(\mathcal{M})$ and $Y_1, \dots Y_k \in \mathbf{O}(\mathcal{M})$.

Proposition 3.0.21. *For all $X \in \mathbf{T}(\mathcal{M})$ and $Y_1, Y_2 \in \mathbf{O}(\mathcal{M})$,*

$$Xh(Y_1, Y_2) = h(\dot{\mathbf{D}}_X Y_1, Y_2) + h(Y_1, \dot{\mathbf{D}}_X Y_2).$$

Proof. Indeed,

$$\begin{aligned} Xh(Y_1, Y_2) &= X\mathbf{g}(Y_1, Y_2) = \mathbf{g}(\mathbf{D}_X Y_1, Y_2) + \mathbf{g}(Y_1, \mathbf{D}_X Y_2) = \mathbf{g}(\dot{\mathbf{D}}_X Y_1, Y_2) + \mathbf{g}(Y_1, \dot{\mathbf{D}}_X Y_2) \\ &= h(\dot{\mathbf{D}}_X Y_1, Y_2) + h(Y_1, \dot{\mathbf{D}}_X Y_2) \end{aligned}$$

as desired. □

We consider tensors $\mathbf{T}_k(\mathcal{M}) \otimes \mathbf{O}_l(\mathcal{M})$, i.e. tensors of the form $U_{\nu_1 \dots \nu_k, a_1 \dots a_l}$ for which we define,

$$\begin{aligned} \dot{\mathbf{D}}_\mu U_{\nu_1 \dots \nu_k, a_1 \dots a_l} &= e_\mu U_{\nu_1 \dots \nu_k, a_1 \dots a_l} - U_{\mathbf{D}_\mu \nu_1 \dots \nu_k, a_1 \dots a_l} - \dots - U_{\nu_1 \dots \mathbf{D}_\mu \nu_k, a_1 \dots a_l} \\ &- U_{\nu_1 \dots \nu_k, \dot{\mathbf{D}}_\mu a_1 \dots a_l} - U_{\nu_1 \dots \nu_k, a_1 \dots \dot{\mathbf{D}}_\mu a_l}. \end{aligned}$$

We are now ready to prove the following.

Proposition 3.0.22. *For a tensor $\Psi \in \mathbf{O}_1(\mathcal{M})$, we have the curvature formula*[4]

$$(\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu - \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\mu)\Psi_a = \dot{\mathbf{R}}_{ab\mu\nu}\Psi^b \tag{3.0.12}$$

where, with connection coefficients $(\Lambda_\alpha)_{\beta\gamma} = \mathbf{g}(\mathbf{D}_\alpha e_\gamma, e_\beta)$,

$$\begin{aligned}
\dot{\mathbf{R}}_{ab\mu\nu} &:= \mathbf{R}_{ab\mu\nu} + \frac{1}{2}\mathbf{B}_{ab\mu\nu} \\
\mathbf{B}_{ab\mu\nu} &:= (\Lambda_\mu)_{3a}(\Lambda_\nu)_{b4} + (\Lambda_\mu)_{4a}(\Lambda_\nu)_{b3} - (\Lambda_\nu)_{3a}(\Lambda_\mu)_{b4} - (\Lambda_\nu)_{4a}(\Lambda_\mu)_{b3}.
\end{aligned} \tag{3.0.13}$$

More generally, for a mixed tensor $\Psi \in \mathbf{T}_1(\mathcal{M}) \otimes \mathbf{O}_1(\mathcal{M})$, we have

$$(\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu - \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\mu)\Psi_{\lambda a} = \mathbf{R}_\lambda{}^\sigma{}_{\mu\nu}\Psi_{\sigma a} + \dot{\mathbf{R}}_a{}^b{}_{\mu\nu}\Psi_{\lambda b}$$

with an immediate generalization to tensors $\Psi \in \mathbf{T}_k(\mathcal{M}) \otimes \mathbf{O}_l(\mathcal{M})$.

Proof. See proof of Proposition 2.1.27 in [GKS-2022]. □

Remark 3.0.23. *Note that the tensor $\mathbf{B}_{ab\mu\nu}$ is anti-symmetric in both $\mu\nu$ and ab.*

Corollary 3.0.24. *Let X, Y be arbitrary vectorfields on $\mathcal{M}$ and $U \in \mathbf{O}_1(\mathcal{M})$ an horizontal tensor. We have*[5]

$$\big(\nabla_X \nabla_Y - \nabla_Y \nabla_X\big)U = \nabla_{[X,Y]}U + \dot{\mathbf{R}}(X,Y)U$$

with an immediate generalization to $U \in \mathbf{O}_l(\mathcal{M})$.

Proof. We have

$$\begin{aligned}
\nabla_Y \nabla_X U_a &= (Y^\lambda \dot{\mathbf{D}}_\lambda)(X^\mu \dot{\mathbf{D}}_\mu)U_a = Y^\lambda X^\mu \dot{\mathbf{D}}_\lambda \dot{\mathbf{D}}_\mu U_a + (Y^\lambda \dot{\mathbf{D}}_\lambda)(X^\mu)\dot{\mathbf{D}}_\mu U_a, \\
\nabla_X \nabla_Y U_a &= X^\mu Y^\lambda \dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\lambda U_a + (X^\mu \dot{\mathbf{D}}_\mu)(Y^\lambda)\dot{\mathbf{D}}_\lambda U_a.
\end{aligned}$$

Hence,

$$\begin{aligned}
\big(\nabla_X \nabla_Y - \nabla_Y \nabla_X\big)U_a &= Y^\lambda X^\mu \big(\dot{\mathbf{D}}_\lambda \dot{\mathbf{D}}_\mu - \dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\lambda\big)U_a + \big(\dot{\mathbf{D}}_X(Y^\mu) - \dot{\mathbf{D}}_Y(X^\mu)\big)\dot{\mathbf{D}}_\mu U_a \\
&= X^\mu Y^\nu \dot{\mathbf{R}}_{ab\mu\nu}U^b + \dot{\mathbf{D}}_{[X,Y]}U_a,
\end{aligned}$$

as stated. □

[4] With an immediate generalization to tensors $\Psi \in \mathbf{O}_l(\mathcal{M})$.

[5] Here $(\dot{\mathbf{R}}(X,Y)U)_a := X^\mu Y^\nu \dot{\mathbf{R}}_{ab\mu\nu}U^b$.

3.0.3 The Gauss equation

Note that in the case of a non-integrable structure, we are missing the traditional Gauss equation which connects the Gauss curvature of a sphere to a Riemann curvature component. In what follows we state a result which is its non-integrable analogue.

Proposition 3.0.25. *The following identity holds true.*

$$\begin{aligned}\nabla_a\nabla_b X_c - \nabla_b\nabla_a X_c = {} & \mathbf{R}_{cdab}X^d + \frac{1}{2}\in_{ab}\Big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\Big)X_c \\ & -\frac{1}{2}\Big(\chi_{ac}\underline{\chi}_{bd} + \underline{\chi}_{ac}\chi_{bd} - \chi_{bc}\underline{\chi}_{ad} - \underline{\chi}_{bc}\chi_{ad}\Big)X^d,\end{aligned} \tag{3.0.14}$$

where $\mathbf{R}_{cdab}$ *denotes the Riemann curvature of* $(\mathcal{M}, \mathbf{g})$.

Proof. See the proof of Proposition 2.1.41 in [GKS-2022]. □

Remark 3.0.26. *We note that* (3.0.14) *can be derived from Corollary 3.0.24 according to which, relative to an arbitrary frame* e_μ,

$$(\nabla_\mu\nabla_\nu - \nabla_\mu\nabla_\nu)X = \nabla_{[e_\mu, e_\nu]}X + \dot{\mathbf{R}}(e_\mu, e_\nu)X$$

with $\dot{\mathbf{R}} = \mathbf{R} + \frac{1}{2}\mathbf{B}$ *and* $\mathbf{B}$ *defined in* (3.1.39). *The Gauss formula follows then easily by evaluating the components* $\mathbf{B}_{cdab}$ *of the tensor* $\mathbf{B}$ *and the term* $\nabla_{[e_a, e_b]}X$.

We now specialize the Gauss equation (3.0.14) to tensors.

Proposition 3.0.27. *The following identities hold true.*

1. *For a scalar* ψ:

$$[\nabla_a, \nabla_b]\psi \;=\; \left(\frac{1}{2}\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi\right)\in_{ab}. \tag{3.0.15}$$

2. *The only non-vanishing component of* $\mathbf{B}_{abcd}$ *is given by*

$$\mathbf{B}_{1212} = -\mathbf{B}_{1221} = \mathbf{B}_{2121} = -\frac{1}{2}tr\,\chi tr\,\underline{\chi} - \frac{1}{2}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \widehat{\chi}\cdot\widehat{\underline{\chi}}. \tag{3.0.16}$$

3. *For* $\psi \in \mathfrak{s}_k$ *for* $k = 1, 2$,

$$[\nabla_a, \nabla_b]\psi \;=\; \left(\frac{1}{2}\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi + k\,{}^{(h)}K\,{}^*\psi\right)\in_{ab} \tag{3.0.17}$$

where

$${}^{(h)}K \;:=\; -\frac{1}{4}tr\,\chi tr\,\underline{\chi} - \frac{1}{4}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}} - \frac{1}{4}\mathbf{R}_{3434}. \tag{3.0.18}$$

Proof. The case of scalars can be easily checked directly.

We consider below the case $\psi \in \mathfrak{s}_2$. From Corollary 3.0.24 applied to $\psi \in \mathfrak{s}_2$, we have

$$
\begin{aligned}
(\nabla_a\nabla_b - \nabla_b\nabla_a)\psi_{st} &= \frac{1}{2}\in_{ab}\left({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\right)\psi_{st} + \frac{1}{2}\mathbf{B}_{sdab}\psi_{dt} + \frac{1}{2}\mathbf{B}_{tdab}\psi_{sd}\\
&+ \mathbf{R}_{sdab}\psi_{dt} + \mathbf{R}_{tdab}\psi_{sd}
\end{aligned}
$$

where, by definition of $\mathbf{B}$ given in (3.1.39),

$$
\mathbf{B}_{cdab} : = \chi_{bc}\underline{\chi}_{ad} + \underline{\chi}_{bc}\chi_{ad} - \chi_{ac}\underline{\chi}_{bd} - \underline{\chi}_{ac}\chi_{bd}. \tag{3.0.19}
$$

Note that by the symmetries of $\mathbf{B}$, all components of $\mathbf{B}_{abcd}$ vanish except for $\mathbf{B}_{1212}$. We have

$$
\begin{aligned}
\mathbf{B}_{1212} &= -\chi_{11}\underline{\chi}_{22} - \underline{\chi}_{11}\chi_{22} + \chi_{21}\underline{\chi}_{12} + \underline{\chi}_{21}\chi_{12}\\
&= -\left(\frac{1}{2}tr\,\chi + \widehat{\chi}_{11}\right)\left(\frac{1}{2}\mathrm{tr}\,\underline{\chi} + \widehat{\underline{\chi}}_{22}\right) - \left(\frac{1}{2}\mathrm{tr}\,\underline{\chi} + \widehat{\underline{\chi}}_{11}\right)\left(\frac{1}{2}tr\,\chi + \widehat{\chi}_{22}\right)\\
&+\left(-\frac{1}{2}{}^{(a)}\mathrm{tr}\chi + \widehat{\chi}_{21}\right)\left(\frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi} + \widehat{\underline{\chi}}_{12}\right) + \left(-\frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi} + \widehat{\underline{\chi}}_{21}\right)\left(\frac{1}{2}{}^{(a)}\mathrm{tr}\chi + \widehat{\chi}_{12}\right)\\
&= -\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \widehat{\chi}_{11}\widehat{\underline{\chi}}_{22} - \widehat{\chi}_{22}\widehat{\underline{\chi}}_{11} + \widehat{\chi}_{21}\widehat{\underline{\chi}}_{12} + \widehat{\chi}_{12}\widehat{\underline{\chi}}_{21}\\
&= -\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \widehat{\chi}\cdot\widehat{\underline{\chi}}.
\end{aligned}
$$

This implies for $\psi \in \mathfrak{s}_2$:

$$
\begin{aligned}
[\nabla_1, \nabla_2]\psi &= \frac{1}{2}\left({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\right)\psi\\
&- \left(\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{2}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \widehat{\chi}\cdot\widehat{\underline{\chi}} + \frac{1}{2}\mathbf{R}_{3434}\right){}^*\psi
\end{aligned}
$$

as stated. The case $\psi \in \mathfrak{s}_1$ can be treated in the same manner. □

Remark 3.0.28. *The quantity* ${}^{(h)}K$ *defined by* (3.0.18) *becomes the standard Gauss curvature in the case of an integrable structure. We note also that the value of* ${}^{(h)}K$ *for the standard non-integrable structure (induced by the standard principal null directions, see Chapter 5) of Kerr is given by the formula*

$$
{}^{(h)}K = \frac{r^4 + a^2r^2\sin^2\theta - 4ma^2r\cos^2\theta - a^4\cos^2\theta}{|q|^6}.
$$

Here is a more general version of Proposition 3.0.27.

Proposition 3.0.29. *The following identity holds true for any horizontal tensor $\psi \in \mathbf{O}_k$ and set of horizontal indices $I = i_1 \dots i_k$*

$$\begin{aligned}[\nabla_a, \nabla_b]\psi_I = &\left(\frac{1}{2}\big({}^{(a)}tr\chi \nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi_I\right) \in_{ab} \\ &+ {}^{(h)}K\Big[\big(g_{i_1 a}g_{tb} - g_{i_1 b}g_{ta}\big)\psi^t{}_{i_2 \dots i_k} + \cdots \big(g_{i_k a}g_{tb} - g_{i_k b}g_{ta}\big)\psi_{i_1 \dots}{}^t\Big]\end{aligned} \tag{3.0.20}$$

with ${}^{(h)}K$ given by (3.0.18).

Proof. The proof is a simple extension of the proof of Proposition 3.0.27, and is left to the reader. □

Remark 3.0.30. *Observe that in the case when the horizontal structure is tangent to a S-foliation, ${}^{(h)}K$ reduces to the Gauss curvature of S. In the integrable case we can calculate directly*[6] *on any surface of integrability S with Gauss curvature K,*

$$[\nabla_a, \nabla_b]\psi_s = K\big(g_{sa}g_{tb} - g_{sb}g_{ta}\big)\psi^t = K\big(g_{sa}\psi_b - g_{sb}\psi_a\big) = K \in_{ab} {}^*\psi_s$$

which coincides with formula (3.0.17) *in this case. Also for $\psi \in \mathbf{O}_2$ (but not necessarily in $\mathfrak{s}_2$),*

$$\begin{aligned}[\nabla_a, \nabla_b]\psi_{s_1 s_2} &= K\big(g_{s_1 a}g_{tb} - g_{s_1 b}g_{ta}\big)\psi^t{}_{s_2} + K\big(g_{s_2 a}g_{tb} - g_{s_2 b}g_{ta}\big)\psi_{s_1}{}^t \\ &= K\big(g_{s_1 a}\psi_{b s_2} - g_{s_1 b}\psi_{a s_2}\big) + K\big(g_{s_2 a}\psi_{s_1 b} - g_{s_2 b}\psi_{s_1 a}\big).\end{aligned}$$

3.0.4 Horizontal Hodge operators

In this section we recall the Hodge operators on 2-spheres as defined in [Ch-Kl] and extend their properties to the case of non-integrable horizontal structure.

We first define the following operators on horizontal tensors.

Definition 3.0.31. *For a given horizontal* 1*-form ξ, we define the frame dependent operators,*

$$div\,\xi = \delta^{ab}\nabla_b\xi_a, \qquad curl\,\xi = \in^{ab} \nabla_a\xi_b, \qquad (\nabla\widehat{\otimes}\xi)_{ba} = \nabla_b\xi_a + \nabla_a\xi_b - \delta_{ab}(div\,\xi).$$

We collect below some Leibniz rules regarding the horizontal Hodge operators.

[6]One can check directly that $g_{sa}\psi_b - g_{sb}\psi_a = \in_{ab} {}^*\psi_s$.

Lemma 3.0.32. *We have for $\xi, \eta \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$,*

$$
\begin{aligned}
(div\, \eta)\xi - (curl\, \eta)\ {}^*\xi &= \xi \cdot \nabla \eta + \xi \cdot\ {}^*\nabla\ {}^*\eta \\
\xi \widehat{\otimes} (div\, u) &= \xi \cdot \nabla u + \xi \cdot\ {}^*\nabla\ {}^*u \\
\xi \cdot (\nabla \widehat{\otimes} \eta) &= \xi \cdot \nabla f - \xi \cdot\ {}^*\nabla\ {}^*\eta.
\end{aligned}
$$

Proof. See the proof of Lemma 2.1.31 in [GKS-2022]. □

Definition 3.0.33. *Given an orthonormal basis of horizontal vectors e_1, e_2 we define the Hodge type operators (recall Definition 3.0.8), as introduced in [Ch-Kl].*

- $\not{\mathcal{D}}_1$ *takes* $\mathfrak{s}_1$ *into*[7] $\mathfrak{s}_0$*:*
$$\not{\mathcal{D}}_1 \xi = (div\, \xi, curl\, \xi),$$
- $\not{\mathcal{D}}_2$ *takes* $\mathfrak{s}_2$ *into* $\mathfrak{s}_1$*:*
$$(\not{\mathcal{D}}_2 \xi)_a = \nabla^b \xi_{ab},$$
- $\not{\mathcal{D}}_1^*$ *takes* $\mathfrak{s}_0$ *into* $\mathfrak{s}_1$*:*
$$\not{\mathcal{D}}_1^*(f, f_*) = -\nabla_a f + \in_{ab} \nabla_b f_*,$$
- $\not{\mathcal{D}}_2^*$ *takes* $\mathfrak{s}_1$ *into* $\mathfrak{s}_2$*:*
$$\not{\mathcal{D}}_2^* \xi = -\frac{1}{2} \nabla \widehat{\otimes} \xi.$$

Lemma 3.0.34. *Note the following pointwise identities:*

1. *Given $(f, f_*) \in \mathfrak{s}_0$, $u \in \mathfrak{s}_1$ we have*
$$\not{\mathcal{D}}_1^*(f, f_*) \cdot u = (f, f_*) \cdot \not{\mathcal{D}}_1 u - \nabla_a \big(f u^a + f_* ({}^*u)^a \big). \tag{3.0.21}$$
2. *Given $f \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$ we have,*
$$(\not{\mathcal{D}}_2^* f) \cdot u = f \cdot (\not{\mathcal{D}}_2 u) - \nabla_a \big(f_b u^{ab} \big). \tag{3.0.22}$$

Proof. To check (3.0.22) we write

$$(\nabla \widehat{\otimes} f) \cdot u = \big(\nabla_a f_b + \nabla_b f_a - \delta_{ab} \mathrm{div}\, f \big) u_{ab} = 2(\nabla_a f_b) u_{ab} = 2\nabla_a (u_{ab} f_b) - 2(\mathrm{div}\, u) \cdot f.$$

□

In the particular case when the horizontal structure is tangent to 2-spheres S these operators are elliptic on S and have the remarkable properties discussed in Chapter 2 of [Ch-Kl] which we recall in the next section.

[7] Recall that $\mathfrak{s}_0$ refers to pairs of scalar functions.

Hodge operators on spheres

The following results were derived in Chapter 2 of [Ch-Kl] in the context of general 2-dimensional compact surfaces S with strictly positive Gauss curvature K which we will refer from now on as a 2-sphere.

Lemma 3.0.35. *Given a 2-sphere S, we have the following:*

- *The kernels of both $\not{\mathcal{D}}_1$ and $\not{\mathcal{D}}_2$ in $L^2(S)$ are trivial while the kernel of $\not{\mathcal{D}}_1^*$ consists of constant pairs in $\mathfrak{s}_0$.*
- *The operators $\not{\mathcal{D}}_1^*$, resp. $\not{\mathcal{D}}_2^*$ are the L^2 adjoints of $\not{\mathcal{D}}_1$, respectively $\not{\mathcal{D}}_2$.*
- *The kernel of $\not{\mathcal{D}}_2^*$ is the space of conformal Killing vectorfields on S.*

Moreover the following identities hold true[8]*, see [Ch-Kl]:*

$$\begin{aligned} \not{\mathcal{D}}_1^* \not{\mathcal{D}}_1 &= -\triangle_1 + K, \qquad & \not{\mathcal{D}}_1 \not{\mathcal{D}}_1^* &= -\triangle_0, \\ \not{\mathcal{D}}_2^* \not{\mathcal{D}}_2 &= -\frac{1}{2}\triangle_2 + K, \qquad & \not{\mathcal{D}}_2 \not{\mathcal{D}}_2^* &= -\frac{1}{2}(\triangle_1 + K). \end{aligned} \tag{3.0.23}$$

Proof. The statements about L^2 adjoints follow immediately by integrating formulas (3.0.21)-(3.0.22) on S. The formulas (3.0.23) follow easily by using the definitions and commuting derivatives. Note also that for $\xi \in \mathfrak{s}_1$

$$\not{\mathcal{D}}_2^* \xi = -\frac{1}{2}\mathcal{L}_\xi \gamma$$

where γ denotes the induced horizontal metric as in Definition 3.0.4. □

As a simple consequence of (3.0.23) one derives the following L^2 estimates.

Proposition 3.0.36. *Let (S, γ) be a compact manifold with Gauss curvature K.*

i.) *The following identity holds for vectorfields f on S:*

$$\int_S \left(|\nabla f|^2 + K|f|^2\right) = \int_S \left(|div\, f|^2 + |curl\, f|^2\right) = \int_S |\not{\mathcal{D}}_1 f|^2 \tag{3.0.24}$$

[8]Here $\triangle_k : \mathfrak{s}_k \to \mathfrak{s}_k$, $k = 0, 1, 2$, is defined by $(\triangle_k U)_A = \nabla^a \nabla_a U_A$.

ii.) *The following identity holds for symmetric, traceless, 2-tensorfields f on S:*

$$\int_S \left(|\nabla f|^2 + 2K|f|^2\right) = 2\int_S |div\, f|^2 = 2\int_S |\,{\mathcal D} \mkern-10mu /\,_2 f|^2 \tag{3.0.25}$$

iii.) *The following identity holds for pairs of functions (f, f_*) on S:*

$$\int_S \left(|\nabla f|^2 + |\nabla f_*|^2\right) = \int_S |-\nabla f + ({}^*\nabla f_*)|^2 = \int_S |\,{\mathcal D} \mkern-10mu /\,_1^*(f, f_*)|^2 \tag{3.0.26}$$

iv.) *The following identity holds for vectors f on S,*

$$\int_S \left(|\nabla f|^2 - K|f|^2\right) = 2\int_S |\,{\mathcal D} \mkern-10mu /\,_2^* f|^2 \tag{3.0.27}$$

Proof. See Chapter 2 in [Ch-Kl]. $\square$

Bochner identities in the non-integrable case

We extend the identities above to the case of non-integrable horizontal structure.

Lemma 3.0.37. *Given a general possibly non-integrable horizontal structure, the Hodge operators and the Laplacians are related by the following relations for $\xi \in \mathfrak{s}_1$ and $u \in \mathfrak{s}_2$:*

$$\begin{aligned}
{\mathcal D} \mkern-10mu /\,_1^* {\mathcal D} \mkern-10mu /\,_1 \xi &= -\triangle_1 \xi - \frac{1}{2} \in_{ab} [\nabla_a, \nabla_b]\ {}^*\xi,\\
{\mathcal D} \mkern-10mu /\,_2\, {\mathcal D} \mkern-10mu /\,_2^* \xi &= -\frac{1}{2}\triangle_1 \xi + \frac{1}{4} \in_{ab} [\nabla_a, \nabla_b]\ {}^*\xi,\\
{\mathcal D} \mkern-10mu /\,_2^* {\mathcal D} \mkern-10mu /\,_2 u &= -\frac{1}{2}\triangle_2 u - \frac{1}{4} \in_{ab} [\nabla_a, \nabla_b]\ {}^*u.
\end{aligned} \tag{3.0.28}$$

Proof. See the proof of Lemma 2.1.36 in [K-S:Kerr].

$\square$

Using the pointwise relations (3.0.21) and (3.0.22) and the above lemma, we can deduce the following pointwise version of the L^2 estimates of Proposition 3.0.36.

Proposition 3.0.38. *Given a not necessarily integrable horizontal structure, the following pointwise relations hold:*

i. *The following identity holds for $f \in \mathfrak{s}_1$:*

$$|\nabla f|^2 - \frac{1}{2} \in_{ab} [\nabla_a, \nabla_b] \, {}^*f \cdot f = | \not{\mathcal{D}}_1 f|^2 + \nabla_a \Big(\nabla^a f \cdot f - (div\, f) f^a - (curl\, f)(\, {}^*f)^a \Big).$$

ii. *The following identity holds for $f \in \mathfrak{s}_2$:*

$$|\nabla f|^2 - \frac{1}{4} \in_{ab} [\nabla_a, \nabla_b] \, {}^*f \cdot f = 2| \not{\mathcal{D}}_2 f|^2 + \nabla_a \Big(\nabla^a f \cdot f - 2(div\, f)_b f^{ab} \Big).$$

iii. *The following identity holds for $f \in \mathfrak{s}_1$:*

$$|\nabla f|^2 + \frac{1}{4} \in_{ab} [\nabla_a, \nabla_b] \, {}^*f \cdot f = 2| \not{\mathcal{D}}_2^* f|^2 + \nabla_a \Big(\nabla^a f \cdot f + 2(\not{\mathcal{D}}_2^* f)^{ab} f_b \Big).$$

Proof. The above relations are obtained by multiplying relations (3.0.28) by f and integrating by parts in the horizontal directions. □

Remark 3.0.39. *In the integrable case the commutator $\in^{ab} \, [\nabla_a, \nabla_b]$ is given by the standard Gauss formula in terms of K. In the non-integrable case it can be computed by using the generalized Gauss equation, see Proposition 3.0.25.*

Observe that in the relations obtained in Proposition 3.0.38, the divergence terms cannot be discarded upon integration because of the absence of an integrable surface. There are various ways to deal with this difficulty, such as to integrate (3.0.29)-(3.0.29) on the entire spacetime manifold $\mathcal{M}$.

Remark 3.0.40. *Note that the divergence terms in Proposition 3.0.38 can be re-expressed in terms of spacetime divergences based on the following lemma.*

Lemma 3.0.41. *For $f \in \mathfrak{s}_1$, we have*[9]

$$\mathbf{D}^\alpha f_\alpha \quad = \quad \nabla^a f_a + (\eta + \underline{\eta}) \cdot f \tag{3.0.29}$$

where $\underline{\eta}_a := \frac{1}{2}\mathbf{g}(e_a, \mathbf{D}_L \underline{L})$ and $\eta_a := \frac{1}{2}\mathbf{g}(e_a, \mathbf{D}_{\underline{L}} L)$, see Definition 3.1.1.

[9]Here, we extend the horizontal 1-form f as a full 1-form on $\mathcal{M}$ by setting $f_3 = f_4 = 0$.

Proof. We have, using (3.1.3),

$$\begin{aligned}
\mathbf{D}^\alpha f_\alpha - \nabla^a f_a &= -\frac{1}{2}\big(\mathbf{D}_3 f_4 + \mathbf{D}_4 f_3\big) = -\frac{1}{2}\big(e_3(f_4) - f_{\mathbf{D}_3 4} + e_4(f_3) - f_{\mathbf{D}_4 3}\big)\\
&= \frac{1}{2}\big(2\eta_a f_a + 2\underline{\eta}_a f_a\big) = (\eta + \underline{\eta})\cdot f
\end{aligned}$$

as stated. □

Using (3.0.17) we can rewrite Proposition 3.0.38 as follows.

Proposition 3.0.42. *Given a not necessarily integrable horizontal structure, the following pointwise relations hold*[10]*:*

i. *The following identity holds for $f \in \mathfrak{s}_1$:*

$$\begin{aligned}
|\nabla f|^2 + {}^{(h)}K|f|^2 &= |\,{\not{\mathcal{D}}}_1 f|^2 + \frac{1}{2}\Big(\Big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\Big)\,{}^*f\Big)\cdot f + div\,[\,{\not{\mathcal{D}}}_1 f],\\
div\,[\,{\not{\mathcal{D}}}_1 f] &:= \nabla_a\Big(\nabla^a f\cdot f - (div\, f)f^a - (curl\, f)(\,{}^*f)^a\Big).
\end{aligned} \tag{3.0.30}$$

ii. *The following identity holds for $f \in \mathfrak{s}_2$:*

$$\begin{aligned}
|\nabla f|^2 + 2\,{}^{(h)}K|f|^2 &= 2|\,{\not{\mathcal{D}}}_2 f|^2 + \frac{1}{2}\Big(\Big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\Big)\,{}^*f\Big)\cdot f + div\,[\,{\not{\mathcal{D}}}_2 f],\\
div\,[\,{\not{\mathcal{D}}}_2 f] &:= \nabla_a\big(\nabla^a f\cdot f - 2(div\, f)_b f^{ab}\big).
\end{aligned} \tag{3.0.31}$$

iii. *The following identity holds for $f \in \mathfrak{s}_1$:*

$$\begin{aligned}
|\nabla f|^2 - {}^{(h)}K|f|^2 &= 2|\,{\not{\mathcal{D}}}_2^* f|^2 - \frac{1}{2}\Big(\Big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\Big)\,{}^*f\Big)\cdot f + div\,[\,{\not{\mathcal{D}}}_2^* f],\\
div\,[\,{\not{\mathcal{D}}}_2^* f] &:= \nabla_a\Big(\nabla^a f\cdot f + 2(\,{\not{\mathcal{D}}}_2^* f)^{ab} f_b\Big).
\end{aligned} \tag{3.0.32}$$

Proof. From (3.0.17), we have for $f \in \mathfrak{s}_1$ and $u \in \mathfrak{s}_2$:

$$\begin{aligned}
\frac{1}{2}\in_{ab}[\nabla_a, \nabla_b]\,{}^*f &= \frac{1}{2}\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\,{}^*f - {}^{(h)}Kf,\\
\frac{1}{2}\in_{ab}[\nabla_a, \nabla_b]\,{}^*u &= \frac{1}{2}\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\,{}^*u - 2\,{}^{(h)}Ku,
\end{aligned}$$

from which we obtain the stated identities. □

[10] Note that according to Lemma 3.0.41, the divergence terms in the proposition can be re-expressed in terms of the spacetime divergences, see Remark 3.0.40.

3.1 Horizontal structures and Einstein equations

We apply the general formalism for non-integrable structures to the case of a spacetime solution to the Einstein vacuum equation. For an application of the formalism to the Einstein-Maxwell equation, see [Giorgi:KN].

3.1.1 Ricci coefficients

Definition 3.1.1. *We define the horizontal* 1*-forms,*

$$
\begin{aligned}
\underline{\eta}(X) &:= \frac{1}{2}\mathbf{g}(X, \mathbf{D}_L \underline{L}), \qquad \eta(X) := \frac{1}{2}\mathbf{g}(X, \mathbf{D}_{\underline{L}} L),\\
\underline{\xi}(X) &:= \frac{1}{2}\mathbf{g}(X, \mathbf{D}_{\underline{L}} \underline{L}), \qquad \xi(X) := \frac{1}{2}\mathbf{g}(X, \mathbf{D}_L L).
\end{aligned}
$$

With these definitions we have,

$$
\begin{aligned}
\nabla_L X &:= {}^{(h)}(\mathbf{D}_L X) = \mathbf{D}_L X - \underline{\eta}(X) L - \xi(X)\,\underline{L},\\
\nabla_{\underline{L}} X &:= {}^{(h)}(\mathbf{D}_{\underline{L}} X) = \mathbf{D}_{\underline{L}} X - \underline{\xi}(X) L - \eta(X)\,\underline{L}.
\end{aligned}
$$

In addition to the horizontal tensor-fields $\chi, \underline{\chi}, \underline{\eta}, \eta, \xi, \underline{\xi}$ introduced above we also define the scalars,

$$
\underline{\omega} := \frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, L), \qquad \omega := \frac{1}{4}\mathbf{g}(\mathbf{D}_L L,\, \underline{L}),
$$

and the horizontal 1-form,

$$
\zeta(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_X L,\, \underline{L}).
$$

We summarize below the definition of the the horizontal 1-forms $\xi, \underline{\xi}, \eta, \underline{\eta}, \zeta \in \mathbf{O}_1$:

$$
\begin{cases}
\xi(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_L L, X), \quad \underline{\xi}(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, X),\\
\eta(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, X), \quad \underline{\eta}(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_L \underline{L}, X),\\
\zeta(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_X L,\, \underline{L}),
\end{cases}
\tag{3.1.1}
$$

and the real scalars

$$
\omega = \frac{1}{4}\mathbf{g}(\mathbf{D}_L L,\, \underline{L}), \qquad \underline{\omega} = \frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, L). \tag{3.1.2}
$$

Definition 3.1.2. *The horizontal tensor-fields* $\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \xi, \underline{\xi}, \omega, \underline{\omega}$ *are called the connection coefficients of the null pair* $(L, \underline{L})$. *Given an arbitrary basis of horizontal vectorfields* e_1, e_2, *we write using the short hand notation* $\mathbf{D}_a = \mathbf{D}_{e_a}$, $a = 1, 2$,

$$
\begin{aligned}
\underline{\chi}_{ab} &= \mathbf{g}(\mathbf{D}_a \underline{L}, e_b), & \chi_{ab} &= \mathbf{g}(\mathbf{D}_a L, e_b),\\
\underline{\xi}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, e_a), & \xi_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_L L, e_a),\\
\underline{\omega} &= \frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, L), & \omega &= \frac{1}{4}\mathbf{g}(\mathbf{D}_L L, \underline{L}),\\
\underline{\eta}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_L \underline{L}, e_a), & \eta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, e_a),\\
\zeta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_a L, \underline{L}).
\end{aligned}
$$

We easily derive the Ricci formulae,

$$
\begin{aligned}
\mathbf{D}_a e_b &= \nabla_a e_b + \frac{1}{2}\chi_{ab} e_3 + \frac{1}{2}\underline{\chi}_{ab} e_4,\\
\mathbf{D}_a e_4 &= \chi_{ab} e_b - \zeta_a e_4,\\
\mathbf{D}_a e_3 &= \underline{\chi}_{ab} e_b + \zeta_a e_3,\\
\mathbf{D}_3 e_a &= \nabla_3 e_a + \eta_a e_3 + \underline{\xi}_a e_4,\\
\mathbf{D}_3 e_3 &= -2\underline{\omega} e_3 + 2\underline{\xi}_b e_b,\\
\mathbf{D}_3 e_4 &= 2\underline{\omega} e_4 + 2\eta_b e_b,\\
\mathbf{D}_4 e_a &= \nabla_4 e_a + \underline{\eta}_a e_4 + \xi_a e_3,\\
\mathbf{D}_4 e_4 &= -2\omega e_4 + 2\xi_b e_b,\\
\mathbf{D}_4 e_3 &= 2\omega e_3 + 2\underline{\eta}_b e_b.
\end{aligned}
\tag{3.1.3}
$$

3.1.2 Curvature and Weyl fields

Assume that $W \in \mathbf{T}^0_4(\mathcal{M})$ is a Weyl field, i.e.

$$
\begin{cases}
W_{\alpha\beta\mu\nu} = -W_{\beta\alpha\mu\nu} = -W_{\alpha\beta\nu\mu} = W_{\mu\nu\alpha\beta},\\
W_{\alpha\beta\mu\nu} + W_{\alpha\mu\nu\beta} + W_{\alpha\nu\beta\mu} = 0,\\
\mathbf{g}^{\beta\nu} W_{\alpha\beta\mu\nu} = 0.
\end{cases}
\tag{3.1.4}
$$

We define the null components of the Weyl field W, $\alpha(W), \underline{\alpha}(W), \varrho(W) \in \mathbf{O}_2(\mathcal{M})$ and $\beta(W), \underline{\beta}(W) \in \mathbf{O}_1(\mathcal{M})$ by the formulas

$$\begin{cases} \alpha(W)(X,Y) = W(L,X,L,Y), \\ \underline{\alpha}(W)(X,Y) = W(\underline{L},X,\underline{L},Y), \\ \beta(W)(X) = \frac{1}{2}W(X,L,\underline{L},L), \\ \underline{\beta}(W)(X) = \frac{1}{2}W(X,\underline{L},\underline{L},L), \\ \varrho(W)(X,Y) = W(X,\underline{L},Y,L). \end{cases} \tag{3.1.5}$$

Recall that if W is a Weyl field its Hodge dual ${}^*W$, defined by ${}^*W_{\alpha\beta\mu\nu} = \frac{1}{2}\in_{\mu\nu}{}^{\rho\sigma}W_{\alpha\beta\rho\sigma}$, is also a Weyl field. We easily check the formulas,

$$\begin{cases} \underline{\alpha}({}^*W) = {}^*\underline{\alpha}(W), \qquad \alpha({}^*W) = -{}^*\alpha(W), \\ \underline{\beta}({}^*W) = {}^*\underline{\beta}(W), \qquad \beta({}^*W) = -{}^*\beta(W), \\ \varrho({}^*W) = {}^*\varrho(W). \end{cases} \tag{3.1.6}$$

It is easy to check that $\alpha, \underline{\alpha}$ are symmetric traceless horizontal tensor-fields. On the other hand the horizontal 2-tensorfield ϱ is neither symmetric nor traceless. It is convenient to express it in terms of the following two scalar quantities,

$$\rho(W) = \frac{1}{4}W(L,\underline{L},L,\underline{L}), \qquad {}^*\rho(W) = \frac{1}{4}\,{}^*W(L,\underline{L},L,\underline{L}). \tag{3.1.7}$$

Observe also that,

$$\rho({}^*W) = {}^*\rho(W), \qquad {}^*\rho({}^*W) = -\rho.$$

Thus,

$$\varrho(X,Y) = \big(-\rho\,\gamma(X,Y) + {}^*\rho \in (X,Y)\big), \qquad \forall\, X,Y \in \mathbf{O}(\mathcal{M}). \tag{3.1.8}$$

We have

$$\begin{aligned} W_{a3b4} &= \varrho_{ab} = (-\rho\delta_{ab} + {}^*\rho \in_{ab}), \\ W_{ab34} &= 2 \in_{ab} {}^*\rho, \\ W_{abcd} &= - \in_{ab}\in_{cd} \rho, \\ W_{abc3} &= \in_{ab} {}^*\underline{\beta}_c, \\ W_{abc4} &= - \in_{ab} {}^*\beta_c. \end{aligned}$$

3.1.3 Pairing transformations

In addition to the Hodge duality we will need to take into account the duality with respect to the interchange of $e_4 = L, e_3 = \underline{L}$, which we call a pairing transformation. Clearly, under this transformation, $\alpha \leftrightarrow \underline{\alpha}$, $\beta \leftrightarrow -\underline{\beta}$, $\rho \leftrightarrow \rho$, ${}^*\rho \leftrightarrow {}^*\rho$, $\varrho \leftrightarrow \check{\varrho}$ with $\check{\varrho}_{ab} := \varrho_{ba}$. One has to be careful however when combining the Hodge dual and pairing transformations. In that case we have, ${}^*\underline{\alpha} \leftrightarrow -{}^*\alpha$, ${}^*\underline{\beta} \leftrightarrow {}^*\beta$. This is due to the fact that under the pairing transformation $\in_{ab} \to -\in_{ab}$ (since $\in_{ab} = \in_{ab34}$). Indeed, for example,

$$
\begin{aligned}
{}^*\underline{\alpha}_{ab} &= \underline{\alpha}({}^*W)_{ab} = {}^*W_{a3b3} = -\in_{a3c4} W_{c3b3} = \in_{ac34} W_{c3b3} = \in_{ac} \underline{\alpha}_{cb}, \\
{}^*\alpha_{ab} &= \alpha({}^*W)_{ab} = {}^*W_{a4b4} = -\in_{a4c3} W_{c4b4} = -\in_{cb34} W_{c4b4} = -\in_{ac} \alpha_{cb}.
\end{aligned}
$$

The reason ${}^*\beta$ transforms to ${}^*\underline{\beta}$ and not $-{}^*\underline{\beta}$ is that in this case there are two sign changes. In the case of ${}^*\rho$ we have

$$
4\,{}^*\rho = {}^*\mathbf{R}_{3434} = \frac{1}{2}\in_{34}{}^{ab}\mathbf{R}_{ab34} \leftrightarrow \frac{1}{2}\in_{43}{}^{ab}\mathbf{R}_{ab43} = 4\,{}^*\rho.
$$

Here is a schematic presentation of all pairing transformations.

$$
\begin{cases}
\widehat{\underline{\chi}}_{ab} \leftrightarrow \widehat{\chi}_{ab} \\
\mathrm{tr}\underline{\chi} \leftrightarrow \mathrm{tr}\chi \\
{}^{(a)}\mathrm{tr}\underline{\chi} \leftrightarrow -{}^{(a)}\mathrm{tr}\chi \\
\underline{\xi}_a \leftrightarrow \xi_a \\
\underline{\omega} \leftrightarrow \omega \\
\underline{\eta}_a \leftrightarrow \eta_a \\
\zeta_a \leftrightarrow -\zeta_a \\
\alpha \leftrightarrow \underline{\alpha} \\
\beta \leftrightarrow -\underline{\beta} \\
\rho \leftrightarrow \rho \\
\varrho \leftrightarrow \check{\varrho}
\end{cases}
\qquad
\begin{cases}
\in_{ab} \leftrightarrow -\in_{ab} \\
{}^*\widehat{\underline{\chi}}_{ab} \leftrightarrow -{}^*\widehat{\chi}_{ab} \\
{}^*\underline{\xi}_a \leftrightarrow -{}^*\xi_a \\
{}^*\underline{\eta}_a \leftrightarrow -{}^*\eta_a \\
{}^*\zeta_a \leftrightarrow {}^*\zeta_a \\
{}^*\alpha \leftrightarrow -{}^*\underline{\alpha} \\
{}^*\beta \leftrightarrow {}^*\underline{\beta} \\
{}^*\rho \leftrightarrow {}^*\rho \\
{}^{*\,(c)}\nabla_a \leftrightarrow -{}^{*\,(c)}\nabla_a \\
curl \leftrightarrow -curl
\end{cases}
$$

The decomposition above for Weyl fields applies in particular to the Riemann curvature tensor $\mathbf{R}$ of a vacuum spacetime.

In the case of a vacuum spacetime, the non-integrable Gauss curvature defined by (3.0.18) becomes

$$
{}^{(h)}K = -\frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{4}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}} - \rho. \tag{3.1.9}
$$

3.1.4 Horizontal tensor B

We calculate below the components of the horizontal curvature tensor $\mathbf{B}$ defined by the formula, see (3.1.39),

$$\mathbf{B}_{ab\mu\nu} := (\Lambda_\mu)_{3a}(\Lambda_\nu)_{b4} + (\Lambda_\mu)_{4a}(\Lambda_\nu)_{b3} - (\Lambda_\nu)_{3a}(\Lambda_\mu)_{b4} - (\Lambda_\nu)_{4a}(\Lambda_\mu)_{b3}.$$

Proposition 3.1.3. *The components of* $\mathbf{B}$ *are given by the following formulas:*

$$\begin{aligned}
\mathbf{B}_{abc3} = -\mathbf{B}_{ab3c} &= 2\big(- \underline{\chi}_{ca}\eta_b + \underline{\chi}_{cb}\eta_a - \chi_{ca}\underline{\xi}_b + \chi_{cb}\underline{\xi}_a\big),\\
\mathbf{B}_{abc4} = -\mathbf{B}_{ab4c} &= 2\big(- \chi_{ca}\underline{\eta}_b + \chi_{cb}\underline{\eta}_a - \underline{\chi}_{ca}\xi_b + \underline{\chi}_{cb}\xi_a\big),\\
\mathbf{B}_{ab34} = -\mathbf{B}_{ab43} &= 4\big(- \underline{\xi}_a\xi_b + \xi_a\underline{\xi}_b - \eta_a\underline{\eta}_b + \underline{\eta}_a\eta_b\big),\\
\mathbf{B}_{abcd} = -\mathbf{B}_{abdc} &= \chi_{bc}\underline{\chi}_{ad} + \underline{\chi}_{bc}\chi_{ad} - \chi_{ac}\underline{\chi}_{bd} - \underline{\chi}_{ac}\chi_{bd}.
\end{aligned} \tag{3.1.10}$$

The above can also be written as

$$\begin{aligned}
\mathbf{B}_{abc3} &= -tr\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - {}^{(a)}tr\underline{\chi}\big(\in_{ca} \eta_b - \in_{cb} \eta_a\big)\\
&\quad + 2\big(- \widehat{\underline{\chi}}_{ca}\eta_b + \widehat{\underline{\chi}}_{cb}\eta_a - \chi_{ca}\underline{\xi}_b + \chi_{cb}\underline{\xi}_a\big),\\
\mathbf{B}_{abc4} &= -tr\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - {}^{(a)}tr\chi\big(\in_{ca} \underline{\eta}_b - \in_{cb} \underline{\eta}_a\big)\\
&\quad + 2\big(- \widehat{\chi}_{ca}\underline{\eta}_b + \widehat{\chi}_{cb}\underline{\eta}_a - \underline{\chi}_{ca}\xi_b + \underline{\chi}_{cb}\xi_a\big).
\end{aligned} \tag{3.1.11}$$

The only non vanishing component of $\mathbf{B}_{abcd}$ *is given by*

$$\mathbf{B}_{1212} = -\mathbf{B}_{1221} = \mathbf{B}_{2121} = -\frac{1}{2}tr\,\chi tr\underline{\chi} - \frac{1}{2}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \widehat{\chi}\cdot\widehat{\underline{\chi}}.$$

Proof. We write recalling the definition $(\Lambda_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha)$ and definition of Ricci coefficients, see Definition 3.1.2,

$$\begin{aligned}
\mathbf{B}_{abc3} &= (\Lambda_c)_{3a}(\Lambda_3)_{b4} + (\Lambda_c)_{4a}(\Lambda_3)_{b3} - (\Lambda_3)_{3a}(\Lambda_c)_{b4} - (\Lambda_3)_{4a}(\Lambda_c)_{b3}\\
&= -2\underline{\chi}_{ca}\eta_b - 2\chi_{ca}\underline{\xi}_b + 2\underline{\xi}_a\chi_{cb} + 2\eta_a\underline{\chi}_{cb}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{B}_{ab34} &= (\Lambda_3)_{3a}(\Lambda_4)_{b4} + (\Lambda_3)_{4a}(\Lambda_4)_{b3} - (\Lambda_4)_{3a}(\Lambda_3)_{b4} - (\Lambda_4)_{4a}(\Lambda_3)_{b3}\\
&= 4(-\underline{\xi}_a)\xi_b + 4(-\eta_a)\underline{\eta}_b - 4(-\underline{\eta}_a)\eta_b - 4(\underline{\eta}_a)\eta_b - (-\xi_a)\underline{\xi}_b\\
&= 4\big(- \underline{\xi}_a\xi_b + \xi_a\underline{\xi}_b - \eta_a\underline{\eta}_b + \underline{\eta}_a\eta_b\big)
\end{aligned}$$

For the remaining formulas see (3.0.19) and (3.0.16). ☐

NEW

Lemma 3.1.4. *We have*

$$\begin{aligned}
\mathbf{B}_{abc3} + \mathbf{B}_{bca3} + \mathbf{B}_{cab3} &= -2\,{}^{(a)}tr\underline{\chi} \in_{[ca}\eta_{b]} - 2\,{}^{(a)}tr\chi \in_{[ca}\underline{\xi}_{b]}\\
\mathbf{B}_{abc4} + \mathbf{B}_{bca4} + \mathbf{B}_{cab4} &= -2\,{}^{(a)}tr\chi \in_{[ca}\underline{\eta}_{b]} - 2\,{}^{(a)}tr\underline{\chi} \in_{[ca}\xi_{b]}\\
\mathbf{B}_{abcd} + \mathbf{B}_{bcad} + \mathbf{B}_{cabd} &= 0.
\end{aligned}$$

Proof. Note that both $\mathbf{B}_{abc3}$ and $\mathbf{B}_{abc4}$ can be written in the form

$$C_{abc} = (U_{ac}W_b - U_{bc}W_a) + (V_{ac}W_b - V_{bc}W_a)$$

with U symmetric and V antisymmetric. Thus

$$C_{abc} + C_{bca} + C_{cab} = U_{[ac}W_{b]} + V_{[ac}W_{b]} = 2V_{[ac}W_{b]}$$

We deduce

$$\begin{aligned}
\mathbf{B}_{[abc]3} &= \mathbf{B}_{abc3} + \mathbf{B}_{bca3} + \mathbf{B}_{cab3} = -2\,{}^{(a)}\mathrm{tr}\underline{\chi} \in_{[ca}\eta_{b]} - 2\,{}^{(a)}\mathrm{tr}\chi \in_{[ca}\underline{\xi}_{b]}\\
\mathbf{B}_{[abc]4} &= \mathbf{B}_{abc4} + \mathbf{B}_{bca4} + \mathbf{B}_{cab4} = -2\,{}^{(a)}\mathrm{tr}\chi \in_{[ca}\underline{\eta}_{b]} - 2\,{}^{(a)}\mathrm{tr}\underline{\chi} \in_{[ca}\xi_{b]}
\end{aligned}$$

Also, since the only non-vanishing components of $\mathbf{B}_{abcd}$ is $\mathbf{B}_{1212}$,

$$\mathbf{B}_{[121]2} = \mathbf{B}_{1212} + \mathbf{B}_{2112} + \mathbf{B}_{1122} = 0.$$

as stated. □

3.1.5 Null structure equations

We state below the null structure equation in the general setting discussed above. We assume given a vacuum spacetime endowed with a general null frame (e_3, e_4, e_1, e_2) relative to which we define our connection and curvature coefficients.

Proposition 3.1.5 (Null structure equations)**.** *The connection coefficients verify the following equations:*

$$\begin{aligned}
\nabla_3 tr\underline{\chi} &= -|\widehat{\underline{\chi}}|^2 - \frac{1}{2}\left(tr\underline{\chi}^2 - {}^{(a)}tr\underline{\chi}^2\right) + 2div\,\underline{\xi} - 2\underline{\omega}tr\underline{\chi} + 2\underline{\xi}\cdot(\eta + \underline{\eta} - 2\zeta),\\
\nabla_3\,{}^{(a)}tr\underline{\chi} &= -tr\underline{\chi}\,{}^{(a)}tr\underline{\chi} + 2curl\,\underline{\xi} - 2\underline{\omega}\,{}^{(a)}tr\underline{\chi} + 2\underline{\xi}\wedge(-\eta + \underline{\eta} + 2\zeta),\\
\nabla_3\widehat{\underline{\chi}} &= -tr\underline{\chi}\,\widehat{\underline{\chi}} + \nabla\widehat{\otimes}\underline{\xi} - 2\underline{\omega}\widehat{\underline{\chi}} + \underline{\xi}\widehat{\otimes}(\eta + \underline{\eta} - 2\zeta) - \underline{\alpha},
\end{aligned}$$

$$
\begin{aligned}
\nabla_3 tr\,\chi &= -\underline{\widehat{\chi}}\cdot\widehat{\chi} - \frac{1}{2}tr\,\underline{\chi}\,tr\,\chi + \frac{1}{2}\,{}^{(a)}tr\underline{\chi}\,{}^{(a)}tr\chi + 2\,div\,\eta + 2\underline{\omega}\,tr\,\chi + 2\big(\xi\cdot\underline{\xi} + |\eta|^2\big) + 2\rho,\\
\nabla_3\,{}^{(a)}tr\chi &= -\underline{\widehat{\chi}}\wedge\widehat{\chi} - \frac{1}{2}\big({}^{(a)}tr\underline{\chi}\,tr\,\chi + tr\,\underline{\chi}\,{}^{(a)}tr\chi\big) + 2\,curl\,\eta + 2\underline{\omega}\,{}^{(a)}tr\chi + 2\underline{\xi}\wedge\xi - 2\,{}^*\rho,\\
\nabla_3\widehat{\chi} &= -\frac{1}{2}\big(tr\,\chi\underline{\widehat{\chi}} + tr\,\underline{\chi}\widehat{\chi}\big) - \frac{1}{2}\big(-\,{}^*\underline{\widehat{\chi}}\,{}^{(a)}tr\chi + {}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi}\big) + \nabla\widehat{\otimes}\eta + 2\underline{\omega}\widehat{\chi}\\
&+ \underline{\xi}\widehat{\otimes}\xi + \eta\widehat{\otimes}\eta,
\end{aligned}
$$

$$
\begin{aligned}
\nabla_4 tr\,\underline{\chi} &= -\widehat{\chi}\cdot\underline{\widehat{\chi}} - \frac{1}{2}tr\,\chi\,tr\,\underline{\chi} + \frac{1}{2}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + 2\,div\,\underline{\eta} + 2\omega\,tr\,\underline{\chi} + 2\big(\xi\cdot\underline{\xi} + |\underline{\eta}|^2\big) + 2\rho,\\
\nabla_4\,{}^{(a)}tr\underline{\chi} &= -\widehat{\chi}\wedge\underline{\widehat{\chi}} - \frac{1}{2}\big({}^{(a)}tr\chi\,tr\,\underline{\chi} + tr\,\chi\,{}^{(a)}tr\underline{\chi}\big) + 2\,curl\,\underline{\eta} + 2\omega\,{}^{(a)}tr\underline{\chi} + 2\xi\wedge\underline{\xi} + 2\,{}^*\rho,\\
\nabla_4\underline{\widehat{\chi}} &= -\frac{1}{2}\big(tr\,\underline{\chi}\widehat{\chi} + tr\,\chi\underline{\widehat{\chi}}\big) - \frac{1}{2}\big(-\,{}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi} + {}^*\underline{\widehat{\chi}}\,{}^{(a)}tr\chi\big) + \nabla\widehat{\otimes}\underline{\eta} + 2\omega\underline{\widehat{\chi}}\\
&+ \xi\widehat{\otimes}\underline{\xi} + \underline{\eta}\widehat{\otimes}\underline{\eta},
\end{aligned}
$$

$$
\begin{aligned}
\nabla_4 tr\,\chi &= -|\widehat{\chi}|^2 - \frac{1}{2}\big(tr\,\chi^2 - {}^{(a)}tr\chi^2\big) + 2\,div\,\xi - 2\omega\,tr\,\chi + 2\xi\cdot(\underline{\eta} + \eta + 2\zeta),\\
\nabla_4\,{}^{(a)}tr\chi &= -tr\,\chi\,{}^{(a)}tr\chi + 2\,curl\,\xi - 2\omega\,{}^{(a)}tr\chi + 2\xi\wedge(-\underline{\eta} + \eta - 2\zeta),\\
\nabla_4\widehat{\chi} &= -tr\,\chi\,\widehat{\chi} + \nabla\widehat{\otimes}\xi - 2\omega\widehat{\chi} + \xi\widehat{\otimes}(\underline{\eta} + \eta + 2\zeta) - \alpha.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\nabla_3\zeta + 2\nabla\underline{\omega} &= -\underline{\widehat{\chi}}\cdot(\zeta+\eta) - \frac{1}{2}tr\underline{\chi}(\zeta+\eta) - \frac{1}{2}\,{}^{(a)}tr\underline{\chi}(\,{}^*\zeta + {}^*\eta) + 2\underline{\omega}(\zeta-\eta)\\
&\quad + \widehat{\chi}\cdot\underline{\xi} + \frac{1}{2}tr\,\chi\,\underline{\xi} + \frac{1}{2}\,{}^{(a)}tr\chi\,{}^*\underline{\xi} + 2\omega\underline{\xi} - \underline{\beta},\\
\nabla_4\zeta - 2\nabla\omega &= \widehat{\chi}\cdot(-\zeta+\underline{\eta}) + \frac{1}{2}tr\,\chi(-\zeta+\underline{\eta}) + \frac{1}{2}\,{}^{(a)}tr\chi(-\,{}^*\zeta + {}^*\underline{\eta}) + 2\omega(\zeta+\underline{\eta})\\
&\quad -\underline{\widehat{\chi}}\cdot\xi - \frac{1}{2}tr\,\underline{\chi}\,\xi - \frac{1}{2}\,{}^{(a)}tr\underline{\chi}\,{}^*\xi - 2\underline{\omega}\xi - \beta,\\
\nabla_3\underline{\eta} - \nabla_4\underline{\xi} &= -\underline{\widehat{\chi}}\cdot(\underline{\eta}-\eta) - \frac{1}{2}tr\underline{\chi}(\underline{\eta}-\eta) + \frac{1}{2}\,{}^{(a)}tr\underline{\chi}(\,{}^*\underline{\eta} - {}^*\eta) - 4\omega\underline{\xi} + \underline{\beta},\\
\nabla_4\eta - \nabla_3\xi &= -\widehat{\chi}\cdot(\eta-\underline{\eta}) - \frac{1}{2}tr\,\chi(\eta-\underline{\eta}) + \frac{1}{2}\,{}^{(a)}tr\chi(\,{}^*\eta - {}^*\underline{\eta}) - 4\underline{\omega}\xi - \beta,
\end{aligned}
$$

and

$$
\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega} - \xi\cdot\underline{\xi} - (\eta-\underline{\eta})\cdot\zeta + \eta\cdot\underline{\eta} \;=\; \rho.
$$

Also,

$$
\begin{aligned}
div\,\widehat{\chi}+\zeta\cdot\widehat{\chi} &= \frac{1}{2}\nabla tr\,\chi+\frac{1}{2}tr\,\chi\zeta-\frac{1}{2}\,{}^*\nabla\,{}^{(a)}tr\chi-\frac{1}{2}\,{}^{(a)}tr\chi\,{}^*\zeta-\,{}^{(a)}tr\chi\,{}^*\eta-\,{}^{(a)}tr\underline{\chi}\,{}^*\xi-\beta,\\
div\,\underline{\widehat{\chi}}-\zeta\cdot\underline{\widehat{\chi}} &= \frac{1}{2}\nabla tr\,\underline{\chi}-\frac{1}{2}tr\,\underline{\chi}\zeta-\frac{1}{2}\,{}^*\nabla\,{}^{(a)}tr\underline{\chi}+\frac{1}{2}\,{}^{(a)}tr\underline{\chi}\,{}^*\zeta-\,{}^{(a)}tr\underline{\chi}\,{}^*\underline{\eta}-\,{}^{(a)}tr\chi\,{}^*\underline{\xi}+\underline{\beta},
\end{aligned}
$$

and[11]

$$
curl\,\zeta = -\frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}+\frac{1}{4}\big(tr\,\chi\,{}^{(a)}tr\underline{\chi}-tr\,\underline{\chi}\,{}^{(a)}tr\chi\big)+\omega\,{}^{(a)}tr\underline{\chi}-\underline{\omega}\,{}^{(a)}tr\chi+\,{}^*\rho.
$$

Proof. Except for the fact that the order of indices in $\chi,\underline{\chi}$ is important, since they are no longer symmetric, the derivation is exactly as in [Ch-Kl]. □

3.1.6 Null Bianchi identities

We state below the equations verified by the null curvature components of an Einstein vacuum space-time.

Proposition 3.1.6 (Null Bianchi identities)**.** *The curvature components verify the following equations:*

$$
\begin{aligned}
\nabla_3\alpha-\nabla\widehat{\otimes}\beta &= -\frac{1}{2}\big(tr\underline{\chi}\alpha+\,{}^{(a)}tr\underline{\chi}\,{}^*\alpha\big)+4\underline{\omega}\alpha+(\zeta+4\eta)\widehat{\otimes}\beta-3(\rho\widehat{\chi}+\,{}^*\rho\,{}^*\widehat{\chi}),\\
\nabla_4\beta-div\,\alpha &= -2(tr\,\chi\beta-\,{}^{(a)}tr\chi\,{}^*\beta)-2\omega\beta+\alpha\cdot(2\zeta+\underline{\eta})+3(\xi\rho+\,{}^*\xi\,{}^*\rho),\\
\nabla_3\beta+div\,\varrho &= -(tr\underline{\chi}\beta+\,{}^{(a)}tr\underline{\chi}\,{}^*\beta)+2\underline{\omega}\,\beta+2\underline{\beta}\cdot\widehat{\chi}+3(\rho\eta+\,{}^*\rho\,{}^*\eta)+\alpha\cdot\underline{\xi},\\
\nabla_4\rho-div\,\beta &= -\frac{3}{2}(tr\,\chi\rho+\,{}^{(a)}tr\chi\,{}^*\rho)+(2\underline{\eta}+\zeta)\cdot\beta-2\xi\cdot\underline{\beta}-\frac{1}{2}\underline{\widehat{\chi}}\cdot\alpha,\\
\nabla_4\,{}^*\rho+curl\,\beta &= -\frac{3}{2}(tr\,\chi\,{}^*\rho-\,{}^{(a)}tr\chi\rho)-(2\underline{\eta}+\zeta)\cdot\,{}^*\beta-2\xi\cdot\,{}^*\underline{\beta}+\frac{1}{2}\underline{\widehat{\chi}}\cdot\,{}^*\alpha,\\
\nabla_3\rho+div\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\rho-\,{}^{(a)}tr\underline{\chi}\,{}^*\rho)-(2\eta-\zeta)\cdot\underline{\beta}+2\underline{\xi}\cdot\beta-\frac{1}{2}\widehat{\chi}\cdot\underline{\alpha},\\
\nabla_3\,{}^*\rho+curl\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\,{}^*\rho+\,{}^{(a)}tr\underline{\chi}\rho)-(2\eta-\zeta)\cdot\,{}^*\underline{\beta}-2\underline{\xi}\cdot\,{}^*\beta-\frac{1}{2}\widehat{\chi}\cdot\,{}^*\underline{\alpha},\\
\nabla_4\underline{\beta}-div\,\check{\varrho} &= -(tr\,\chi\underline{\beta}+\,{}^{(a)}tr\chi\,{}^*\underline{\beta})+2\omega\,\underline{\beta}+2\beta\cdot\underline{\widehat{\chi}}-3(\rho\underline{\eta}-\,{}^*\rho\,{}^*\underline{\eta})-\underline{\alpha}\cdot\xi,\\
\nabla_3\underline{\beta}+div\,\underline{\alpha} &= -2(tr\,\underline{\chi}\,\underline{\beta}-\,{}^{(a)}tr\underline{\chi}\,{}^*\underline{\beta})-2\underline{\omega}\underline{\beta}-\underline{\alpha}\cdot(-2\zeta+\eta)-3(\underline{\xi}\rho-\,{}^*\underline{\xi}\,{}^*\rho),\\
\nabla_4\underline{\alpha}+\nabla\widehat{\otimes}\underline{\beta} &= -\frac{1}{2}\big(tr\,\chi\underline{\alpha}+\,{}^{(a)}tr\chi\,{}^*\underline{\alpha}\big)+4\omega\underline{\alpha}+(\zeta-4\underline{\eta})\widehat{\otimes}\underline{\beta}-3(\rho\underline{\widehat{\chi}}-\,{}^*\rho\,{}^*\underline{\widehat{\chi}}).
\end{aligned}
$$

[11]Note that this equation follows from expanding $\mathbf{R}_{34ab}$.

Here,

$$div\, \varrho = -(\nabla\rho + {}^*\nabla\, {}^*\rho), \qquad div\, \check{\varrho} = -(\nabla\rho - {}^*\nabla\, {}^*\rho). \tag{3.1.12}$$

Proof. The proof follows line by line from the derivation in [Ch-Kl] except, once more, for keeping track of the lack of symmetry for $\chi, \underline{\chi}$. Note also that $\check{\varrho}_{ab} = \varrho_{ba}$ and that $(\mathrm{d}iv\, \varrho)_b = \nabla^a \varrho_{ab}$. □

Remark 3.1.7. *Note that both the null structure and null Bianchi equations are invariant with respect to the pairing transformations of section 3.1.3.*

3.1.7 Null Bianchi Equations using Hodge Operators

The special structure of the null structure equations is more apparent if we make use of the Hodge operators $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_1^*$, $\mathcal{D}_2^*$. In doing this it is important to remember that $\mathcal{D}_1$ takes $\mathfrak{s}_1$ to $\mathfrak{s}_0$ and that these latter are pairs of. scalars. It is for this reason that

Proposition 3.1.8 (Null structure equations using Hodge operators)**.**

$$\begin{aligned}
\nabla_3\alpha + 2\,\mathcal{D}_2^*\beta &= -\frac{1}{2}\big(tr\,\underline{\chi}\alpha + {}^{(a)}tr\underline{\chi}\,{}^*\alpha\big) + 4\underline{\omega}\alpha + (\zeta + 4\eta)\widehat{\otimes}\beta - 3(\rho\widehat{\chi} + {}^*\rho\,{}^*\widehat{\chi}),\\
\nabla_4\beta - \mathcal{D}_2\alpha &= -2(tr\,\chi\beta - {}^{(a)}tr\chi\,{}^*\beta) - 2\omega\beta + \alpha\cdot(2\zeta + \underline{\eta}) + 3(\xi\rho + {}^*\xi\,{}^*\rho),\\
\nabla_3\beta - \mathcal{D}_1^*(-\rho, {}^*\rho) &= -(tr\,\underline{\chi}\beta + {}^{(a)}tr\underline{\chi}\,{}^*\beta) + 2\underline{\omega}\,\beta + 2\underline{\beta}\cdot\widehat{\chi} + 3(\rho\eta + {}^*\rho\,{}^*\eta) + \alpha\cdot\underline{\xi},\\
\nabla_4(-\rho, {}^*\rho) + \mathcal{D}_1\beta &= -\frac{3}{2}tr\,\chi(-\rho, {}^*\rho) - \frac{3}{2}{}^{(a)}tr\chi(-{}^*\rho, -\rho) - (2\underline{\eta} + \zeta)\cdot(\beta, {}^*\beta)\\
&\quad + 2\xi\cdot(\underline{\beta}, -{}^*\underline{\beta}) + \frac{1}{2}\widehat{\underline{\chi}}\cdot(\alpha, {}^*\alpha),\\
\nabla_3(\rho, {}^*\rho) + \mathcal{D}_1\underline{\beta} &= -\frac{3}{2}tr\,\underline{\chi}(\rho, {}^*\rho) - \frac{3}{2}{}^{(a)}tr\underline{\chi}(-{}^*\rho, \rho) - (2\eta - \zeta)\cdot(\underline{\beta}, {}^*\underline{\beta})\\
&\quad + 2\underline{\xi}\cdot(\beta, -{}^*\beta) - \frac{1}{2}\widehat{\chi}\cdot(\underline{\alpha}, {}^*\underline{\alpha}),\\
\nabla_4\underline{\beta} - \mathcal{D}_1^*(\rho, {}^*\rho) &= -(tr\,\chi\underline{\beta} + {}^{(a)}tr\chi\,{}^*\underline{\beta}) + 2\omega\,\underline{\beta} + 2\beta\cdot\widehat{\underline{\chi}} - 3(\rho\underline{\eta} - {}^*\rho\,{}^*\underline{\eta}) - \underline{\alpha}\cdot\xi,\\
\nabla_3\underline{\beta} + \mathcal{D}_2\underline{\alpha} &= -2(tr\,\underline{\chi}\,\underline{\beta} - {}^{(a)}tr\underline{\chi}\,{}^*\underline{\beta}) - 2\underline{\omega}\underline{\beta} - \underline{\alpha}\cdot(-2\zeta + \eta) - 3(\underline{\xi}\rho - {}^*\underline{\xi}\,{}^*\rho),\\
\nabla_4\underline{\alpha} - 2\,\mathcal{D}_2^*\underline{\beta} &= -\frac{1}{2}\big(tr\,\chi\underline{\alpha} + {}^{(a)}tr\chi\,{}^*\underline{\alpha}\big) + 4\omega\underline{\alpha} + (\zeta - 4\underline{\eta})\widehat{\otimes}\underline{\beta} - 3(\rho\widehat{\underline{\chi}} - {}^*\rho\,{}^*\widehat{\underline{\chi}}).
\end{aligned}$$

Remark 3.1.9. *As we shall see later further simplification can be obtained by introducing complex horizontal tensors.*

3.1.8 Main equations using conformally invariant derivatives

Consider frame transformations of the form

$$e_3' = \lambda^{-1} e_3, \qquad e_4' = \lambda e_4, \qquad e_a' = e_a.$$

Note that under the above mentioned frame transformation we have

$$\begin{aligned}
\mathrm{tr}\,\underline{\chi}' &= \lambda^{-1}\mathrm{tr}\,\underline{\chi}, \quad {}^{(a)}\mathrm{tr}\underline{\chi}' = \lambda^{-1}\,{}^{(a)}\mathrm{tr}\underline{\chi}, \quad \mathrm{tr}\,\chi' = \lambda \mathrm{tr}\,\chi, \quad {}^{(a)}\mathrm{tr}\chi' = \lambda\,{}^{(a)}\mathrm{tr}\chi,\\
\xi' &= \lambda^2 \xi, \quad \eta' = \eta, \quad \underline{\eta}' = \underline{\eta}, \quad \underline{\xi}' = \lambda^{-2}\underline{\xi},\\
\alpha' &= \lambda^2 \alpha, \quad \beta' = \lambda\beta, \quad \rho' = \rho, \quad {}^*\rho' = {}^*\rho, \quad \underline{\beta}' = \lambda^{-1}\underline{\beta}, \quad \underline{\alpha}' = \lambda^{-2}\underline{\alpha},
\end{aligned}$$

and

$$\underline{\omega}' = \lambda^{-1}\left(\underline{\omega} + \frac{1}{2} e_3(\log\lambda)\right), \quad \omega' = \lambda\left(\omega - \frac{1}{2} e_4(\log\lambda)\right), \quad \zeta' = \zeta - \nabla(\log\lambda).$$

Definition 3.1.10 (s-conformally invariants)**.** *We say that a horizontal tensor f is s-conformally invariant if, under the conformal frame transformation above, it changes as $f' = \lambda^s f$.*

Remark 3.1.11. *Note that in the case when f is a Ricci or curvature coefficient s corresponds precisely to the signature, as define in Chapter 5 of [Ch-Kl].*

Remark 3.1.12. *If f s-conformal invariant, then $\nabla_3 f, \nabla_4 f, \nabla_a f$ are not conformal invariant.*

We correct the lacking of being conformal invariant by making the following definition.

Lemma 3.1.13. *If f is s-conformal invariant, then*

1. ${}^{(c)}\nabla_3 f := \nabla_3 f - 2s\underline{\omega} f$ *is $(s-1)$-conformally invariant.*
2. ${}^{(c)}\nabla_4 f := \nabla_4 f + 2s\omega f$ *is $(s+1)$-conformally invariant.*
3. ${}^{(c)}\nabla_A f := \nabla_A f + s\zeta_A f$ *is s-conformally invariant.*

Proof. Immediate verification. □

Remark 3.1.14. *Note that s is precisely what in [Ch-Kl] is called the signature of the tensor. In GHP formalism [GHP], the signature is related to the spin and the boost weights of the complex scalars.*

Using these definitions we rewrite the main equations as follows.

Proposition 3.1.15. *We have*

$$
\begin{aligned}
{}^{(c)}\nabla_3 tr\underline{\chi} &= -|\widehat{\underline{\chi}}|^2 - \frac{1}{2}\big(tr\underline{\chi}^2 - {}^{(a)}tr\underline{\chi}^2\big) + 2\,{}^{(c)}div\,\underline{\xi} + 2\underline{\xi}\cdot(\eta+\underline{\eta}),\\
{}^{(c)}\nabla_3\,{}^{(a)}tr\underline{\chi} &= -tr\underline{\chi}\,{}^{(a)}tr\underline{\chi} + 2\,{}^{(c)}curl\,\underline{\xi} + 2\underline{\xi}\wedge(-\eta+\underline{\eta}),\\
{}^{(c)}\nabla_3\widehat{\underline{\chi}} &= -tr\underline{\chi}\,\widehat{\underline{\chi}} + {}^{(c)}\nabla\widehat{\otimes}\underline{\xi} + \underline{\xi}\widehat{\otimes}(\eta+\underline{\eta}) - \underline{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_3 tr\,\chi &= -\widehat{\underline{\chi}}\cdot\widehat{\chi} - \frac{1}{2}tr\underline{\chi}tr\,\chi + \frac{1}{2}\,{}^{(a)}tr\underline{\chi}\,{}^{(a)}tr\chi + 2\,{}^{(c)}div\,\eta + 2\big(\xi\cdot\underline{\xi} + |\eta|^2\big) + 2\rho,\\
{}^{(c)}\nabla_3\,{}^{(a)}tr\chi &= -\widehat{\underline{\chi}}\wedge\widehat{\chi} - \frac{1}{2}\big({}^{(a)}tr\underline{\chi}tr\,\chi + tr\underline{\chi}\,{}^{(a)}tr\chi\big) + 2\,{}^{(c)}curl\,\eta + 2\underline{\xi}\wedge\xi - 2\,{}^*\rho,\\
{}^{(c)}\nabla_3\widehat{\chi} &= -\frac{1}{2}\big(tr\,\chi\widehat{\underline{\chi}} + tr\underline{\chi}\widehat{\chi}\big) - \frac{1}{2}\big(-\,{}^*\widehat{\underline{\chi}}\,{}^{(a)}tr\chi + {}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi}\big) + {}^{(c)}\nabla\widehat{\otimes}\eta + \underline{\xi}\widehat{\otimes}\xi + \eta\widehat{\otimes}\eta,
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_4 tr\underline{\chi} &= -\widehat{\chi}\cdot\widehat{\underline{\chi}} - \frac{1}{2}tr\,\chi tr\underline{\chi} + \frac{1}{2}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + 2\,{}^{(c)}div\,\underline{\eta} + 2\big(\xi\cdot\underline{\xi} + |\underline{\eta}|^2\big) + 2\rho,\\
{}^{(c)}\nabla_4\,{}^{(a)}tr\underline{\chi} &= -\widehat{\chi}\wedge\widehat{\underline{\chi}} - \frac{1}{2}\big({}^{(a)}tr\chi tr\underline{\chi} + tr\,\chi\,{}^{(a)}tr\underline{\chi}\big) + 2\,{}^{(c)}curl\,\underline{\eta} + 2\xi\wedge\underline{\xi} + 2\,{}^*\rho,\\
{}^{(c)}\nabla_4\widehat{\underline{\chi}} &= -\frac{1}{2}\big(tr\underline{\chi}\widehat{\chi} + tr\,\chi\widehat{\underline{\chi}}\big) - \frac{1}{2}\big(-\,{}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi} + {}^*\widehat{\underline{\chi}}\,{}^{(a)}tr\chi\big) + {}^{(c)}\nabla\widehat{\otimes}\underline{\eta} + \xi\widehat{\otimes}\underline{\xi} + \underline{\eta}\widehat{\otimes}\underline{\eta},
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_4 tr\,\chi &= -|\widehat{\chi}|^2 - \frac{1}{2}\big(tr\,\chi^2 - {}^{(a)}tr\chi^2\big) + 2\,{}^{(c)}div\,\xi + 2\xi\cdot(\underline{\eta}+\eta),\\
{}^{(c)}\nabla_4\,{}^{(a)}tr\chi &= -tr\,\chi\,{}^{(a)}tr\chi + 2\,{}^{(c)}curl\,\xi + 2\xi\wedge(-\underline{\eta}+\eta),\\
{}^{(c)}\nabla_4\widehat{\chi} &= -tr\,\chi\,\widehat{\chi} + {}^{(c)}\nabla\widehat{\otimes}\xi + \xi\widehat{\otimes}(\underline{\eta}+\eta) - \alpha,
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_3\underline{\eta} - {}^{(c)}\nabla_4\underline{\xi} &= -\widehat{\underline{\chi}}\cdot(\underline{\eta}-\eta) - \frac{1}{2}tr\underline{\chi}(\underline{\eta}-\eta) + \frac{1}{2}\,{}^{(a)}tr\underline{\chi}\big({}^*\underline{\eta} - {}^*\eta\big) + \underline{\beta},\\
{}^{(c)}\nabla_4\eta - {}^{(c)}\nabla_3\xi &= -\widehat{\chi}\cdot(\eta-\underline{\eta}) - \frac{1}{2}tr\,\chi(\eta-\underline{\eta}) + \frac{1}{2}\,{}^{(a)}tr\chi\big({}^*\eta - {}^*\underline{\eta}\big) - \beta.
\end{aligned}
$$

Also,

$$
\begin{aligned}
{}^{(c)}div\,\widehat{\chi} &= \frac{1}{2}\,{}^{(c)}\nabla(tr\,\chi) - \frac{1}{2}\,{}^{*\,(c)}\nabla\big({}^{(a)}tr\chi\big) - {}^{(a)}tr\chi\,{}^*\eta - {}^{(a)}tr\underline{\chi}\,{}^*\xi - \beta,\\
{}^{(c)}div\,\widehat{\underline{\chi}} &= \frac{1}{2}\,{}^{(c)}\nabla(tr\underline{\chi}) - \frac{1}{2}\,{}^{*\,(c)}\nabla\big({}^{(a)}tr\underline{\chi}\big) - {}^{(a)}tr\underline{\chi}\,{}^*\underline{\eta} - {}^{(a)}tr\chi\,{}^*\underline{\xi} + \underline{\beta}.
\end{aligned}
$$

Proposition 3.1.16. *We have*

$$
\begin{aligned}
{}^{(c)}\nabla_3\alpha - {}^{(c)}\nabla\widehat{\otimes}\beta &= -\frac{1}{2}\big(tr\underline{\chi}\alpha + {}^{(a)}tr\underline{\chi}\ {}^*\alpha\big) + 4\eta\widehat{\otimes}\beta - 3(\rho\widehat{\chi} + {}^*\rho\ {}^*\widehat{\chi}),\\
{}^{(c)}\nabla_4\beta - {}^{(c)}div\,\alpha &= -2(tr\,\chi\beta - {}^{(a)}tr\chi\ {}^*\beta) + \alpha\cdot\underline{\eta} + 3(\xi\rho + {}^*\xi\ {}^*\rho),\\
{}^{(c)}\nabla_3\beta + {}^{(c)}div\,\varrho &= -(tr\underline{\chi}\beta + {}^{(a)}tr\underline{\chi}\ {}^*\beta) + 2\underline{\beta}\cdot\widehat{\chi} + 3(\rho\eta + {}^*\rho\ {}^*\eta) + \alpha\cdot\underline{\xi},\\
{}^{(c)}\nabla_4\rho - {}^{(c)}div\,\beta &= -\frac{3}{2}(tr\,\chi\rho + {}^{(a)}tr\chi\ {}^*\rho) + 2\underline{\eta}\cdot\beta - 2\xi\cdot\underline{\beta} - \frac{1}{2}\widehat{\underline{\chi}}\cdot\alpha,\\
{}^{(c)}\nabla_4\ {}^*\rho + {}^{(c)}curl\,\beta &= -\frac{3}{2}(tr\,\chi\ {}^*\rho - {}^{(a)}tr\chi\rho) - 2\underline{\eta}\cdot\ {}^*\beta - 2\xi\cdot\ {}^*\underline{\beta} + \frac{1}{2}\widehat{\underline{\chi}}\cdot\ {}^*\alpha,\\
{}^{(c)}\nabla_3\rho + {}^{(c)}div\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\rho - {}^{(a)}tr\underline{\chi}\ {}^*\rho) - 2\eta\cdot\underline{\beta} + 2\underline{\xi}\cdot\beta - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha},\\
{}^{(c)}\nabla_3\ {}^*\rho + {}^{(c)}curl\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\ {}^*\rho + {}^{(a)}tr\underline{\chi}\rho) - 2\eta\cdot\ {}^*\underline{\beta} - 2\underline{\xi}\cdot\ {}^*\beta - \frac{1}{2}\widehat{\chi}\cdot\ {}^*\underline{\alpha},\\
{}^{(c)}\nabla_4\underline{\beta} - {}^{(c)}div\,\check{\varrho} &= -(tr\,\chi\underline{\beta} + {}^{(a)}tr\chi\ {}^*\underline{\beta}) + 2\beta\cdot\widehat{\underline{\chi}} - 3(\rho\underline{\eta} - {}^*\rho\ {}^*\underline{\eta}) - \underline{\alpha}\cdot\xi,\\
{}^{(c)}\nabla_3\underline{\beta} + {}^{(c)}div\,\underline{\alpha} &= -2(tr\underline{\chi}\,\underline{\beta} - {}^{(a)}tr\underline{\chi}\ {}^*\underline{\beta}) - \underline{\alpha}\cdot\eta - 3(\underline{\xi}\rho - {}^*\underline{\xi}\ {}^*\rho),\\
{}^{(c)}\nabla_4\underline{\alpha} + {}^{(c)}\nabla\widehat{\otimes}\underline{\beta} &= -\frac{1}{2}\big(tr\,\chi\underline{\alpha} + {}^{(a)}tr\chi\ {}^*\underline{\alpha}\big) - 4\underline{\eta}\widehat{\otimes}\underline{\beta} - 3(\rho\widehat{\underline{\chi}} - {}^*\rho\ {}^*\widehat{\underline{\chi}}).
\end{aligned}
$$

3.1.9 Spacetimes of Petrov type D

Consider an Einstein vacuum spacetime $(\mathcal{M}, \mathbf{g})$. A spacetime is said to be of type D if there exists an horizontal structure for which $\alpha, \beta, \underline{\beta}, \underline{\alpha}$ vanish identically. The main example is provided by the Kerr family. Using the first and last equations in proposition 3.1.6 we deduce, for any spacetime of type D that $\frac{3}{2}(\rho\widehat{\chi} + {}^*\rho\ {}^*\widehat{\chi})$ and $\frac{3}{2}(\rho\widehat{\underline{\chi}} - {}^*\rho\ {}^*\widehat{\underline{\chi}})$ from which we easily deduce that $\widehat{\chi} = \widehat{\underline{\chi}} = 0$. must vanish. Similarly, using the second and second to last equations of the same proposition, we deduce that $\frac{3}{2}(\xi\rho + {}^*\xi\ {}^*\rho) = 0$ and $\frac{3}{2}(\underline{\xi}\rho - {}^*\underline{\xi}\ {}^*\rho) = 0$ from which we also infer that $\xi = \underline{\xi} = 0$. We obtain the well known Penrose-Saks theorem according to which, if a spacetime is of type D we must have, relative to the correspondinh horizontal structure,

$$
\widehat{\chi} = \widehat{\underline{\chi}} = \xi = \underline{\xi} = 0 \tag{3.1.13}
$$

3.1.10 Commutation formulas

Lemma 3.1.17. *Let $U_A = U_{a_1\ldots a_k}$ be a general k-horizontal tensorfield.*

1. *We have*

$$[\nabla_3, \nabla_b]U_A = -\underline{\chi}_{bc}\nabla_c U_A + (\eta_b - \zeta_b)\nabla_3 U_A + \underline{\xi}_b \nabla_4 U_A + \dot{\mathbf{R}}_{a_i c 3 b} U_{a_1}{}^{c}{}_{a_k}. \quad (3.1.14)$$

2. *We have*

$$[\nabla_4, \nabla_b]U_A = -\chi_{bc}\nabla_c U_a + (\underline{\eta}_b + \zeta_b)\nabla_4 U_a + \xi_b \nabla_3 U_a + \dot{\mathbf{R}}_{a_i c 4 b} U_{a_1}{}^{c}{}_{a_k} \quad (3.1.15)$$

3. *We have*

$$[\nabla_4, \nabla_3]U_A = 2(\underline{\eta}_b - \eta_b)\nabla_b U_A + 2\omega\nabla_3 U_A - 2\underline{\omega}\nabla_4 U_A + \dot{\mathbf{R}}_{a_i c 43} U_{a_1}{}^{c}{}_{a_k} \quad (3.1.16)$$

where, recall Proposition 3.1.3 and that $\dot{\mathbf{R}}_{ab\mu\nu} := \mathbf{R}_{ab\mu\nu} + \frac{1}{2}\mathbf{B}_{ab\mu\nu}$,

$$\begin{aligned}
\dot{\mathbf{R}}_{ac3b} =& - \in_{a_c} \ {}^*\underline{\beta}_b + \frac{1}{2} tr\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) + \frac{1}{2}\,{}^{(a)}tr\underline{\chi}\big(\in_{ca} \eta_b - \in_{cb} \eta_a\big) \\
& - \big(- \widehat{\chi}_{ca}\underline{\eta}_b + \frac{1}{2}\widehat{\chi}_{cb}\underline{\eta}_a - \frac{1}{2}\underline{\chi}_{ca}\xi_b + \underline{\chi}_{cb}\xi_a\big), \\
\dot{\mathbf{R}}_{ac4b} =& \in_{a_c} \ {}^*\beta_b + \frac{1}{2} tr\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) + \frac{1}{2}\,{}^{(a)}tr\chi\big(\in_{ca} \underline{\eta}_b - \in_{cb} \underline{\eta}_a\big) \\
& - \big(- \widehat{\chi}_{ca}\underline{\eta}_b + \widehat{\chi}_{cb}\underline{\eta}_a - \underline{\chi}_{ca}\xi_b + \underline{\chi}_{cb}\xi_a\big), \\
\dot{\mathbf{R}}_{ab43} =& -2 \in_{ab} \ {}^*\rho - 2\big(- \eta_a\underline{\eta}_b + \underline{\eta}_a\eta_b - \underline{\xi}_a\xi_b + \xi_a\underline{\xi}_b\big).
\end{aligned} \quad (3.1.17)$$

Proof. See the proof of Lemma 2.2.7 in [GKS-2022]. As a corollary we derive □

Lemma 3.1.18. *The following commutation formulas hold true:*

1. *Given* $f \in \mathfrak{s}_0$, *we have*

$$\begin{aligned}
[\nabla_3, \nabla_a]f =& -\frac{1}{2}\left(tr\underline{\chi}\nabla_a f + {}^{(a)}tr\underline{\chi}\ {}^*\nabla_a f\right) + (\eta_a - \zeta_a)\nabla_3 f - \underline{\widehat{\chi}}_{ab}\nabla_b f \\
& + \underline{\xi}_a \nabla_4 f, \\
[\nabla_4, \nabla_a]f =& -\frac{1}{2}\left(tr\,\chi\nabla_a f + {}^{(a)}tr\chi\ {}^*\nabla_a f\right) + (\underline{\eta}_a + \zeta_a)\nabla_4 f - \widehat{\chi}_{ab}\nabla_b f \\
& + \xi_a \nabla_3 f, \\
[\nabla_4, \nabla_3]f =& \ 2(\underline{\eta} - \eta)\cdot\nabla f + 2\omega\nabla_3 f - 2\underline{\omega}\nabla_4 f.
\end{aligned} \quad (3.1.18)$$

2. *Given $u \in \mathfrak{s}_1$, we have*

$$\begin{aligned}
[\nabla_3, \nabla_a]u_b &= -\frac{1}{2}tr\underline{\chi}\big(\nabla_a u_b + \eta_b u_a - \delta_{ab}\eta\cdot u\big)\\
&- \frac{1}{2}{}^{(a)}tr\underline{\chi}\big({}^*\nabla_a u_b + \eta_b {}^*u_a - \in_{ab}\eta\cdot u\big)\\
&+ (\eta-\zeta)_a\nabla_3 u_b + Err_{3ab}[u],\\
Err_{3ab}[u] &= -{}^*\underline{\beta}_a {}^*u_b + \underline{\xi}_a\nabla_4 u_b - \underline{\xi}_b\chi_{ac}u_c + \chi_{ab}\underline{\xi}\cdot u - \widehat{\underline{\chi}}_{ac}\nabla_c u_b - \eta_b\widehat{\underline{\chi}}_{ac}u_c\\
&+ \widehat{\underline{\chi}}_{ab}\eta\cdot u,
\end{aligned} \tag{3.1.19}$$

$$\begin{aligned}
[\nabla_4, \nabla_a]u_b &= -\frac{1}{2}tr\,\chi\big(\nabla_a u_b + \underline{\eta}_b u_a - \delta_{ab}\underline{\eta}\cdot u\big)\\
&- \frac{1}{2}{}^{(a)}tr\chi\big({}^*\nabla_a u_b + \underline{\eta}_b {}^*u_a - \in_{ab}\underline{\eta}\cdot u\big) + (\underline{\eta}+\zeta)_a\nabla_4 u_b\\
&+ Err_{4ab}[u],\\
Err_{4ab}[u] &= {}^*\beta_a {}^*u_b + \xi_a\nabla_3 u_b - \xi_b\underline{\chi}_{ac}u_c + \underline{\chi}_{ab}\xi\cdot u - \widehat{\chi}_{ac}\nabla_c u_b - \underline{\eta}_b\widehat{\chi}_{ac}u_c\\
&+ \widehat{\chi}_{ab}\underline{\eta}\cdot u,
\end{aligned} \tag{3.1.20}$$

$$\begin{aligned}
[\nabla_4, \nabla_3]u_a &= 2\omega\nabla_3 u_a - 2\underline{\omega}\nabla_4 u_a + 2(\underline{\eta}_b - \eta_b)\nabla_b u_a + 2(\underline{\eta}\cdot u)\eta_a - 2(\eta\cdot u)\underline{\eta}_a\\
&- 2\,{}^*\rho\,{}^*u_a + Err_{43a}[u],\\
Err_{43a}[u] &= 2\big(\underline{\xi}_a\xi_b - \xi_a\underline{\xi}_b\big)u^b.
\end{aligned} \tag{3.1.21}$$

3. *Given $u \in \mathfrak{s}_2$, we have*

$$\begin{aligned}
[\nabla_3, \nabla_a]u_{bc} &= -\frac{1}{2}tr\underline{\chi}\,(\nabla_a u_{bc} + \eta_b u_{ac} + \eta_c u_{ab} - \delta_{ab}(\eta\cdot u)_c - \delta_{ac}(\eta\cdot u)_b)\\
&- \frac{1}{2}{}^{(a)}tr\underline{\chi}\,({}^*\nabla_a u_{bc} + \eta_b {}^*u_{ac} + \eta_c {}^*u_{ab} - \in_{ab}(\eta\cdot u)_c - \in_{ac}(\eta\cdot u)_b)\\
&+ (\eta_a - \zeta_a)\nabla_3 u_{bc} + Err_{3abc}[u],\\
Err_{3abc}[u] &= -2\,{}^*\underline{\beta}_a {}^*u_{bc} + \underline{\xi}_a\nabla_4 u_{bc} - \underline{\xi}_b\chi_{ad}u_{dc} - \underline{\xi}_c\chi_{ad}u_{bd} + \chi_{ab}\underline{\xi}_d u_{dc}\\
&+ \chi_{ac}\underline{\xi}_d u_{bd} - \widehat{\underline{\chi}}_{ad}\nabla_d u_{bc} - \eta_b\widehat{\underline{\chi}}_{ad}u_{dc} - \eta_c\widehat{\underline{\chi}}_{ad}u_{bd} + \widehat{\underline{\chi}}_{ab}\eta_d u_{dc} + \widehat{\underline{\chi}}_{ac}\eta_d u_{bd},
\end{aligned} \tag{3.1.22}$$

$$\begin{aligned}
[\nabla_4, \nabla_a]u_{bc} &= -\frac{1}{2}tr\,\chi\,(\nabla_a u_{bc} + \underline{\eta}_b u_{ac} + \underline{\eta}_c u_{ab} - \delta_{ab}(\underline{\eta}\cdot u)_c - \delta_{ac}(\underline{\eta}\cdot u)_b)\\
&- \frac{1}{2}{}^{(a)}tr\chi\,({}^*\nabla_a u_{bc} + \underline{\eta}_b {}^*u_{ac} + \underline{\eta}_c {}^*u_{ab} - \in_{ab}(\underline{\eta}\cdot u)_c - \in_{ac}(\underline{\eta}\cdot u)_b)\\
&+ (\underline{\eta}_a + \zeta_a)\nabla_4 u_{bc} + Err_{4abc}[u],\\
Err_{4abc}[u] &= 2\,{}^*\beta_a {}^*u_{bc} + \xi_a\nabla_3 u_{bc} - \xi_b\underline{\chi}_{ad}u_{dc} - \xi_c\underline{\chi}_{ad}u_{bd} + \underline{\chi}_{ab}\xi_d u_{dc} + \underline{\chi}_{ac}\xi_d u_{bd}\\
&- \widehat{\chi}_{ad}\nabla_d u_{bc} - \underline{\eta}_b\widehat{\chi}_{ad}u_{dc} - \underline{\eta}_c\widehat{\chi}_{ad}u_{bd} + \widehat{\chi}_{ab}\underline{\eta}_d u_{dc} + \widehat{\chi}_{ac}\underline{\eta}_d u_{bd},
\end{aligned} \tag{3.1.23}$$

$$
\begin{aligned}
[\nabla_4, \nabla_3]u_{ab} &= 2\omega\nabla_3 u_{ab} - 2\underline{\omega}\nabla_4 u_{ab} + 2(\underline{\eta}_c - \eta_c)\nabla_c u_{ab} \\
&\quad - 2\underline{\eta}_a\eta_c u_{bc} - 2\underline{\eta}_b\eta_c u_{ac} + 2\eta_a\underline{\eta}_c u_{bc} + 2\eta_b\underline{\eta}_c u_{ac} - 4\ {}^*\rho\ {}^*u_{ab} + Err_{43ab}[u] \\
&= 2\omega\nabla_3 u_{ab} - 2\underline{\omega}\nabla_4 u_{ab} + 2(\underline{\eta}_c - \eta_c)\nabla_c u_{ab} + 4\eta\widehat{\otimes}(\underline{\eta}\cdot u) \\
&\quad - 4\underline{\eta}\widehat{\otimes}(\eta\cdot u) - 4\ {}^*\rho\ {}^*u_{ab} + Err_{43ab}[u], \\
Err_{43ab}[u] &= 2\big(\underline{\xi}_a\xi_c - \xi_a\underline{\xi}_c\big)u^c{}_b + 2\big(\underline{\xi}_b\xi_c - \xi_b\underline{\xi}_c\big)u_a{}^c.
\end{aligned}
\tag{3.1.24}
$$

We deduce the following corollary.

Corollary 3.1.19. *The following commutation formulas hold true:*

1. *Given $u \in \mathfrak{s}_1$, we have*

$$
\begin{aligned}
[\nabla_3, \,div\,]u &= -\frac{1}{2}tr\underline{\chi}\left(div\,u - \eta\cdot u\right) + \frac{1}{2}{}^{(a)}tr\underline{\chi}\left(div\ {}^*u - \eta\cdot\ {}^*u\right) + (\eta-\zeta)\cdot\nabla_3 u \\
&\quad + Err_{3div}[u], \\
Err_{3div}[u] &= -\ {}^*\underline{\beta}\cdot\ {}^*u + \underline{\xi}\cdot\nabla_4 u - \underline{\xi}\cdot\widehat{\chi}\cdot u - \widehat{\underline{\chi}}\cdot\nabla u - \eta\cdot\widehat{\underline{\chi}}\cdot u, \\
[\nabla_4, \,div\,]u &= -\frac{1}{2}tr\,\chi\left(div\,u - \underline{\eta}\cdot u\right) + \frac{1}{2}{}^{(a)}tr\chi\left(div\ {}^*u - \underline{\eta}\cdot\ {}^*u\right) + (\underline{\eta}+\zeta)\cdot\nabla_4 u \\
&\quad + Err_{4div}[u], \\
Err_{4div}[u] &= \ {}^*\beta\cdot\ {}^*u + \xi\cdot\nabla_3 u - \xi\cdot\widehat{\underline{\chi}}\cdot u - \widehat{\chi}\cdot\nabla u - \underline{\eta}\cdot\widehat{\chi}\cdot u.
\end{aligned}
\tag{3.1.25}
$$

Also,

$$
\begin{aligned}
[\nabla_3, \nabla\widehat{\otimes}]u &= -\frac{1}{2}tr\underline{\chi}\left(\nabla\widehat{\otimes}u + \eta\widehat{\otimes}u\right) - \frac{1}{2}{}^{(a)}tr\underline{\chi}\ {}^*\left(\nabla\widehat{\otimes}u + \underline{\eta}\widehat{\otimes}u\right) + (\eta-\zeta)\widehat{\otimes}\nabla_3 u \\
&\quad + Err_{3\widehat{\otimes}}[u], \\
Err_{3\widehat{\otimes}}[u] &= -\ {}^*\underline{\beta}\widehat{\otimes}\ {}^*u + \underline{\xi}\widehat{\otimes}\nabla_4 u - \underline{\xi}\widehat{\otimes}(\chi\cdot u) + \widehat{\chi}\,(\underline{\xi}\cdot u) - \widehat{\underline{\chi}}\cdot\nabla u - \eta\widehat{\otimes}(\widehat{\underline{\chi}}\cdot u) \\
&\quad + \widehat{\underline{\chi}}(\eta\cdot u), \\
[\nabla_4, \nabla\widehat{\otimes}]u &= -\frac{1}{2}tr\,\chi\left(\nabla\widehat{\otimes}u + \underline{\eta}\widehat{\otimes}u\right) - \frac{1}{2}{}^{(a)}tr\chi\ {}^*\left(\nabla\widehat{\otimes}u + \underline{\eta}\widehat{\otimes}u\right) + (\underline{\eta}+\zeta)\widehat{\otimes}\nabla_4 u \\
&\quad + Err_{4\widehat{\otimes}}[u], \\
Err_{4\widehat{\otimes}}[u] &= \ {}^*\beta\widehat{\otimes}\ {}^*u + \xi\widehat{\otimes}\nabla_3 u - \xi\widehat{\otimes}(\underline{\chi}\cdot u) + \widehat{\underline{\chi}}\,(\xi\cdot u) - \widehat{\chi}\cdot\nabla u - \underline{\eta}\widehat{\otimes}(\widehat{\chi}\cdot u) \\
&\quad + \widehat{\chi}(\underline{\eta}\cdot u).
\end{aligned}
\tag{3.1.26}
$$

2. Given $u \in \mathfrak{s}_2$, we have

$$
\begin{aligned}
[\nabla_3, div\,]u &= -\frac{1}{2}tr\underline{\chi}\big(div\, u - 2\eta\cdot u\big) + \frac{1}{2}\,{}^{(a)}tr\underline{\chi}\big(div\ {}^*u - 2\eta\cdot\ {}^*u\big)\\
&\quad + (\eta-\zeta)\cdot\nabla_3 u + Err_{3_{div}}[u],\\
Err_{3_{div}}[u] &= -2\ {}^*\underline{\beta}\cdot\ {}^*u + \underline{\xi}\cdot\nabla_4 u - \underline{\xi}\cdot\chi\cdot u - (\chi\cdot u)\underline{\xi} + \underline{\xi}\cdot u\cdot\chi - \widehat{\underline{\chi}}\cdot\nabla u\\
&\quad - \eta\cdot\widehat{\underline{\chi}}\cdot u - (\widehat{\underline{\chi}}\cdot u)\eta + \eta\cdot u\cdot\widehat{\underline{\chi}},\\
[\nabla_4, div\,]u &= -\frac{1}{2}tr\,\chi\big(div\, u - 2\underline{\eta}\cdot u\big) + \frac{1}{2}\,{}^{(a)}tr\chi\big(div\ {}^*u - 2\underline{\eta}\cdot\ {}^*u\big)\\
&\quad + (\underline{\eta}+\zeta)\cdot\nabla_4 u + Err_{4_{div}}[u],\\
Err_{4_{div}}[u] &= 2\ {}^*\beta\cdot\ {}^*u + \xi\cdot\nabla_3 u - \xi\cdot\underline{\chi}\cdot u - (\underline{\chi}\cdot u)\xi + \xi\cdot u\cdot\underline{\chi} - \widehat{\chi}\cdot\nabla u\\
&\quad - \underline{\eta}\cdot\widehat{\chi}\cdot u - (\widehat{\chi}\cdot u)\underline{\eta} + \underline{\eta}\cdot u\cdot\widehat{\chi}.
\end{aligned}
\tag{3.1.27}
$$

Proof. Straightforward. See also section 2.2.7 in [GKS-2022]. □

3.1.11 Commutation formulas for conformal derivatives

Lemma 3.1.20. *Let $U_A = U_{a_1\ldots a_k}$ be a general k-horizontal tensorfield of signature s..*

1. We have

$$
\begin{aligned}
[\,{}^{(c)}\nabla_3,\ {}^{(c)}\nabla_b]U_A &= -\underline{\chi}_{bc}\,{}^{(c)}\nabla_c U_A + \eta_b\,{}^{(c)}\nabla_3 U_A + \underline{\xi}_b\,{}^{(c)}\nabla_4 U_A + \sum_{i=1}^k \dot{\mathbf{R}}_{a_i c 3 b} U_{a_1}{}^{c}{}_{a_k}\\
&\quad - s\big(\underline{\chi}\cdot\eta - \chi\cdot\underline{\xi} + \underline{\beta}\big)U_A
\end{aligned}
\tag{3.1.28}
$$

2. We have

$$
\begin{aligned}
[\,{}^{(c)}\nabla_4,\ {}^{(c)}\nabla_b]U_A &= -\chi_{bc}\,{}^{(c)}\nabla_c U_a + \underline{\eta}_b\,{}^{(c)}\nabla_4 U_a + \xi_b\,{}^{(c)}\nabla_3 U_a + \sum_{i=1}^k \dot{\mathbf{R}}_{a_i c 4 b} U_{a_1}{}^{c}{}_{a_k}\\
&\quad - s\big(\chi\cdot\eta - \underline{\chi}\cdot\xi - \beta\big)U_A
\end{aligned}
\tag{3.1.29}
$$

3. We have

$$
\begin{aligned}
[\,{}^{(c)}\nabla_4,\ {}^{(c)}\nabla_3]U_A &= 2(\underline{\eta}_b - \eta_b)\,{}^{(c)}\nabla_b U_A + \sum_{i=1}^k \dot{\mathbf{R}}_{a_i b 43} U_{a_1}{}^{b}{}_{a_k}\\
&\quad - 2s\big(\rho - \eta\cdot\underline{\eta} + \xi\cdot\underline{\xi}\big)U
\end{aligned}
\tag{3.1.30}
$$

with the terms in $\dot{\mathbf{R}}$ given by formula 3.1.17.

Proof. We first deduce, using[12] the definition of conformal derivative, since $\mathrm{sign}(U) = s$, $\mathrm{sign}({}^{(c)}\nabla_3 U) = s - 1$,

$$[{}^{(c)}\nabla_3, {}^{(c)}\nabla_b]U_A = [\nabla_3, \nabla_b]U_A + s\big(\nabla_3\zeta_b + 2\nabla_b\underline{\omega}\big)U_A + \zeta_b\,{}^{(c)}\nabla_3 U_A \tag{3.1.31}$$

Hence,

$$\begin{aligned}
[{}^{(c)}\nabla_3, {}^{(c)}\nabla_b]U_A &= -\underline{\chi}_{bc}\nabla_c U_A + (\eta_b - \zeta_b)\nabla_3 U_A + \underline{\xi}_b\nabla_4 U_A + \dot{\mathbf{R}}_{a_i c 3 b} U_{a_1}{}^{c}{}_{a_k}\\
&\quad +\zeta_b\,{}^{(c)}\nabla_3 U_A + s\big(\nabla_3\zeta_b + 2\nabla_b\underline{\omega}\big)U_A\\
&= -\underline{\chi}_{bc}\big({}^{(c)}\nabla_c U_A - s\zeta_c U_A\big) + (\eta_b - \zeta_b)\big({}^{(c)}\nabla_3 U_A + 2s\underline{\omega}U_A\big)\\
&\quad +\underline{\xi}_b\big({}^{(c)}\nabla_4 - 2s\omega\big)U_A + s\big(\nabla_3\zeta_b + 2\nabla_b\underline{\omega}\big)U_A + \dot{\mathbf{R}}_{a_i c 3 b} U_{a_1}{}^{c}{}_{a_k}\\
&= -\underline{\chi}_{bc}\,{}^{(c)}\nabla_c U_A + \underline{\eta}\,{}^{(c)}\nabla_3 U_A + \underline{\xi}_b\,{}^{(c)}\nabla_4 U_A + \dot{\mathbf{R}}_{a_i c 3 b} U_{a_1}{}^{c}{}_{a_k}\\
&\quad +s\zeta_c\underline{\chi}_{bc}U_A + 2s\underline{\omega}(\eta - \zeta)_b U_A - 2s\underline{\omega}\underline{\xi}_b U_A + s\big(\nabla_3\zeta_b + 2\nabla_b\underline{\omega}\big)U_A.
\end{aligned}$$

Therefore,

$$\begin{aligned}
[{}^{(c)}\nabla_3, {}^{(c)}\nabla_b]U_A &= -\underline{\chi}_{bc}\,{}^{(c)}\nabla_c U_A + \underline{\eta}\,{}^{(c)}\nabla_3 U_A + \underline{\xi}_b\,{}^{(c)}\nabla_4 U_A + \dot{\mathbf{R}}_{a_i c 3 b} U_{a_1}{}^{c}{}_{a_k}\\
&\quad +s\big(\nabla_3\zeta_b + 2\nabla_b\underline{\omega} + 2\underline{\omega}(\eta - \zeta)_b - 2\underline{\omega}\underline{\xi}_b\big)U_A
\end{aligned}$$

In view of the null structure equation, see Proposition 3.1.5,

$$\nabla_3\zeta + 2\nabla\underline{\omega} = -\underline{\chi}\cdot(\zeta + \eta) + 2\underline{\omega}(\zeta - \eta) + \chi\cdot\underline{\xi} + 2\omega\underline{\xi} - \underline{\beta}$$

we deduce

$$\begin{aligned}
[{}^{(c)}\nabla_3, {}^{(c)}\nabla_b]U_A &= -\underline{\chi}_{bc}\,{}^{(c)}\nabla_c U_A + \underline{\eta}\,{}^{(c)}\nabla_3 U_A + \underline{\xi}_b\,{}^{(c)}\nabla_4 U_A + \dot{\mathbf{R}}_{a_i c 3 b} U_{a_1}{}^{c}{}_{a_k}\\
&\quad +s\big(-\underline{\chi}\cdot(\zeta + \eta) + \chi\cdot\underline{\xi} - \underline{\beta}\big)U_A
\end{aligned}$$

as stated. The second formula can be deduced in the same manner.

[12] Indeed on the commutator on the left

$$\begin{aligned}
&= {}^{(c)}\nabla_3({}^{(c)}\nabla U) - {}^{(c)}\nabla({}^{(c)}\nabla_3 U) = \nabla_3({}^{(c)}\nabla U) - 2s\underline{\omega}({}^{(c)}\nabla U) - (\nabla + (s-1)\zeta)({}^{(c)}\nabla_3 U)\\
&= \nabla_3\big(\nabla U + s\zeta U\big) - 2s\underline{\omega}\big(\nabla U + s\zeta U\big) - (\nabla + (s-1)\zeta)\big(\nabla_3 U - 2s\underline{\omega}U\big)\\
&= [\nabla_3, \nabla]U + s\nabla_3(\zeta U) - 2\underline{\omega}\nabla U - s\zeta\nabla_3 U + 2s\nabla(\underline{\omega}U) + 2s\zeta\underline{\omega}U\\
&= [\nabla_3, \nabla]U + s\big(\nabla_3\zeta + 2\nabla\underline{\omega}\big)U + \zeta\big(\nabla_3 U - 2s\underline{\omega}U\big)
\end{aligned}$$

To derive the last formula we first obtain from the definitions of ${}^{(c)}\nabla_3$, ${}^{(c)}\nabla_4$,

$$[{}^{(c)}\nabla_4, {}^{(c)}\nabla_3]U = [\nabla_4, \nabla_3]U - 2\omega\, {}^{(c)}\nabla_3 U + 2\underline{\omega}\, {}^{(c)}\nabla_4 U - 2s\big(\nabla_3\omega + \nabla_4\underline{\omega}\big)U.$$

Using the last commutator formula in Lemma 3.1.17 we deduce

$$\begin{aligned}
[{}^{(c)}\nabla_4, {}^{(c)}\nabla_3]U_A &= 2(\underline{\eta}_b - \eta_b)\nabla_b U_A + 2\omega\nabla_3 U_A - 2\underline{\omega}\nabla_4 U_A + \dot{\mathbf{R}}_{a_i c 43} U_{a_1}{}^{c}{}_{a_k}\\
&\quad -2\omega\, {}^{(c)}\nabla_3 U_A + 2\omega\, {}^{(c)}\nabla_4 U_A - 2s\big(\nabla_3\omega + \nabla_4\underline{\omega}\big)U_A\\
&= 2(\underline{\eta}_b - \eta_b)\big({}^{(c)}\nabla_b - s\zeta_b\big)U_A - 2s\big(\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega}\big)U_A\\
&= 2(\underline{\eta}_b - \eta_b)\, {}^{(c)}\nabla_b U_A - 2s\big(\nabla_4\underline{\omega} + \nabla_3\omega - 4\omega\underline{\omega} + (\underline{\eta} - \eta)\cdot\zeta\big)U_A
\end{aligned}$$

Making us of the null structure equation, see Proposition 3.1.5,

$$\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega} - (\eta - \underline{\eta})\cdot\zeta = \rho - \eta\cdot\underline{\eta} + \xi\cdot\underline{\xi}.$$

Therefore

$$[{}^{(c)}\nabla_4, {}^{(c)}\nabla_3]U_A = 2(\underline{\eta}_b - \eta_b)\, {}^{(c)}\nabla_b U_A - 2s\big(\rho - \eta\cdot\underline{\eta} + \xi\cdot\underline{\xi}\big)U_A + \dot{\mathbf{R}}_{a_i c 43} U_{a_1}{}^{c}{}_{a_k}$$

as stated.

□

3.1.12 Commutation formulas with horizontal Lie derivatives

Recall that the Lie derivative of a k-covariant tensor U relative to a vectorfield X is given by

$$\mathcal{L}_X(Y_1, \ldots, Y_k) = XU(Y_1, \ldots, Y_k) - U(\mathcal{L}_X Y_1, \ldots, Y_k) - U(Y_1, \ldots, \mathcal{L}_X Y_k),$$

where $\mathcal{L}_X Y = [X, Y]$. In components relative to an arbitrary frame

$$\mathcal{L}_X U_{\alpha_1\ldots\alpha_k} := \mathbf{D}_X U_{\alpha_1\ldots\alpha_k} + \mathbf{D}_{\alpha_1} X^\beta U_{\beta\alpha_1\ldots\alpha_k} + \cdots \mathbf{D}_{\alpha_k} X^\beta U_{\alpha_1\ldots\beta}.$$

Recall also the general commutation Lemma, see chapter 7 in [Ch-Kl].

Lemma 3.1.21. *The following formula*[13] *for a vectorfield X and a k-covariant tensor-field U holds true:*

$$\mathbf{D}_\beta(\mathcal{L}_X U_{\alpha_1\ldots\alpha_k}) - \mathcal{L}_X(\mathbf{D}_\beta U_{\alpha_1\ldots\alpha_k}) = \sum_{j=1}^{k} {}^{(X)}\Gamma_{\alpha_j\beta\rho} U_{\alpha_1\ldots}{}^{\rho}{}_{\ldots\alpha_k}, \tag{3.1.32}$$

[13] This holds true for an arbitrary pseudo-riemannian space $(\mathcal{M}, \mathbf{g})$.

where

$$
{}^{(X)}\Gamma_{\alpha\beta\mu} \;=\; \frac{1}{2}(\mathbf{D}_\alpha\, {}^{(X)}\pi_{\beta\mu} + \mathbf{D}_\beta\, {}^{(X)}\pi_{\alpha\mu} - \mathbf{D}_\mu\, {}^{(X)}\pi_{\alpha\beta}). \tag{3.1.33}
$$

Proof. We check the result for for a 1-tensor U_α.

$$
\begin{aligned}
\mathcal{L}_X U_\alpha &= X^\lambda \mathbf{D}_\lambda U_\alpha + (\mathbf{D}_\alpha X^\lambda)U_\lambda \\
\mathbf{D}_\beta(\mathcal{L}_X V_\alpha) &= X^\lambda \mathbf{D}_\beta \mathbf{D}_\lambda U_\alpha + (\mathbf{D}_\beta X^\lambda)\mathbf{D}_\lambda U_\alpha + (\mathbf{D}_\beta \mathbf{D}_\alpha X^\lambda)U_\lambda + (\mathbf{D}_\alpha X^\lambda)\mathbf{D}_\beta U_\lambda \\
\mathcal{L}_X(\mathbf{D}_\beta U_\alpha) &= X^\lambda \mathbf{D}_\lambda \mathbf{D}_\beta U_\alpha + (\mathbf{D}_\alpha X^\lambda)\mathbf{D}_\beta U_\lambda + (\mathbf{D}_\beta X^\lambda)\mathbf{D}_\lambda U_\alpha.
\end{aligned}
$$

Hence, in view of Lemma A.1.4 stated and proved below,

$$
\begin{aligned}
\mathbf{D}_\beta(\mathcal{L}_X V_\alpha) - \mathcal{L}_X(\mathbf{D}_\beta U_\alpha) &= X^\lambda\big(\mathbf{D}_\beta \mathbf{D}_\lambda - \mathbf{D}_\lambda \mathbf{D}_\beta\big)U_\alpha + (\mathbf{D}_\beta \mathbf{D}_\alpha X^\lambda)U_\lambda \\
&= X^\lambda \mathbf{R}_{\alpha\rho\beta\lambda} U^\rho + (\mathbf{D}_\beta \mathbf{D}_\alpha X^\lambda)U_\lambda = {}^{(X)}\Gamma_{\alpha\beta\rho} U^\rho
\end{aligned}
$$

□

The proof of the Lemma was given in [Ch-Kl], see Lemma 7.1.3, based on the following

Lemma 3.1.22. *Given an arbitrary vectorfield X we have the identity*

$$
\mathbf{D}_\mu \mathbf{D}_\nu X_\beta = \mathbf{R}_{\beta\mu\nu\gamma} X^\gamma + {}^{(X)}\Gamma_{\mu\nu\beta}.
$$

Proof.

$$
\begin{aligned}
2\,{}^{(X)}\Gamma_{\alpha\beta\lambda} &= \mathbf{D}_\beta\, {}^{(X)}\pi_{\alpha\lambda} + \mathbf{D}_\alpha\, {}^{(X)}\pi_{\beta\lambda} - \mathbf{D}_\lambda\, {}^{(X)}\pi_{\alpha\beta} \\
&= \mathbf{D}_\beta(\mathbf{D}_\alpha X_\lambda + \mathbf{D}_\lambda X_\alpha) + \mathbf{D}_\alpha(\mathbf{D}_\beta X_\lambda + \mathbf{D}_\lambda X_\beta) - \mathbf{D}_\lambda(\mathbf{D}_\alpha X_\beta + \mathbf{D}_\beta X_\alpha) \\
&= \mathbf{D}_\alpha \mathbf{D}_\beta X_\lambda + \mathbf{D}_\beta \mathbf{D}_\alpha X_\lambda + (\mathbf{D}_\alpha \mathbf{D}_\lambda - \mathbf{D}_\lambda \mathbf{D}_\alpha)X_\beta + (\mathbf{D}_\beta \mathbf{D}_\lambda - \mathbf{D}_\lambda \mathbf{D}_\beta)X_\alpha \\
&= 2\mathbf{D}_\beta \mathbf{D}_\alpha X_\lambda + (\mathbf{D}_\alpha \mathbf{D}_\beta - \mathbf{D}_\beta \mathbf{D}_\alpha)X_\lambda + (\mathbf{D}_\alpha \mathbf{D}_\lambda - \mathbf{D}_\lambda \mathbf{D}_\alpha)X_\beta + (\mathbf{D}_\beta \mathbf{D}_\lambda - \mathbf{D}_\lambda \mathbf{D}_\beta)X_\alpha \\
&= 2\mathbf{D}_\beta \mathbf{D}_\alpha X_\lambda + \mathbf{R}_{\lambda\sigma\alpha\beta} X^\sigma + \mathbf{R}_{\beta\sigma\alpha\lambda} X^\sigma + \mathbf{R}_{\alpha\sigma\beta\lambda} X^\sigma \\
&= 2\mathbf{D}_\beta \mathbf{D}_\alpha X_\lambda - \big(\mathbf{R}_{\sigma\lambda\alpha\beta} + \mathbf{R}_{\sigma\beta\alpha\lambda} + \mathbf{R}_{\sigma\alpha\beta\lambda}\big)X^\sigma \\
&= 2\mathbf{D}_\beta \mathbf{D}_\alpha X_\lambda - \big(\mathbf{R}_{\sigma\lambda\alpha\beta} - \mathbf{R}_{\sigma\beta\alpha\lambda} + \mathbf{R}_{\sigma\alpha\beta\lambda}\big)X^\sigma - 2\mathbf{R}_{\sigma\beta\alpha\lambda} X^\lambda \\
&= 2\mathbf{D}_\beta \mathbf{D}_\alpha X_\lambda - \big(\mathbf{R}_{\sigma\lambda\alpha\beta} + \mathbf{R}_{\sigma\beta\lambda\alpha} + \mathbf{R}_{\sigma\alpha\beta\lambda}\big)X^\sigma - 2\mathbf{R}_{\sigma\beta\alpha\lambda} X^\sigma \\
&= 2\mathbf{D}_\beta \mathbf{D}_\alpha X_\lambda - 2\mathbf{R}_{\sigma\beta\alpha\lambda} X^\lambda
\end{aligned}
$$

Therefore,

$$
\mathbf{D}_\beta \mathbf{D}_\alpha X_\lambda \;=\; \mathbf{R}_{\sigma\beta\alpha\lambda} X^\sigma + {}^{(X)}\Gamma_{\alpha\beta\lambda} = \mathbf{R}_{\alpha\lambda\sigma\beta} X^\sigma + {}^{(X)}\Gamma_{\alpha\beta\lambda} = \mathbf{R}_{\lambda\alpha\beta\sigma} X^\sigma + {}^{(X)}\Gamma_{\alpha\beta\lambda}
$$

as stated.

As an alternative proof one could consider the tensor $A_{\mu\nu\beta} = \mathbf{D}_\mu \mathbf{D}_\nu X_\beta - \mathbf{R}_{\beta\mu\nu\gamma} X^\gamma - {}^{(X)}\Gamma_{\mu\nu\beta}$ and observe that it verifies the symmetries

$$A_{\mu\nu\beta} = A_{\nu\mu\beta} = -A_{\mu\beta\nu}.$$

The proof of Lemma A.1.4 follows by observing that any such tensor must vanish identically[14]. □

Proof. We check the result for for a 1-tensor U_α.

$$\begin{aligned}
\mathcal{L}_X U_\alpha &= X^\lambda \mathbf{D}_\lambda U_\alpha + (\mathbf{D}_\alpha X^\lambda) V_\lambda \\
\mathbf{D}_\beta(\mathcal{L}_X V_\alpha) &= X^\lambda \mathbf{D}_\beta \mathbf{D}_\lambda U_\alpha + (\mathbf{D}_\beta X^\lambda)\mathbf{D}_\lambda U_\alpha + (\mathbf{D}_\beta \mathbf{D}_\alpha X^\lambda) U_\lambda + (\mathbf{D}_\alpha X^\lambda)\mathbf{D}_\beta U_\lambda \\
\mathcal{L}_X(\mathbf{D}_\beta U_\alpha) &= X^\lambda \mathbf{D}_\lambda \mathbf{D}_\beta U_\alpha + (\mathbf{D}_\alpha X^\lambda)\mathbf{D}_\beta U_\lambda + (\mathbf{D}_\beta X^\lambda)\mathbf{D}_\lambda U_\alpha.
\end{aligned}$$

Hence, in view of Lemma A.1.4,

$$\begin{aligned}
\mathbf{D}_\beta(\mathcal{L}_X V_\alpha) - \mathcal{L}_X(\mathbf{D}_\beta U_\alpha) &= X^\lambda \big(\mathbf{D}_\beta \mathbf{D}_\lambda - \mathbf{D}_\lambda \mathbf{D}_\beta\big) U_\alpha + (\mathbf{D}_\beta \mathbf{D}_\alpha X^\lambda) U_\lambda \\
&= X^\lambda \mathbf{R}_{\alpha\rho\beta\lambda} U^\rho + (\mathbf{D}_\beta \mathbf{D}_\alpha X^\lambda) U_\lambda = {}^{(X)}\Gamma_{\alpha\beta\rho} U^\rho
\end{aligned}$$

□

We are now ready to define the horizontal Lie derivative operator $\not{\mathcal{L}}$ as follows.

Definition 3.1.23 (Horizontal Lie derivatives)**.** *Given vectorfields X, Y, the horizontal Lie derivative $\not{\mathcal{L}}_X Y$ is given by*

$$\not{\mathcal{L}}_X Y := \mathcal{L}_X Y + \frac{1}{2}\mathbf{g}(\mathcal{L}_X Y, e_3) e_4 + \frac{1}{2}\mathbf{g}(\mathcal{L}_X Y, e_4) e_3.$$

Given a horizontal covariant k-tensor U, the horizontal Lie derivative $\not{\mathcal{L}}_X U$ is defined to be the projection of $\mathcal{L}_X U$ to the horizontal space. Thus, for horizontal indices $A = a_1 \ldots a_k$,

$$(\not{\mathcal{L}}_X U)_A : = \nabla_X U_A + \mathbf{D}_{a_1} X^b U_{b \ldots a_k} + \cdots + \mathbf{D}_{a_k} X^b U_{a_1 \ldots b}. \tag{3.1.34}$$

[14] Indeed $A_{\mu\nu\beta} = -A_{\mu\beta\nu} = -A_{\beta\mu\nu} = A_{\beta\nu\mu} = A_{\nu\beta\mu} = -A_{\nu\mu\beta} = -A_{\mu\nu\beta}$.

Lemma 3.1.24. *The following commutation formulas hold true for a horizontal covariant k-tensor U and a vectorfield X*

$$\begin{aligned} \nabla_b(\not{\mathcal{L}}_X U_A) - \not{\mathcal{L}}_X(\nabla_b U_A) &= \sum_{j=1}^k {}^{(X)}\not{\Gamma}_{a_j bc} U_{a_1\ldots}{}^{c}{}_{\ldots a_k}, \\ \nabla_4(\not{\mathcal{L}}_X U_A) - \not{\mathcal{L}}_X(\nabla_4 U_A) + \nabla_{\not{\mathcal{L}}_X e_4} U_A &= \sum_{j=1}^k {}^{(X)}\not{\Gamma}_{a_j 4c} U_{a_1\ldots}{}^{c}{}_{\ldots a_k}, \\ \nabla_3(\not{\mathcal{L}}_X U_A) - \not{\mathcal{L}}_X(\nabla_3 U_A) + \nabla_{\not{\mathcal{L}}_X e_3} U_A &= \sum_{j=1}^k {}^{(X)}\not{\Gamma}_{a_j 3c} U_{a_1\ldots}{}^{c}{}_{\ldots a_k}, \end{aligned} \tag{3.1.35}$$

with[15]

$$\begin{aligned} {}^{(X)}\not{\Gamma}_{abc} &= \frac{1}{2}(\nabla_a {}^{(X)}\pi_{bc} + \nabla_b {}^{(X)}\pi_{ac} - \nabla_c {}^{(X)}\pi_{ab}), \\ {}^{(X)}\not{\Gamma}_{a4b} &= \frac{1}{2}(\nabla_a {}^{(X)}\pi_{4b} + \nabla_4 {}^{(X)}\pi_{ab} - \nabla_b {}^{(X)}\pi_{a4}), \\ {}^{(X)}\not{\Gamma}_{a3b} &= \frac{1}{2}(\nabla_a {}^{(X)}\pi_{3b} + \nabla_3 {}^{(X)}\pi_{ab} - \nabla_b {}^{(X)}\pi_{a3}). \end{aligned} \tag{3.1.36}$$

Proof. Follows easily by projecting formula (3.1.32) in Lemma 3.1.21, see also Lemma 9.1 in [Chr-BH]. Below we also give a more direct proof based on the following analogue of Lemma □

We now extend the definition of horizontal Lie derivative to any $U \in \mathbf{T}_k(\mathcal{M}) \otimes \mathbf{O}_l(\mathcal{M})$.

Definition 3.1.25. *We define the general horizontal derivatives as follows.*

1. *Given $X \in \mathbf{T}(\mathcal{M})$ and a general, horizontal tensor-field $U \in \mathbf{O}_k(\mathcal{M})$, we define*

$$\dot{\mathcal{L}}_X U \;:=\; \not{\mathcal{L}}_X U.$$

2. *Given a tensor in $U \in \mathbf{T}_k(\mathcal{M}) \otimes \mathbf{O}_l(\mathcal{M})$ and $X \in T(\mathcal{M})$ we define, for $Z = Z_1, \ldots, Z_k \in O(\mathbf{M})$ and $Y = Y_1, \ldots Y_l \in \mathbf{O}_1(\mathcal{M})$*

$$\begin{aligned} \dot{\mathcal{L}}_X U\big(Z, Y\big) = XU\big(Z, Y\big) \quad & -U\big(\mathcal{L}_X Z_1, \cdots Z_k, Y\big) - \ldots - U\big(Z_1, \cdots \mathcal{L}_X Z_k, Y\big) \\ & -U\big(Z, \dot{\mathcal{L}}_X Y_1, \ldots, Y_l\big) - \ldots - U\big(Z, Y_1, \ldots, \dot{\mathcal{L}}_X Y_l\big). \end{aligned}$$

[15]Here, ${}^{(X)}\pi_{ab}$ is treated as a horizontal symmetric 2-tensor, and ${}^{(X)}\pi_{a4}$, ${}^{(X)}\pi_{a3}$, as horizontal 1-forms.

3. *We have*

$$\dot{\mathcal{L}}_X(U\otimes V)=\dot{\mathcal{L}}_X U\otimes V+U\otimes\dot{\mathcal{L}}_X V.$$

4. *The definition can be extended by duality to any mixed tensors tensors in* $\mathbf{T}^{k_1}_{k_2}(\mathcal{M})\otimes\mathbf{O}^{l_1}_{l_2}(\mathcal{M})$.

Lemma 3.1.26. *The following commutation formulas hold true*[16] *for* $U\in\mathbf{O}_k(\mathcal{M})$ *and* $X\in\mathbf{T}(\mathcal{M})$,

$$\dot{\mathbf{D}}_\mu(\dot{\mathcal{L}}_X U_{a_1\dots a_k})-\dot{\mathcal{L}}_X(\dot{\mathbf{D}}_\mu U_{a_1\dots a_k}) \;=\; \sum_{j=1}^{k}{}^{(X)}\not{\Gamma}_{a_j\mu c}U_{a_1\dots}{}^{c}{}_{\dots a_k}.$$

The following commutation formula holds true for $U\in\mathbf{T}(\mathcal{M})\otimes\mathbf{O}_k(\mathcal{M})$ *and* $X\in\mathbf{T}(\mathcal{M})$,

$$\dot{\mathbf{D}}_\mu(\dot{\mathcal{L}}_X U_{\gamma a_1\dots a_k})-\dot{\mathcal{L}}_X(\dot{\mathbf{D}}_\mu U_{\gamma a_1\dots a_k})={}^{(X)}\Gamma_{\gamma\mu\rho}U^{\rho}{}_{a_1\dots\ \dots a_k}+\sum_{j=1}^{k}{}^{(X)}\not{\Gamma}_{a_j\mu c}U_{\gamma a_1\dots}{}^{c}{}_{\dots a_k}.$$

Proof. Follows easily by projecting formula (3.1.32) in Lemma 3.1.21, see also Lemma 9.1 in [Chr-BH]. □

Below we also give a more direct proof based on the following analogue of Lemma A.1.4.

Lemma 3.1.27. *Given an arbitrary vectorfield* X *the following identities hold true*

- *We have*

$$\dot{\mathbf{D}}_a\dot{\mathbf{D}}_b X_c=\mathbf{R}_{\beta\mu\nu\gamma}X^\gamma+{}^{(X)}\dot{\Gamma}_{\mu\nu\beta}. \tag{3.1.37}$$

$$\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu X_\beta=\mathbf{R}_{\beta\mu\nu\gamma}X^\gamma+{}^{(X)}\dot{\Gamma}_{\mu\nu\beta}.$$

where

$${}^{(X)}\dot{\Gamma}_{\mu\nu\beta} \;=\; \frac{1}{2}(\dot{\mathbf{D}}_\alpha\,{}^{(X)}\dot{\pi}_{\beta\mu}+\dot{\mathbf{D}}_\beta\,{}^{(X)}\dot{\pi}_{\alpha\mu}-\dot{\mathbf{D}}_\mu\,{}^{(X)}\dot{\pi}_{\alpha\beta})$$

and

$${}^{(X)}\dot{\pi}_{\alpha\beta} \;=\; \dot{\mathbf{D}}_\alpha X_\beta+\dot{\mathbf{D}}_\beta X_\alpha.$$

[16] With $\not{\Gamma}_X$ defined in (3.1.36).

Proof.

$$\begin{aligned}
2\,{}^{(X)}\dot{\Gamma}_{\alpha\beta\lambda} &= \dot{\mathbf{D}}_\beta\,{}^{(X)}\dot{\pi}_{\alpha\lambda} + \dot{\mathbf{D}}_\alpha\,{}^{(X)}\dot{\pi}_{\beta\lambda} - \dot{\mathbf{D}}_\lambda\,{}^{(X)}\dot{\pi}_{\alpha\beta}\\
&= \dot{\mathbf{D}}_\beta(\dot{\mathbf{D}}_\alpha X_\lambda + \dot{\mathbf{D}}_\lambda X_\alpha) + \dot{\mathbf{D}}_\alpha(\dot{\mathbf{D}}_\beta X_\lambda + \dot{\mathbf{D}}_\lambda X_\beta) - \dot{\mathbf{D}}_\lambda(\dot{\mathbf{D}}_\alpha X_\beta + \dot{\mathbf{D}}_\beta X_\alpha)\\
&= 2\dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\alpha X_\lambda + (\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\beta - \dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\alpha)X_\lambda + (\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\lambda - \dot{\mathbf{D}}_\lambda\dot{\mathbf{D}}_\alpha)X_\beta + (\dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\lambda - \dot{\mathbf{D}}_\lambda\dot{\mathbf{D}}_\beta)X_\alpha
\end{aligned}$$

To calculate ${}^{(X)}\dot{\Gamma}_{abc}$ we make use of the commutation formulas

$$\begin{aligned}
(\dot{\mathbf{D}}_a\dot{\mathbf{D}}_b - \dot{\mathbf{D}}_b\dot{\mathbf{D}}_a)X_c &= \dot{\mathbf{R}}_{c\lambda ab}X^\lambda\\
(\dot{\mathbf{D}}_a\dot{\mathbf{D}}_c - \dot{\mathbf{D}}_c\dot{\mathbf{D}}_a)X_b &= \dot{\mathbf{R}}_{b\lambda ac}X^\lambda\\
(\dot{\mathbf{D}}_b\dot{\mathbf{D}}_c - \dot{\mathbf{D}}_c\dot{\mathbf{D}}_b)X_a &= \dot{\mathbf{R}}_{a\lambda bc}X^\lambda
\end{aligned}$$

Hence

$$\begin{aligned}
2\,{}^{(X)}\dot{\Gamma}_{abc} &= 2\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c + (\dot{\mathbf{D}}_a\dot{\mathbf{D}}_b - \dot{\mathbf{D}}_b\dot{\mathbf{D}}_a)X_c + (\dot{\mathbf{D}}_a\dot{\mathbf{D}}_c - \dot{\mathbf{D}}_c\dot{\mathbf{D}}_a)X_b + (\dot{\mathbf{D}}_b\dot{\mathbf{D}}_c - \dot{\mathbf{D}}_c\dot{\mathbf{D}}_b)X_a\\
&= 2\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c + \Big(\dot{\mathbf{R}}_{c\lambda ab} + \dot{\mathbf{R}}_{b\lambda ac} + \dot{\mathbf{R}}_{a\lambda bc}\Big)X^\lambda\\
&= 2\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c - \Big(\dot{\mathbf{R}}_{\lambda cab} + \dot{\mathbf{R}}_{\lambda bac} + \dot{\mathbf{R}}_{\lambda abc}\Big)X^\lambda\\
&= 2\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c - \Big(\dot{\mathbf{R}}_{\lambda cab} - \dot{\mathbf{R}}_{\lambda bac} + \dot{\mathbf{R}}_{\lambda abc}\Big)X^\lambda - 2\dot{\mathbf{R}}_{\lambda bac}X^\lambda\\
&= 2\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c - \Big(\dot{\mathbf{R}}_{\lambda cab} + \dot{\mathbf{R}}_{\lambda bca} + \dot{\mathbf{R}}_{\lambda abc}\Big)X^\lambda - 2\dot{\mathbf{R}}_{\lambda bac}X^\lambda\\
&= 2\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c - \Big(\mathbf{B}_{\lambda cab} + \mathbf{B}_{\lambda bca} + \mathbf{B}_{\lambda abc}\Big)X^\lambda - 2\dot{\mathbf{R}}_{\lambda bac}X^\lambda
\end{aligned}$$

In view of Lemma 3.1.4 we deduce

$$\begin{aligned}
2\,{}^{(X)}\dot{\Gamma}_{abc} &= 2\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c - \Big(\mathbf{B}_{\lambda cab} + \mathbf{B}_{\lambda bca} + \mathbf{B}_{\lambda abc}\Big)X^\lambda - 2\dot{\mathbf{R}}_{\lambda bac}X^\lambda\\
&= 2\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c - \Big(\mathbf{B}_{3cab} + \mathbf{B}_{3bca} + \mathbf{B}_{3abc}\Big)X^3 - \Big(\mathbf{B}_{4cab} + \mathbf{B}_{4bca} + \mathbf{B}_{4abc}\Big)X^4\\
&\quad -2\dot{\mathbf{R}}_{\lambda bac}X^\lambda\\
&= 2\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c + \Big(\mathbf{B}_{abc3} + \mathbf{B}_{cab3} + \mathbf{B}_{bca3}\Big)X^3 + \Big(\mathbf{B}_{abc4} + \mathbf{B}_{cab4} + \mathbf{B}_{bca4}\Big)X^4\\
&\quad -2\dot{\mathbf{R}}_{\lambda bac}X^\lambda - 2\,{}^{(X)}\mathrm{Err}_{abc}
\end{aligned}$$

with

$$
{}^{(X)}\mathrm{Err}_{abc} = \Big({}^{(a)}\mathrm{tr}\underline{\chi}\in_{[ca}\eta_{b]} + {}^{(a)}\mathrm{tr}\chi\in_{[ca}\underline{\xi}_{b]}\Big)X^3 + \Big({}^{(a)}\mathrm{tr}\chi\in_{[ca}\underline{\eta}_{b]} + {}^{(a)}\mathrm{tr}\underline{\chi}\in_{[ca}\xi_{b]}\Big)X^4
$$

Therefore

$$\dot{\mathbf{D}}_b\dot{\mathbf{D}}_a X_c \;=\; {}^{(X)}\dot{\Gamma}_{abc} + \dot{\mathbf{R}}_{\lambda bac}X^\lambda + {}^{(X)}\mathrm{Err}_{abc}$$

$$\mathbf{D}_\mu\mathbf{D}_\nu X_\beta = {}^{(X)}\Gamma_{\mu\nu\beta} + \mathbf{R}_{\beta\mu\nu\gamma}X^\gamma.$$

$$\begin{aligned}
\mathbf{B}_{abc3} + \mathbf{B}_{bca3} + \mathbf{B}_{cab3} &= -2\,{}^{(a)}\mathrm{tr}\underline{\chi} \in_{[ca}\eta_{b]} - 2\,{}^{(a)}\mathrm{tr}\chi \in_{[ca}\underline{\xi}_{b]}\\
\mathbf{B}_{abc4} + \mathbf{B}_{bca4} + \mathbf{B}_{cab4} &= -2\,{}^{(a)}\mathrm{tr}\chi \in_{[ca}\underline{\eta}_{b]} - 2\,{}^{(a)}\mathrm{tr}\underline{\chi} \in_{[ca}\xi_{b]}\\
\mathbf{B}_{abcd} + \mathbf{B}_{bcad} + \mathbf{B}_{cabd} &= 0.
\end{aligned}$$

$\square$

Proposition 3.1.28. *For a tensor $\Psi \in \mathbf{O}_1(\mathcal{M})$, we have the curvature formula*[17]

$$(\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu - \dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\mu)\Psi_a = \dot{\mathbf{R}}_{ab\mu\nu}\Psi^b \tag{3.1.38}$$

where, with connection coefficients $(\Lambda_\alpha)_{\beta\gamma} = \mathbf{g}(\mathbf{D}_\alpha e_\gamma, e_\beta)$,

$$\begin{aligned}
\dot{\mathbf{R}}_{ab\mu\nu} &:= \mathbf{R}_{ab\mu\nu} + \frac{1}{2}\mathbf{B}_{ab\mu\nu}\\
\mathbf{B}_{ab\mu\nu} &:= (\Lambda_\mu)_{3a}(\Lambda_\nu)_{b4} + (\Lambda_\mu)_{4a}(\Lambda_\nu)_{b3} - (\Lambda_\nu)_{3a}(\Lambda_\mu)_{b4} - (\Lambda_\nu)_{4a}(\Lambda_\mu)_{b3}.
\end{aligned} \tag{3.1.39}$$

The above can also be written as

$$\begin{aligned}
\mathbf{B}_{abc3} &= -\mathrm{tr}\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\in_{ca}\eta_b - \in_{cb}\eta_a\big)\\
&\quad + 2\big(- \underline{\widehat{\chi}}_{ca}\eta_b + \underline{\widehat{\chi}}_{cb}\eta_a - \chi_{ca}\underline{\xi}_b + \chi_{cb}\underline{\xi}_a\big),\\
\mathbf{B}_{abc4} &= -\mathrm{tr}\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - {}^{(a)}\mathrm{tr}\chi\big(\in_{ca}\underline{\eta}_b - \in_{cb}\underline{\eta}_a\big)\\
&\quad + 2\big(- \widehat{\chi}_{ca}\underline{\eta}_b + \widehat{\chi}_{cb}\underline{\eta}_a - \underline{\chi}_{ca}\xi_b + \underline{\chi}_{cb}\xi_a\big).
\end{aligned} \tag{3.1.40}$$

The only non vanishing component of $\mathbf{B}_{abcd}$ is given by

$$\mathbf{B}_{1212} = -\mathbf{B}_{1221} = \mathbf{B}_{2121} = -\frac{1}{2}\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \widehat{\chi}\cdot\underline{\widehat{\chi}}$$

Calculate

$$\begin{aligned}
&\mathbf{B}_{abc3} + \mathbf{B}_{bca3} + \mathbf{B}_{cab3}\\
=\;& -\mathrm{tr}\,\underline{\chi}\Big(\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) + \big(\delta_{ab}\eta_c - \delta_{ac}\eta_b\big) + \big(\delta_{bc}\eta_a - \delta_{ba}\eta_c\big)\Big)\\
& - {}^{(a)}\mathrm{tr}\underline{\chi}\Big(\big(\in_{ca}\eta_b - \in_{cb}\eta_a\big) + \big(\in_{ab}\eta_c - \in_{ac}\eta_b\big) + \big(\in_{bc}\eta_a - \in_{ba}\eta_c\big)\Big)\\
& +2\big(- \underline{\widehat{\chi}}_{ca}\eta_b + \underline{\widehat{\chi}}_{cb}\eta_a - \chi_{ca}\underline{\xi}_b + \chi_{cb}\underline{\xi}_a\big)\\
=\;&
\end{aligned}$$

[17] With an immediate generalization to tensors $\Psi \in \mathbf{O}_l(\mathcal{M})$.

3.1.13 Frame transformations

3.1.14 General null frame transformations

Lemma 3.1.29. *Given a null frame (e_3, e_4, e_1, e_2), a general null transformation from the null frame (e_3, e_4, e_1, e_2) to another null frame (e'_3, e'_4, e'_1, e'_2) can be written in the form,*

$$
\begin{aligned}
e'_4 &= \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right),\\
e'_a &= \left(\delta_{ab} + \frac{1}{2}\underline{f}_a f_b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3, \qquad a = 1, 2, \\
e'_3 &= \lambda^{-1}\left(\left(1 + \frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b + \frac{1}{4}|\underline{f}|^2 e_4\right),
\end{aligned}
\tag{3.1.41}
$$

where λ is a scalar, f and $\underline{f}$ are horizontal 1-forms. The dot product and magnitude $|\cdot|$ are taken with respect to the standard euclidian norm of $\mathbb{R}^2$. We call $(f, \underline{f}, \lambda)$ the transition coefficients of the change of frame.

Remark 3.1.30. *Note that we have in particular the following identities*

$$
\begin{aligned}
e'_a &= e_a + \frac{1}{2}\underline{f}_a\lambda^{-1}e'_4 + \frac{1}{2}f_a e_3,\\
e'_3 &= \lambda^{-1}\left(e_3 + \underline{f}^a e'_a - \frac{1}{4}|\underline{f}|^2\lambda^{-1}e'_4\right).
\end{aligned}
$$

Proof. Clearly e'_4 is null. Also, we have

$$
\begin{aligned}
\lambda^{-1}\mathbf{g}(e'_4, e'_a) &= \mathbf{g}\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3, \left(\delta_a^c + \frac{1}{2}\underline{f}_a f^c\right) e_c + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3\right)\\
&= f^b\left(\delta_a^c + \frac{1}{2}\underline{f}_a f^c\right)\delta_{bc} - 2\left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) - \frac{1}{4}|f|^2\underline{f}_a\\
&= f_a + \frac{1}{2}|f|^2\underline{f}_a - f_a - \frac{1}{4}|f|^2\underline{f}_a - \frac{1}{4}|f|^2\underline{f}_a = 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbf{g}(e'_a, e'_b) &= \left(\delta_a^c + \frac{1}{2}\underline{f}_a f^c\right)\left(\delta_b^d + \frac{1}{2}\underline{f}_b f^d\right)\delta_{cd} - \underline{f}_a\left(\frac{1}{2}f_b + \frac{1}{8}|f|^2\underline{f}_b\right)\\
&\quad -\left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)\underline{f}_b = \delta_{ab}
\end{aligned}
$$

and

$$\mathbf{g}(e_3', e_4') = \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) f_b - 2\left(1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) - \frac{1}{8}|\underline{f}|^2|f|^2 = -2.$$

Also, we have

$$\begin{aligned}
\lambda \mathbf{g}(e_3', e_a') &= \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right)\left(\delta_a^c + \frac{1}{2}\underline{f}_a f^c\right)\delta_{bc} - \left(1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right)\underline{f}_a \\
&\quad -\frac{1}{2}|\underline{f}|^2\left(\frac{1}{2} f_a + \frac{1}{8}|f|^2 \underline{f}_a\right) \\
&= \underline{f}_a + \frac{1}{4}|\underline{f}|^2 f_a + \left(f \cdot \underline{f} + \frac{1}{4}|\underline{f}|^2|f|^2\right)\frac{1}{2}\underline{f}_a \\
&\quad - \left(1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right)\underline{f}_a - \frac{1}{2}|\underline{f}|^2\left(\frac{1}{2} f_a + \frac{1}{8}|f|^2 \underline{f}_a\right) = 0.
\end{aligned}$$

Finally

$$\begin{aligned}
\lambda^2 \mathbf{g}(e_3', e_3') &= \left|\underline{f} + \frac{1}{4}|\underline{f}|^2 f\right|^2 - |\underline{f}|^2\left(1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) \\
&= |\underline{f}|^2 + \frac{1}{2}|\underline{f}|^2 f \cdot \underline{f} + \frac{1}{16}|\underline{f}|^4|f|^2 - |\underline{f}|^2\left(1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) = 0.
\end{aligned}$$

This concludes the proof of the lemma. □

3.1.15 Transformation formulas for Ricci and Curvature coefficients

While we only need the transformation formulas for χ, $\underline{\chi}$, ζ and ρ for this paper, we nevertheless derive below the transformation formulas for all connection coefficients and curvature components for completeness.

Proposition 3.1.31. *Under a general transformation of type* (3.1.41)*, the Ricci coefficients transform as follows:*

- *The transformation formula for ξ is given by*

$$\begin{aligned}
\lambda^{-2}\xi' &= \xi + \frac{1}{2}\lambda^{-1}\nabla_4' f + \frac{1}{4}(tr\,\chi f - {}^{(a)}tr\chi\, {}^* f) + \omega f + Err(\xi, \xi'), \\
Err(\xi, \xi') &= \frac{1}{2} f \cdot \widehat{\chi} + \frac{1}{4}|f|^2 \eta + \frac{1}{2}(f \cdot \zeta) f - \frac{1}{4}|f|^2 \underline{\eta} \\
&\quad + \lambda^{-2}\left(\frac{1}{2}(f \cdot \xi')\underline{f} + \frac{1}{2}(f \cdot \underline{f})\xi'\right) + \ l.o.t.
\end{aligned} \tag{3.1.42}$$

- *The transformation formula for $\underline{\xi}$ is given by*

$$\begin{aligned}\lambda^2\underline{\xi}' &= \underline{\xi} + \frac{1}{2}\lambda\nabla_3'\underline{f} + \underline{\omega}\,\underline{f} + \frac{1}{4}tr\underline{\chi}\,\underline{f} - \frac{1}{4}{}^{(a)}tr\underline{\chi}\,{}^*\underline{f} + Err(\underline{\xi},\underline{\xi}'),\\ Err(\underline{\xi},\underline{\xi}') &= \frac{1}{2}\underline{f}\cdot\widehat{\underline{\chi}} - \frac{1}{2}(\underline{f}\cdot\zeta)\underline{f} + \frac{1}{4}|\underline{f}|^2\underline{\eta} - \frac{1}{4}|\underline{f}|^2\eta' + \text{ l.o.t.}\end{aligned} \tag{3.1.43}$$

- *The transformation formulas for χ are given by*

$$\begin{aligned}\lambda^{-1}tr\,\chi' &= tr\,\chi + div'f + f\cdot\eta + f\cdot\zeta + Err(tr\,\chi, tr\,\chi')\\ Err(tr\,\chi, tr\,\chi') &= \underline{f}\cdot\xi + \frac{1}{4}\underline{f}\cdot\big(f\,tr\,\chi - {}^*f\,{}^{(a)}tr\chi\big) + \omega(f\cdot\underline{f}) - \underline{\omega}|f|^2\\ &\quad - \frac{1}{4}|f|^2tr\underline{\chi} - \frac{1}{4}(f\cdot\underline{f})\lambda^{-1}tr\,\chi' + \frac{1}{4}(\underline{f}\wedge f)\lambda^{-1}\,{}^{(a)}tr\chi' + \text{ l.o.t.},\end{aligned} \tag{3.1.44}$$

$$\begin{aligned}\lambda^{-1}\,{}^{(a)}tr\chi' &= {}^{(a)}tr\chi + curl'f + f\wedge\eta + f\wedge\zeta + Err({}^{(a)}tr\chi, {}^{(a)}tr\chi'),\\ Err({}^{(a)}tr\chi, {}^{(a)}tr\chi') &= \underline{f}\wedge\xi + \frac{1}{4}\big(\underline{f}\wedge f\,tr\,\chi + (f\cdot\underline{f})\,{}^{(a)}tr\chi\big) + \omega f\wedge\underline{f}\\ &\quad - \frac{1}{4}|f|^2\,{}^{(a)}tr\underline{\chi} - \frac{1}{4}(f\cdot\underline{f})\lambda^{-1}\,{}^{(a)}tr\chi' + \frac{1}{4}\lambda^{-1}(f\wedge\underline{f})tr\,\chi' + \text{ l.o.t.},\end{aligned} \tag{3.1.45}$$

$$\begin{aligned}\lambda^{-1}\widehat{\chi}' &= \widehat{\chi} + \nabla'\widehat{\otimes}f + f\widehat{\otimes}\eta + f\widehat{\otimes}\zeta + Err(\widehat{\chi},\widehat{\chi}'),\\ Err(\widehat{\chi},\widehat{\chi}') &= \underline{f}\widehat{\otimes}\xi + \frac{1}{4}\underline{f}\widehat{\otimes}\big(f\,tr\,\chi - {}^*f\,{}^{(a)}tr\chi\big) + \omega f\widehat{\otimes}\underline{f} - \underline{\omega}f\widehat{\otimes}f - \frac{1}{4}|f|^2\,{}^{(a)}tr\underline{\chi}\\ &\quad + \frac{1}{4}(f\widehat{\otimes}\underline{f})\lambda^{-1}tr\,\chi' + \frac{1}{4}({}^*f\widehat{\otimes}\underline{f})\lambda^{-1}\,{}^{(a)}tr\chi' + \frac{1}{2}\underline{f}\widehat{\otimes}(f\cdot\lambda^{-1}\widehat{\chi}') + \text{ l.o.t.}\end{aligned} \tag{3.1.46}$$

- *The transformation formulas for $\underline{\chi}$ are given by*

$$\begin{aligned}\lambda tr\,\underline{\chi}' &= tr\,\underline{\chi} + div'\underline{f} + \underline{f}\cdot\underline{\eta} - \underline{f}\cdot\zeta + Err(tr\,\underline{\chi}, tr\,\underline{\chi}'),\\ Err(tr\,\underline{\chi}, tr\,\underline{\chi}') &= \frac{1}{2}(f\cdot\underline{f})tr\,\underline{\chi} + f\cdot\underline{\xi} - |\underline{f}|^2\omega + (f\cdot\underline{f})\underline{\omega} - \frac{1}{4}|\underline{f}|^2\lambda^{-1}tr\,\chi' + \text{ l.o.t.},\end{aligned} \tag{3.1.47}$$

$$\begin{aligned}\lambda\,{}^{(a)}tr\underline{\chi}' &= {}^{(a)}tr\underline{\chi} + curl'\underline{f} + \underline{f}\wedge\underline{\eta} - \zeta\wedge\underline{f} + Err({}^{(a)}tr\underline{\chi}, {}^{(a)}tr\underline{\chi}'),\\ Err({}^{(a)}tr\underline{\chi}, {}^{(a)}tr\underline{\chi}') &= \frac{1}{2}(f\cdot\underline{f})\,{}^{(a)}tr\underline{\chi} + f\wedge\underline{\xi} + (f\wedge\underline{f})\underline{\omega} - \frac{1}{4}|\underline{f}|^2\lambda^{-1}\,{}^{(a)}tr\chi' + \text{ l.o.t.},\end{aligned} \tag{3.1.48}$$

$$\begin{aligned}\lambda\widehat{\underline{\chi}}' &= \widehat{\underline{\chi}} + \nabla'\widehat{\otimes}\underline{f} + \underline{f}\widehat{\otimes}\underline{\eta} - \underline{f}\widehat{\otimes}\zeta + Err(\widehat{\underline{\chi}},\widehat{\underline{\chi}}'),\\ Err(\widehat{\underline{\chi}},\widehat{\underline{\chi}}') &= \frac{1}{2}(f\widehat{\otimes}\underline{f})tr\,\underline{\chi} + f\widehat{\otimes}\underline{\xi} - (\underline{f}\widehat{\otimes}\underline{f})\omega + (f\widehat{\otimes}\underline{f})\underline{\omega} - \frac{1}{4}|\underline{f}|^2\lambda^{-1}\widehat{\chi}' + \text{ l.o.t.}\end{aligned} \tag{3.1.49}$$

- *The transformation formula for ζ is given by*

$$
\begin{aligned}
\zeta' &= \zeta - \nabla'(\log\lambda) - \frac{1}{4}tr\underline{\chi}f + \frac{1}{4}{}^{(a)}tr\underline{\chi}\,{}^*f + \omega\underline{f} - \underline{\omega}f + \frac{1}{4}\underline{f}tr\,\chi\\
&+ \frac{1}{4}\,{}^*\underline{f}\,{}^{(a)}tr\chi + Err(\zeta,\zeta'),\\
Err(\zeta,\zeta') &= -\frac{1}{2}\widehat{\underline{\chi}}\cdot f + \frac{1}{2}(f\cdot\zeta)\underline{f} - \frac{1}{2}(f\cdot\underline{\eta})\underline{f} + \frac{1}{4}\underline{f}(f\cdot\eta) + \frac{1}{4}\underline{f}(f\cdot\zeta)\\
&+ \frac{1}{4}\,{}^*\underline{f}(f\wedge\eta) + \frac{1}{4}\,{}^*\underline{f}(f\wedge\zeta) + \frac{1}{4}\underline{f}\,div'f + \frac{1}{4}\,{}^*\underline{f}\,curl'f + \frac{1}{2}\lambda^{-1}\underline{f}\cdot\widehat{\chi}'\\
&- \frac{1}{16}(f\cdot\underline{f})\underline{f}\lambda^{-1}tr\,\chi' + \frac{1}{16}(\underline{f}\wedge f)\underline{f}\lambda^{-1}\,{}^{(a)}tr\chi' - \frac{1}{16}\,{}^*\underline{f}(f\cdot\underline{f})\lambda^{-1}\,{}^{(a)}tr\chi'\\
&+ \frac{1}{16}\,{}^*\underline{f}\lambda^{-1}(f\wedge\underline{f})tr\,\chi' + \ l.o.t.
\end{aligned}
\tag{3.1.50}
$$

- *The transformation formula for η is given by*

$$
\begin{aligned}
\eta' &= \eta + \frac{1}{2}\lambda\nabla_3'f + \frac{1}{4}\underline{f}tr\,\chi - \frac{1}{4}\,{}^*\underline{f}\,{}^{(a)}tr\chi - \underline{\omega}\,f + Err(\eta,\eta'),\\
Err(\eta,\eta') &= \frac{1}{2}(f\cdot\underline{f})\eta + \frac{1}{2}\underline{f}\cdot\widehat{\chi} + \frac{1}{2}f(\underline{f}\cdot\zeta) - (\underline{f}\cdot f)\eta' + \frac{1}{2}\underline{f}(f\cdot\eta') + \ l.o.t.
\end{aligned}
\tag{3.1.51}
$$

- *The transformation formula for $\underline{\eta}$ is given by*

$$
\begin{aligned}
\underline{\eta}' &= \underline{\eta} + \frac{1}{2}\lambda^{-1}\nabla_4'\underline{f} + \frac{1}{4}tr\underline{\chi}f - \frac{1}{4}{}^{(a)}tr\underline{\chi}\,{}^*f - \omega\underline{f} + Err(\underline{\eta},\underline{\eta}'),\\
Err(\underline{\eta},\underline{\eta}') &= \frac{1}{2}f\cdot\widehat{\underline{\chi}} + \frac{1}{2}(f\cdot\underline{\eta})\underline{f} - \frac{1}{4}(f\cdot\zeta)\underline{f} - \frac{1}{4}|\underline{f}|^2\lambda^{-2}\xi' + \ l.o.t.
\end{aligned}
\tag{3.1.52}
$$

- *The transformation formula for ω is given by*

$$
\begin{aligned}
\lambda^{-1}\omega' &= \omega - \frac{1}{2}\lambda^{-1}e_4'(\log\lambda) + \frac{1}{2}f\cdot(\zeta-\underline{\eta}) + Err(\omega,\omega'),\\
Err(\omega,\omega') &= -\frac{1}{4}|f|^2\underline{\omega} - \frac{1}{8}tr\underline{\chi}|f|^2 + \frac{1}{2}\lambda^{-2}\underline{f}\cdot\xi' + \ l.o.t.
\end{aligned}
\tag{3.1.53}
$$

- *The transformation formula for $\underline{\omega}$ is given by*

$$
\begin{aligned}
\lambda\underline{\omega}' &= \underline{\omega} + \frac{1}{2}\lambda e_3'(\log\lambda) - \frac{1}{2}\underline{f}\cdot\zeta - \frac{1}{2}\underline{f}\cdot\eta + Err(\underline{\omega},\underline{\omega}'),\\
Err(\underline{\omega},\underline{\omega}') &= f\cdot\underline{f}\,\underline{\omega} - \frac{1}{4}|\underline{f}|^2\omega + \frac{1}{2}f\cdot\underline{\xi} + \frac{1}{8}(f\cdot\underline{f})tr\underline{\chi} + \frac{1}{8}(\underline{f}\wedge f)\,{}^{(a)}tr\underline{\chi}\\
&- \frac{1}{8}|\underline{f}|^2tr\,\chi - \frac{1}{4}\lambda\underline{f}\cdot\nabla_3'f + \frac{1}{2}(\underline{f}\cdot f)(\underline{f}\cdot\eta') - \frac{1}{4}|\underline{f}|^2(f\cdot\eta') + \ l.o.t.
\end{aligned}
\tag{3.1.54}
$$

where, for the transformation formulas of the Ricci coefficients above, l.o.t. denote expressions of the type

$$l.o.t. \;= O((f,\underline{f})^3)\Gamma + O((f,\underline{f})^2)\check{\Gamma}$$

containing no derivatives of f, $\underline{f}$, Γ *and* $\check{\Gamma}$.

Also, the curvature components transform as follows

- *The transformation formula for* $\alpha, \underline{\alpha}$ *are given by*

$$\begin{aligned}\lambda^{-2}\alpha' &= \alpha + Err(\alpha,\alpha'),\\ Err(\alpha,\alpha') &= \big(f\widehat{\otimes}\beta - {}^*f\widehat{\otimes}{}^*\beta\big) + \left(f\widehat{\otimes}f - \frac{1}{2}{}^*f\widehat{\otimes}{}^*f\right)\rho + \frac{3}{2}\big(f\widehat{\otimes}{}^*f\big){}^*\rho + \; l.o.t.,\end{aligned} \tag{3.1.55}$$

$$\begin{aligned}\lambda^{2}\underline{\alpha}' &= \underline{\alpha} + Err(\alpha,\alpha'),\\ Err(\underline{\alpha},\underline{\alpha}') &= -\big(\underline{f}\widehat{\otimes}\underline{\beta} - {}^*\underline{f}\widehat{\otimes}{}^*\underline{\beta}\big) + \big(\underline{f}\widehat{\otimes}\underline{f} - \frac{1}{2}{}^*\underline{f}\widehat{\otimes}{}^*\underline{f}\big)\rho + \frac{3}{2}\big(\underline{f}\widehat{\otimes}{}^*\underline{f}\big){}^*\rho + \; l.o.t.\end{aligned} \tag{3.1.56}$$

- *The transformation formula for* $\beta, \underline{\beta}$ *are given by*

$$\begin{aligned}\lambda^{-1}\beta' &= \beta + \frac{3}{2}\big(f\rho + {}^*f\,{}^*\rho\big) + Err(\beta,\beta'),\\ Err(\beta,\beta') &= \frac{1}{2}\alpha\cdot\underline{f} + \; l.o.t.,\end{aligned} \tag{3.1.57}$$

$$\begin{aligned}\lambda\underline{\beta}' &= \underline{\beta} - \frac{3}{2}\big(\underline{f}\rho + {}^*\underline{f}\,{}^*\rho\big) + Err(\underline{\beta},\underline{\beta}'),\\ Err(\underline{\beta},\underline{\beta}') &= -\frac{1}{2}\underline{\alpha}\cdot f + \; l.o.t.\end{aligned} \tag{3.1.58}$$

- *The transformation formula for* ρ *and* ${}^*\rho$ *are given by*

$$\begin{aligned}\rho' &= \rho + Err(\rho,\rho'),\\ Err(\rho,\rho') &= \underline{f}\cdot\beta - f\cdot\underline{\beta} + \frac{3}{2}\rho(f\cdot\underline{f}) - \frac{3}{2}{}^*\rho(f\wedge\underline{f}) + \; l.o.t.\end{aligned} \tag{3.1.59}$$

$$\begin{aligned}{}^*\rho' &= {}^*\rho + Err({}^*\rho,{}^*\rho'),\\ Err({}^*\rho,{}^*\rho') &= -\underline{f}\cdot{}^*\beta - f\cdot{}^*\underline{\beta} + \frac{3}{2}{}^*\rho(f\cdot\underline{f}) + \frac{3}{2}\rho(f\wedge\underline{f}) + \; l.o.t.\end{aligned} \tag{3.1.60}$$

where, for the transformation formulas of the curvature components above, l.o.t. denote expressions of the type

$$l.o.t. \;\; = O((f,\underline{f})^3)(\rho, \ {}^*\rho) + O((f,\underline{f})^2)(\alpha,\beta,\underline{\alpha},\beta)$$

containing no derivatives of f, $\underline{f}$, α, β, $(\rho, \ {}^*\rho)$, $\underline{\beta}$, and $\underline{\alpha}$.

Proof. See Appendix A1 in [K-S:GCM1] □

3.2 Wave operators

Consider a spacetime $(\mathcal{M}, \mathbf{g})$ with a horizontal structure induced by a null pair (e_3, e_4).

Definition 3.2.1. *We define the wave operator for tensor-fields $\psi \in \mathbf{O}_k(\mathcal{M})$ to be*

$$\dot{\square}_k\psi := \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi. \tag{3.2.1}$$

3.2.1 Commutation with $\dot{\mathcal{L}}_X$ and $\dot{\mathbf{D}}_X$

Recalling the definition of ${}^{(X)}\Gamma, {}^{(X)}\not\Gamma$ in section 3.1.12 we have:

Proposition 3.2.2. *The following commutation formula[18] holds true for $\psi \in \mathfrak{s}_2$ and $X \in \mathbf{T}(\mathcal{M})$,*

$$\begin{aligned}
[\dot{\mathcal{L}}_X, \dot{\square}_2]\psi_{ab} \;\; &= \;\; -\,{}^{(X)}\pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi_{ab} - {}^{(X)}\Gamma^\mu{}_{\mu\rho}\dot{\mathbf{D}}^\rho\psi_{ab} - 2{}^{(X)}\not\Gamma_{a\mu c}\dot{\mathbf{D}}^\mu\psi^c{}_b - 2{}^{(X)}\not\Gamma_{b\mu c}\dot{\mathbf{D}}^\mu\psi_a{}^c\\
&\quad -\dot{\mathbf{D}}^\nu({}^{(X)}\not\Gamma_{a\nu c})\psi^c{}_b - \dot{\mathbf{D}}^\nu({}^{(X)}\not\Gamma_{b\nu c})\psi_a{}^c.
\end{aligned}$$

Proof. See proof of Proposition 2.3.2 in [GKS-2022]. □

Lemma 3.2.3. *We have in a vacuum spacetime*

$$\begin{aligned}
\dot{\square}(X^\beta\dot{\mathbf{D}}_\beta U_a) - X^\beta\dot{\mathbf{D}}_\beta\dot{\square}U_a \;\; &= \;\; \pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu U_a + \big(\mathbf{D}^\mu\pi_\mu{}^\beta - \frac{1}{2}\mathbf{D}^\beta tr\pi\big)\dot{\mathbf{D}}_\beta U_a\\
&\quad -2X^\beta\mathbf{R}_{ac\beta\mu}\dot{\mathbf{D}}^\mu U_c + \mathbf{D}^\beta X^\mu\mathbf{R}_{ac\beta\mu}U^c\\
&\quad -X^\beta\mathbf{B}_{ac\beta\mu}\dot{\mathbf{D}}^\mu U_c + \frac{1}{2}\mathbf{D}^\beta X^\mu\mathbf{B}_{ac\beta\mu}U^c + \frac{1}{2}X^\beta\mathbf{D}^\mu\mathbf{B}_{ac\mu\beta}U^c.
\end{aligned}$$

Proof. Straightforward computation using Lemma A.1.4 and Proposition 3.1.28. See Lemma 2.3.3 in [GKS-2022]. □

[18] Recall that $\dot{\mathcal{L}}$ has been introduced in Definition 3.1.25.

3.2.2 Killing tensor and commutation with second order operators

Recall that the deformation tensor of a vectorfield ${}^{(X)}\pi$ is defined as

$$ {}^{(X)}\pi_{\mu\nu} := \mathbf{D}_{(\mu}X_{\nu)} = \mathbf{D}_\mu X_\nu + \mathbf{D}_\nu X_\mu. $$

The vectorfield is said to be Killing if ${}^{(X)}\pi \equiv 0$. The Kerr spacetime has, in addition to the symmetries generated by its two linearly independent Killing vectorfields $\mathbf{T}$ and $\mathbf{Z}$, a higher order symmetry defined by a Killing tensor.

Definition 3.2.4. *A symmetric 2-tensor $K_{\mu\nu}$ is said to be a Killing tensor if its deformation 3-tensor Π, defined below, vanishes identically.*

$$ \Pi_{\mu\nu\rho} := \mathbf{D}_{(\mu}K_{\nu\rho)} = \mathbf{D}_\mu K_{\nu\rho} + \mathbf{D}_\nu K_{\rho\mu} + \mathbf{D}_\rho K_{\mu\nu}. \tag{3.2.2} $$

Remark 3.2.5. *Observe that if X, Y are Killing vectorfields then the symmetric 2-tensor $K = \frac{1}{2}(X \otimes Y + Y \otimes X)$ is a Killing tensor.*

We define the second order differential operator associated to a tensor-field $\psi \in \mathfrak{s}_k$.

Definition 3.2.6. *Given a symmetric tensor K its associated second order differential operator $\mathcal{K}$ applied to a tensor $\psi \in \mathfrak{s}_k$ is defined as*

$$ \mathcal{K}(\psi) = \dot{\mathbf{D}}_\mu(K^{\mu\nu}\dot{\mathbf{D}}_\nu(\psi)). \tag{3.2.3} $$

We now compute the commutators of $\mathcal{K}$ with $\square_{\mathbf{g}}$ in terms of the symmetric tensor Π.

Proposition 3.2.7. *In a vacuum spacetime, the commutator between the differential operator $\mathcal{K}$ and the $\square_{\mathbf{g}}$ operator applied to a scalar function ϕ is given by*

$$ [\mathcal{K}, \square_{\mathbf{g}}]\phi \quad = \quad Err[\Pi](\phi) $$

where $Err[\Pi](\phi)$ denotes terms involving Π given by

$$ \begin{aligned} Err[\Pi](\phi) \quad := \quad & \mathbf{D}^\mu\left(\left(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\dot{\mathbf{D}}^\nu\phi - 2\Pi_{\mu\alpha\nu}\dot{\mathbf{D}}^\alpha\dot{\mathbf{D}}^\nu\phi\right) \\ & -2\left(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu}\right)\dot{\mathbf{D}}^\mu\dot{\mathbf{D}}^\nu\phi. \end{aligned} $$

Proof. See the proof of Proposition 2.3.7 in [GKS-2022]. □

3.2.3 A class of spin-k wave operators with potential

The following class of *spin*-k wave operators play a very important role in our analysis.

$$\dot{\square}_k\psi - V\psi = N, \tag{3.2.4}$$

where $\psi \in \mathfrak{s}_k$ and V is a real potential. The equation is variational with Lagrangian

$$\mathcal{L}[\psi] = \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu\psi \cdot \dot{\mathbf{D}}_\nu\psi + V\psi\cdot\psi.$$

where the dot product here denotes full contraction with respect to the horizontal indices.

The corresponding energy-momentum tensor associated to (3.2.4) is given by

$$\mathcal{Q}_{\mu\nu} := \dot{\mathbf{D}}_\mu\psi\cdot\dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\Big(\dot{\mathbf{D}}_\lambda\psi\cdot\dot{\mathbf{D}}^\lambda\psi + V\psi\cdot\psi\Big) = \dot{\mathbf{D}}_\mu\psi\cdot\dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{L}[\psi]. \tag{3.2.5}$$

Lemma 3.2.8. *Given a solution $\psi \in \mathfrak{s}_k$ of equation* (3.2.4) *we have*

$$\mathbf{D}^\nu\mathcal{Q}_{\mu\nu} = \dot{\mathbf{D}}_\mu\psi\cdot\big(\dot{\square}_k\psi - V\psi\big) + \dot{\mathbf{D}}^\nu\psi^A\dot{\mathbf{R}}_{AB\nu\mu}\psi^B - \frac{1}{2}\mathbf{D}_\mu V|\psi|^2.$$

Proof. We have, making us of Proposition 3.1.28

$$\begin{aligned}
\mathbf{D}^\nu\mathcal{Q}_{\mu\nu} &= \dot{\mathbf{D}}^\nu\dot{\mathbf{D}}_\nu\psi\cdot\dot{\mathbf{D}}_\mu\psi + \dot{\mathbf{D}}^\nu\psi\cdot\Big(\dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\mu - \dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\Big)\psi - V\mathbf{D}_\mu\psi\cdot\psi - \frac{1}{2}\mathbf{D}_\mu V\psi\cdot\psi\\
&= \dot{\mathbf{D}}_\mu\psi\cdot\dot{\mathbf{D}}^\nu\dot{\mathbf{D}}_\nu\psi + \dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b - V\mathbf{D}_\mu\psi\cdot\psi - \frac{1}{2}\mathbf{D}_\mu V\psi\cdot\psi\\
&= \dot{\mathbf{D}}_\mu\psi\cdot\big(\dot{\square}_k\psi - V\psi\big) + \dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b - \frac{1}{2}\mathbf{D}_\mu V|\psi|^2.
\end{aligned}$$

$\square$

Proposition 3.2.9. *[Standard calculation for generalized currents] Let $\psi \in \mathfrak{s}_k$ be a solution of* (3.2.4) *and X be a vectorfield. Then,*

1. *The* 1*-form $\mathcal{P}_\mu = \mathcal{Q}_{\mu\nu}X^\nu$ verifies*

$$\mathbf{D}^\mu\mathcal{P}_\mu = \frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi + X(\psi)\cdot\big(\dot{\square}_k\psi - V\psi\big) - \frac{1}{2}X(V)|\psi|^2 + X^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b.$$

2. *Let X as above, w a scalar and M a one form. Define*

$$\mathcal{P}_\mu[X,w,M] := \mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\psi\cdot\dot{\mathbf{D}}_\mu\psi - \frac{1}{4}|\psi|^2\partial_\mu w + \frac{1}{4}|\psi|^2 M_\mu.$$

Then,

$$\begin{aligned}\mathbf{D}^\mu \mathcal{P}_\mu[X,w,M] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}} w + \frac{1}{4}Div(|\psi|^2 M)\\ &+ X^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b + \left(X(\psi)+\frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_k\psi - V\psi\right).\end{aligned}$$

Proof. Immediate verification. See also the proof of Proposition 4.7.2 in [GKS-2022]. □

3.2.4 Decomposition of $\dot{\square}_k$ in null frames

Lemma 3.2.10. *The wave operator for $\psi \in \mathfrak{s}_k$ is given by*

$$\begin{aligned}\dot{\square}_k\psi &= -\frac{1}{2}\big(\nabla_3\nabla_4\psi + \nabla_4\nabla_3\psi\big) + \left(\underline{\omega} - \frac{1}{2}tr\,\underline{\chi}\right)\nabla_4\psi + \left(\omega - \frac{1}{2}tr\,\chi\right)\nabla_3\psi\\ &+ \triangle_k\psi + (\eta+\underline{\eta})\cdot\nabla\psi,\end{aligned} \tag{3.2.6}$$

where $\triangle = \nabla^a\nabla_a$ denotes the horizontal Laplacian for k-tensors. Moreover If ψ is also 0-conformal invariant we also have

$$\begin{aligned}\dot{\square}_k\psi &= -\frac{1}{2}\big({}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\psi + {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\psi\big) - \frac{1}{2}tr\,\underline{\chi}\,{}^{(c)}\nabla_4\psi - \frac{1}{2}tr\,\chi\,{}^{(c)}\nabla_3\psi\\ &+ \triangle_k^c\psi + (\eta+\underline{\eta})\cdot{}^{(c)}\nabla\psi,\end{aligned} \tag{3.2.7}$$

Proof. See the proof of Lemma 4.7.4 in [GKS-2022] for the first part. The second part follows then easily from first and the definition of the conformal derivatives. □

Corollary 3.2.11. *The wave operator for 0-conformally invariant $\psi \in \mathfrak{s}_k(\mathbb{C})$ is given by the formula (with $A = a_1 \ldots a_k$.)*

$$\begin{aligned}\dot{\square}_k\psi_A &= -\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\psi_A - \frac{1}{2}tr\,\underline{\chi}\,{}^{(c)}\nabla_4\psi_A - \frac{1}{2}tr\,\chi\,{}^{(c)}\nabla_3\psi_A + {}^{(c)}\triangle_2\psi_A + 2\underline{\eta}\cdot{}^{(c)}\nabla\psi_A\\ &+ \sum_{i=1}^k \dot{\mathbf{R}}_{a_i c 43}\psi_{a_1}{}^{c}{}_{a_k}\end{aligned} \tag{3.2.8}$$

Proof. It follows easily from the commutator formula for $[\,{}^{(c)}\nabla_4, {}^{(c)}\nabla_3]$ of 3.1.20 applied to signature $s = 0$. □

3.3 Integrable S-foliations

We consider the case of a given foliation of our spacetime by compact two dimensional surfaces S. At every point p of a given S surface we take e_3, e_4 to be orthogonal to S. We are thus in a situation when our horizontal structure is integrable, i.e. ${}^{(a)}\mathrm{tr}\chi = {}^{(a)}\mathrm{tr}\underline{\chi} = 0$. Thus, in this case the second fundamental forms $\chi, \underline{\chi}$ are symmetric. The null structure and Bianchi equations simplify considerably.

Proposition 3.3.1. *We have*

$$\begin{aligned}
{}^{(c)}\nabla_3 tr\,\underline{\chi} &= -|\widehat{\underline{\chi}}|^2 - tr\,\underline{\chi}^2 + 2\,{}^{(c)}div\,\underline{\xi} + 2\underline{\xi}\cdot(\eta+\underline{\eta}),\\
0 &= 2\,{}^{(c)}curl\,\underline{\xi} + 2\underline{\xi}\wedge(-\eta+\underline{\eta}),\\
{}^{(c)}\nabla_3\widehat{\underline{\chi}} &= -tr\,\underline{\chi}\,\widehat{\underline{\chi}} + {}^{(c)}\nabla\widehat{\otimes}\underline{\xi} + \underline{\xi}\widehat{\otimes}(\eta+\underline{\eta}) - \underline{\alpha},
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_3 tr\,\chi &= -\widehat{\underline{\chi}}\cdot\widehat{\chi} - \frac{1}{2}tr\,\underline{\chi}\,tr\,\chi + 2\,{}^{(c)}div\,\eta + 2\big(\xi\cdot\underline{\xi} + |\eta|^2\big) + 2\rho,\\
0 &= 2\,{}^{(c)}curl\,\eta - \widehat{\underline{\chi}}\wedge\widehat{\chi} + 2\underline{\xi}\wedge\xi - 2\,{}^*\rho,\\
{}^{(c)}\nabla_3\widehat{\chi} &= -\frac{1}{2}\big(tr\,\chi\widehat{\underline{\chi}} + tr\,\underline{\chi}\widehat{\chi}\big) + {}^{(c)}\nabla\widehat{\otimes}\eta + \underline{\xi}\widehat{\otimes}\xi + \eta\widehat{\otimes}\eta,
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_4 tr\,\underline{\chi} &= -\widehat{\chi}\cdot\widehat{\underline{\chi}} - \frac{1}{2}tr\,\chi\,tr\,\underline{\chi} + 2\,{}^{(c)}div\,\underline{\eta} + 2\big(\xi\cdot\underline{\xi} + |\underline{\eta}|^2\big) + 2\rho,\\
0 &= 2\,{}^{(c)}curl\,\underline{\eta} - \widehat{\chi}\wedge\widehat{\underline{\chi}} + 2\xi\wedge\underline{\xi} + 2\,{}^*\rho,\\
{}^{(c)}\nabla_4\widehat{\underline{\chi}} &= -\frac{1}{2}\big(tr\,\underline{\chi}\widehat{\chi} + tr\,\chi\widehat{\underline{\chi}}\big) + {}^{(c)}\nabla\widehat{\otimes}\underline{\eta} + \xi\widehat{\otimes}\underline{\xi} + \underline{\eta}\widehat{\otimes}\underline{\eta},
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_4 tr\,\chi &= -|\widehat{\chi}|^2 - \frac{1}{2}tr\,\chi^2 + 2\,{}^{(c)}div\,\xi + 2\xi\cdot(\underline{\eta}+\eta),\\
0 &= 2\,{}^{(c)}curl\,\xi - tr\,\chi\,{}^{(a)}tr\chi + 2\xi\wedge(-\underline{\eta}+\eta),\\
{}^{(c)}\nabla_4\widehat{\chi} &= -tr\,\chi\,\widehat{\chi} + {}^{(c)}\nabla\widehat{\otimes}\xi + \xi\widehat{\otimes}(\underline{\eta}+\eta) - \alpha,
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_3\underline{\eta} - {}^{(c)}\nabla_4\underline{\xi} &= -\widehat{\underline{\chi}}\cdot(\underline{\eta}-\eta) - \frac{1}{2}tr\,\underline{\chi}(\underline{\eta}-\eta) + \underline{\beta},\\
{}^{(c)}\nabla_4\eta - {}^{(c)}\nabla_3\xi &= -\widehat{\chi}\cdot(\eta-\underline{\eta}) - \frac{1}{2}tr\,\chi(\eta-\underline{\eta}) - \beta.
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}div\,\widehat{\chi} &= \frac{1}{2}\,{}^{(c)}\nabla(tr\,\chi) - \beta,\\
{}^{(c)}div\,\widehat{\underline{\chi}} &= \frac{1}{2}\,{}^{(c)}\nabla(tr\,\underline{\chi}) + \underline{\beta}.
\end{aligned}$$

Proposition 3.3.2. *The Bianchi equations take the form*

$$
\begin{aligned}
{}^{(c)}\nabla_3\alpha - {}^{(c)}\nabla\widehat{\otimes}\beta &= -\frac{1}{2}tr\,\underline{\chi}\alpha + 4\eta\widehat{\otimes}\beta - 3(\rho\widehat{\chi} + {}^*\rho\,{}^*\widehat{\chi}),\\
{}^{(c)}\nabla_4\beta - {}^{(c)}div\,\alpha &= -2tr\,\chi\beta + \alpha\cdot\underline{\eta} + 3(\xi\rho + {}^*\xi\,{}^*\rho),\\
{}^{(c)}\nabla_3\beta + {}^{(c)}div\,\varrho &= -tr\,\underline{\chi}\beta + 2\underline{\beta}\cdot\widehat{\chi} + 3(\rho\eta + {}^*\rho\,{}^*\eta) + \alpha\cdot\underline{\xi},\\
{}^{(c)}\nabla_4\rho - {}^{(c)}div\,\beta &= -\frac{3}{2}tr\,\chi\rho + 2\underline{\eta}\cdot\beta - 2\xi\cdot\underline{\beta} - \frac{1}{2}\widehat{\underline{\chi}}\cdot\alpha,\\
{}^{(c)}\nabla_4\,{}^*\rho + {}^{(c)}curl\,\beta &= -\frac{3}{2}tr\,\chi\,{}^*\rho - 2\underline{\eta}\cdot{}^*\beta - 2\xi\cdot{}^*\underline{\beta} + \frac{1}{2}\widehat{\underline{\chi}}\cdot{}^*\alpha,\\
{}^{(c)}\nabla_3\rho + {}^{(c)}div\,\underline{\beta} &= -\frac{3}{2}tr\,\underline{\chi}\rho - 2\eta\cdot\underline{\beta} + 2\underline{\xi}\cdot\beta - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha},\\
{}^{(c)}\nabla_3\,{}^*\rho + {}^{(c)}curl\,\underline{\beta} &= -\frac{3}{2}tr\,\underline{\chi}\,{}^*\rho - 2\eta\cdot{}^*\underline{\beta} - 2\underline{\xi}\cdot{}^*\beta - \frac{1}{2}\widehat{\chi}\cdot{}^*\underline{\alpha},\\
{}^{(c)}\nabla_4\underline{\beta} - {}^{(c)}div\,\check{\varrho} &= -tr\,\chi\underline{\beta} + 2\beta\cdot\widehat{\underline{\chi}} - 3(\rho\underline{\eta} - {}^*\rho\,{}^*\underline{\eta}) - \underline{\alpha}\cdot\xi,
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_3\underline{\beta} + {}^{(c)}div\,\underline{\alpha} &= -2tr\,\underline{\chi}\,\underline{\beta} - \underline{\alpha}\cdot\eta - 3(\underline{\xi}\rho - {}^*\underline{\xi}\,{}^*\rho),\\
{}^{(c)}\nabla_4\underline{\alpha} + {}^{(c)}\nabla\widehat{\otimes}\underline{\beta} &= -\frac{1}{2}tr\,\chi\underline{\alpha} - 4\underline{\eta}\widehat{\otimes}\underline{\beta} - 3(\rho\widehat{\underline{\chi}} - {}^*\rho\,{}^*\widehat{\underline{\chi}}).
\end{aligned}
$$

3.3.1 Diez operators

Definition 3.3.3. *Define the rank of an horizontal tensor ψ to be the negative of its scale. Thus curvature components $\alpha, \beta, \ldots$ have scale -2 and rank $k = 2$ while Ricci coefficients have scale -1 and rank $k = 1$. Note that the scale of the metric is 0 and its derivative of it lowers the scale by 1.*

Definition 3.3.4. *If f is a horizontal tensor of signature s and scale k we define*

$$
\begin{aligned}
\nabla_3^{\#}\psi &= {}^{(c)}\nabla_3\psi + \frac{1}{2}(1 - s + k)tr\,\underline{\chi}\psi\\
\nabla_4^{\#}\psi &= {}^{(c)}\nabla_3\psi + \frac{1}{2}(1 + s + k)tr\,\chi\psi
\end{aligned}
\tag{3.3.1}
$$

Remark 3.3.5. *According to this definition*

$$
\begin{aligned}
\nabla_3^{\#}\alpha &= {}^{(c)}\nabla_3\alpha + \frac{1}{2}(1-2+2)tr\underline{\chi}\alpha = {}^{(c)}\nabla_3\alpha + \frac{1}{2}tr\underline{\chi}\alpha\\
\nabla_4^{\#}\alpha &= {}^{(c)}\nabla_4\alpha + \frac{1}{2}(1+2+2)tr\,\chi\alpha = {}^{(c)}\nabla_4\alpha + tr\,\chi\alpha\\
\nabla_3^{\#}\beta &= {}^{(c)}\nabla_3\beta + \frac{1}{2}(1-1+2)tr\underline{\chi}\beta = {}^{(c)}\nabla_3\beta + tr\underline{\chi}\beta\\
\nabla_4^{\#}\beta &= {}^{(c)}\nabla_4\beta + \frac{1}{2}(1+1+2)tr\underline{\chi}\beta = {}^{(c)}\nabla_4\beta + 2tr\,\chi\beta\\
\nabla_3^{\#}\rho &= {}^{(c)}\nabla_3\rho + \frac{1}{2}(1-0+2)tr\underline{\chi}\rho = {}^{(c)}\nabla_3\rho + \frac{3}{2}tr\underline{\chi}\rho\\
\nabla_4^{\#}\rho &= {}^{(c)}\nabla_4\rho + \frac{1}{2}(1+0+2)tr\,\chi\rho = {}^{(c)}\nabla_4\rho + \frac{3}{2}tr\,\chi\rho\\
\nabla_3^{\#}\ {}^*\rho &= {}^{(c)}\nabla_3\ {}^*\rho + \frac{1}{2}(1-0+2)tr\underline{\chi}\ {}^*\rho = {}^{(c)}\nabla_3\ {}^*\rho + \frac{3}{2}tr\underline{\chi}\ {}^*\rho\\
\nabla_4^{\#}\ {}^*\rho &= {}^{(c)}\nabla_4\ {}^*\rho + \frac{1}{2}(1+0+2)tr\,\chi\ {}^*\rho = {}^{(c)}\nabla_4\ {}^*\rho + \frac{3}{2}tr\,\chi\ {}^*\rho
\end{aligned}
$$

Also, since $\nabla_3^{\#}\alpha$ has signature $s=1$ and rank $k=3$

$$
\begin{aligned}
\nabla_3^{\#}\nabla_3^{\#}\alpha &= {}^{(c)}\nabla_3(\nabla_3^{\#}\alpha) + \frac{1}{2}(1-1+3)tr\underline{\chi}\nabla_3^{\#}\alpha = {}^{(c)}\nabla_3(\nabla_3^{\#}\alpha) + \frac{3}{2}tr\underline{\chi}\nabla_3^{\#}\alpha\\
\nabla_4^{\#}\nabla_3^{\#}\alpha &= {}^{(c)}\nabla_4(\nabla_3^{\#}\alpha) + \frac{1}{2}(1+1+3)tr\underline{\chi}\nabla_3^{\#}\alpha = {}^{(c)}\nabla_4(\nabla_3^{\#}\alpha) + \frac{5}{2}tr\underline{\chi}\nabla_3^{\#}\alpha
\end{aligned}
$$

and similarly for all other second and higher derivatives.

Thus Proposition 3.3.2 takes the form

$$
\begin{aligned}
\nabla_3^\# \alpha &= {}^{(c)}\nabla\widehat{\otimes}\beta + 4\eta\widehat{\otimes}\beta - 3(\rho\widehat{\chi} + {}^*\rho\,{}^*\widehat{\chi}),\\
\nabla_4^\# \beta &= {}^{(c)}\mathrm{d}iv\,\alpha + \alpha\cdot\underline{\eta} + 3(\xi\rho + {}^*\xi\,{}^*\rho),\\
\nabla_3^\# \beta &= -{}^{(c)}\mathrm{d}iv\,\varrho + 2\underline{\beta}\cdot\widehat{\chi} + 3(\rho\eta + {}^*\rho\,{}^*\eta) + \alpha\cdot\underline{\xi},\\
\nabla_4^\# \rho &= {}^{(c)}\mathrm{d}iv\,\beta + 2\underline{\eta}\cdot\beta - 2\xi\cdot\underline{\beta} - \frac{1}{2}\underline{\widehat{\chi}}\cdot\alpha,\\
\nabla_4^\# \,{}^*\rho &= -{}^{(c)}curl\,\beta - 2\underline{\eta}\cdot{}^*\beta - 2\xi\cdot{}^*\underline{\beta} + \frac{1}{2}\underline{\widehat{\chi}}\cdot{}^*\alpha,\\
\\
\nabla_3^\# \rho &= -{}^{(c)}\mathrm{d}iv\,\underline{\beta} - 2\eta\cdot\underline{\beta} + 2\underline{\xi}\cdot\beta - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha},\\
\nabla_3^\# \,{}^*\rho &= -{}^{(c)}curl\,\underline{\beta} - 2\eta\cdot{}^*\underline{\beta} - 2\underline{\xi}\cdot{}^*\beta - \frac{1}{2}\widehat{\chi}\cdot{}^*\underline{\alpha},\\
{}^{(c)}\nabla_4\underline{\beta} &= {}^{(c)}\mathrm{d}iv\,\check{\varrho} + 2\beta\cdot\underline{\widehat{\chi}} - 3(\rho\underline{\eta} - {}^*\rho\,{}^*\underline{\eta}) - \underline{\alpha}\cdot\xi,\\
\nabla_3^\# \underline{\beta} &= -{}^{(c)}\mathrm{d}iv\,\underline{\alpha} - \underline{\alpha}\cdot\eta - 3(\underline{\xi}\rho - {}^*\underline{\xi}\,{}^*\rho),\\
\nabla_4^\# \underline{\alpha} &= -{}^{(c)}\nabla\widehat{\otimes}\underline{\beta} - 4\underline{\eta}\widehat{\otimes}\underline{\beta} - 3(\rho\underline{\widehat{\chi}} - {}^*\rho\,{}^*\underline{\widehat{\chi}}).
\end{aligned}
$$

Using the definition of the Hodge operators, see Lemma 3.0.33 and recalling the formulas $-\mathrm{d}iv\,\varrho = (\nabla\rho + {}^*\nabla\,{}^*\rho) = \mathcal{D}_1^*(-\rho, {}^*\rho)$ and $-\mathrm{d}iv\,\check{\varrho} = (\nabla\rho - {}^*\nabla\,{}^*\rho) = -\mathcal{D}_1^*(\rho, {}^*\rho)$, see (3.1.12), we deduce

Proposition 3.3.6. *The Bianchi equations take the form*

$$
\begin{aligned}
\nabla_3^\# \alpha &= 2\,\mathcal{D}_2^*\beta + 4\eta\widehat{\otimes}\beta - 3(\rho\widehat{\chi} + {}^*\rho\,{}^*\widehat{\chi}),\\
\nabla_4^\# \beta &= \mathcal{D}_2\alpha + \alpha\cdot\underline{\eta} + 3(\xi\rho + {}^*\xi\,{}^*\rho),\\
\\
\nabla_3^\# \beta &= \mathcal{D}_1^*(-\rho, {}^*\rho) - tr\,\underline{\chi}\beta + 2\underline{\beta}\cdot\widehat{\chi} + 3(\rho\eta + {}^*\rho\,{}^*\eta) + \alpha\cdot\underline{\xi},\\
\nabla_4^\#(-\rho, {}^*\rho) &= -\mathcal{D}_1\beta - 2\underline{\eta}\cdot(\beta, {}^*\underline{\beta}) + 2\xi\cdot(\underline{\beta}, -{}^*\underline{\beta}) + \frac{1}{2}\underline{\widehat{\chi}}\cdot(\alpha, {}^*\alpha),\\
\\
\nabla_3^\#(\rho, {}^*\rho) &= -\mathcal{D}_1\underline{\beta} - 2\eta\cdot(\underline{\beta}, {}^*\underline{\beta}) + 2\underline{\xi}\cdot(\beta, -{}^*\beta) - \frac{1}{2}\widehat{\chi}\cdot(\alpha, {}^*\alpha)\\
{}^{(c)}\nabla_4\underline{\beta} &= \mathcal{D}_1^*(\rho, {}^*\rho) + 2\beta\cdot\underline{\widehat{\chi}} - 3(\rho\underline{\eta} - {}^*\rho\,{}^*\underline{\eta}) - \underline{\alpha}\cdot\xi,\\
\\
\nabla_3^\# \underline{\beta} &= -\mathcal{D}_2\underline{\alpha} - \underline{\alpha}\cdot\eta - 3(\underline{\xi}\rho - {}^*\underline{\xi}\,{}^*\rho),\\
\nabla_4^\# \underline{\alpha} &= 2\,\mathcal{D}_2^*\underline{\beta} - \frac{1}{2}tr\,\chi\underline{\alpha} - 4\underline{\eta}\widehat{\otimes}\underline{\beta} - 3(\rho\underline{\widehat{\chi}} - {}^*\rho\,{}^*\underline{\widehat{\chi}}).
\end{aligned}
$$

Remark 3.3.7. *The division in Bianchi pairs is important as we shall see later.*

3.3.2 Commutator formulas for the diez operators

Lemma 3.3.8 (Commutator Formulas)**.** *Given U of signature s and rank k we have*

1. *We have*

$$
\begin{aligned}
[\,{}^{(c)}\nabla_3^\#,\,{}^{(c)}\nabla_b]U_A &= -\underline{\widehat{\chi}}_{bc}\,{}^{(c)}\nabla_c U_A + \eta_b\,{}^{(c)}\nabla_3 U_A + \underline{\xi}_b\,{}^{(c)}\nabla_4 U_A + \sum_{i=1}^k \dot{\mathbf{R}}_{a_i c 3 b} U_{a_1}{}^{c}{}_{a_k} \\
&\quad - s\big(\underline{\chi}\cdot\eta - \chi\cdot\underline{\xi} + \underline{\beta}\big)U_A - \frac{1}{2}(1-s+k)\nabla tr\underline{\chi} U_A
\end{aligned}
\tag{3.3.2}
$$

2. *We have*

$$
\begin{aligned}
[\,{}^{(c)}\nabla_4^\#,\,{}^{(c)}\nabla_b]U_A &= -\widehat{\chi}_{bc}\,{}^{(c)}\nabla_c U_a + \underline{\eta}_b\,{}^{(c)}\nabla_4 U_a + \xi_b\,{}^{(c)}\nabla_3 U_a + \sum_{i=1}^k \dot{\mathbf{R}}_{a_i c 4 b} U_{a_1}{}^{c}{}_{a_k} \\
&\quad - s\big(\chi\cdot\eta - \underline{\chi}\cdot\xi - \beta\big)U_A - \frac{1}{2}(1-s+k)\nabla tr\chi U_A
\end{aligned}
\tag{3.3.3}
$$

3. *We have*

$$
\begin{aligned}
[\,{}^{(c)}\nabla_4^\#,\,{}^{(c)}\nabla_3^\#]U_A &= 2(\underline{\eta}_b - \eta_b)\,{}^{(c)}\nabla_b U_A + \sum_{i=1}^k \dot{\mathbf{R}}_{a_i b 4 3} U_{a_1}{}^{b}{}_{a_k} - s\big(4\rho - \frac{1}{2}tr\,\underline{\chi} tr\,\chi - 2\eta\cdot\underline{\eta}\big) \\
&\quad + s\big(\underline{\widehat{\chi}}\cdot\widehat{\chi} - 4\xi\cdot\underline{\xi}\big)U \\
&\quad + \Big((1-s+k)\big({}^{(c)}div\,\underline{\eta} + |\underline{\eta}|^2\big) - (1+s+k)\big({}^{(c)}div\,\eta + |\eta|^2\big)\Big)U
\end{aligned}
\tag{3.3.4}
$$

Proof. Since U has rank k and signature s, ∇U has signature s and rank $k+1$, ${}^{(c)}\nabla_3^\# U$ has signature $s-1$ and rank $k+1$ and ${}^{(c)}\nabla_4^\# U$ has signature $s+1$ and rank $k+1$. Therefore

$$
\begin{aligned}
& [\,{}^{(c)}\nabla_3^\#,\,{}^{(c)}\nabla_b]U = {}^{(c)}\nabla_3^\#\big({}^{(c)}\nabla_b U\big) - {}^{(c)}\nabla_b\,{}^{(c)}\nabla_3^\# \\
= & \left({}^{(c)}\nabla_3 + \frac{1}{2}(1-s+k+1)\mathrm{tr}\underline{\chi}\right){}^{(c)}\nabla_b U - {}^{(c)}\nabla_b\left({}^{(c)}\nabla_3 U + \mathrm{tr}\,\underline{\chi} U\right) \\
= & [\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_b]U + \frac{1}{2}(1-s+k+1)\mathrm{tr}\underline{\chi}\,{}^{(c)}\nabla_b U - \frac{1}{2}(1-s+k)\mathrm{tr}\underline{\chi}\,{}^{(c)}\nabla_b U \\
& -\frac{1}{2}(1-s+k)\nabla\mathrm{tr}\underline{\chi} U \\
= & [\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_b]U - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_b U - \frac{1}{2}(1-s+k)\nabla\mathrm{tr}\underline{\chi} U
\end{aligned}
$$

Hence, in view of Lemma 3.1.20 and ${}^{(a)}\mathrm{tr}\underline{\chi}=0$,

$$[\,{}^{(c)}\nabla_3^\#,\,{}^{(c)}\nabla_b]U=-\widehat{\underline{\chi}}_{bc}\,{}^{(c)}\nabla_c U_A+\eta_b\,{}^{(c)}\nabla_3 U_A+\underline{\xi}_b\,{}^{(c)}\nabla_4 U_A+\sum_{i=1}^k \dot{\mathbf{R}}_{a_i c 3 b}U_{a_1}{}^{c}{}_{a_k}$$
$$-s\big(\underline{\chi}\cdot\eta-\chi\cdot\underline{\xi}+\underline{\beta}\big)U_A-\frac{1}{2}(1-s+k)\nabla\mathrm{tr}\underline{\chi}U$$

as stated. The commutator formula for $[\,{}^{(c)}\nabla_3^\#,\,{}^{(c)}\nabla_b]U$ is derived in the same manner.

Using the definitions of ${}^{(c)}\nabla_3^\#$, ${}^{(c)}\nabla_4^\#$ we deduce

$$\Big[{}^{(c)}\nabla_4^\#,\,{}^{(c)}\nabla_3^\#\Big]U \;=\; [\,{}^{(c)}\nabla_4,\,{}^{(c)}\nabla_3]U-\frac{1}{2}\Big((1+s+k)(\,{}^{(c)}\nabla_3\mathrm{tr}\chi)-(1-s+k)(\,{}^{(c)}\nabla_4\mathrm{tr}\underline{\chi})\Big)U$$

In view of the null structure equations, see Proposition 3.1.15, since ${}^{(a)}\mathrm{tr}\chi={}^{(a)}\mathrm{tr}\underline{\chi}=0$,

$$\begin{aligned}
{}^{(c)}\nabla_3\mathrm{tr}\,\chi &= 2\rho-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi+2\big(\,{}^{(c)}\mathrm{div}\,\eta+|\eta|^2\big)-\widehat{\underline{\chi}}\cdot\widehat{\chi}+2\xi\cdot\underline{\xi}\\
{}^{(c)}\nabla_4\mathrm{tr}\,\underline{\chi} &= 2\rho-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi+2\big(\,{}^{(c)}\mathrm{div}\,\underline{\eta}+|\underline{\eta}|^2\big)-\widehat{\chi}\cdot\widehat{\underline{\chi}}+2\xi\cdot\underline{\xi}
\end{aligned}$$

we deduce

$$\begin{aligned}
\Big[{}^{(c)}\nabla_4^\#,\,{}^{(c)}\nabla_3^\#\Big]U &= \big[\,{}^{(c)}\nabla_4,\,{}^{(c)}\nabla_3\big]\,U-s\Big(2\rho-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi-\widehat{\underline{\chi}}\cdot\widehat{\chi}+2\xi\cdot\underline{\xi}\Big)U\\
&\quad+(1-s+k)\big(\,{}^{(c)}\mathrm{div}\,\underline{\eta}+|\underline{\eta}|^2\big)-(1+s+k)\big(\,{}^{(c)}\mathrm{div}\,\eta+|\eta|^2\big)\\
&= 2(\underline{\eta}_b-\eta_b)\,{}^{(c)}\nabla_b U_A+\sum_{i=1}^k\dot{\mathbf{R}}_{a_i b 4 3}U_{a_1}{}^{b}{}_{a_k}-s\big(2\rho-2\eta\cdot\underline{\eta}+2\xi\cdot\underline{\xi}\big)U\\
&\quad-s\Big(2\rho-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi-\widehat{\underline{\chi}}\cdot\widehat{\chi}+2\xi\cdot\underline{\xi}\Big)\\
&\quad+(1-s+k)\big(\,{}^{(c)}\mathrm{div}\,\underline{\eta}+|\underline{\eta}|^2\big)-(1+s+k)\big(\,{}^{(c)}\mathrm{div}\,\eta+|\eta|^2\big)\\
&= 2(\underline{\eta}_b-\eta_b)\,{}^{(c)}\nabla_b U_A+\sum_{i=1}^k\dot{\mathbf{R}}_{a_i b 4 3}U_{a_1}{}^{b}{}_{a_k}\\
&\quad-s\big(4\rho-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi-2\eta\cdot\underline{\eta}-\widehat{\underline{\chi}}\cdot\widehat{\chi}+4\xi\cdot\underline{\xi}\big)U\\
&\quad+\Big((1-s+k)\big(\,{}^{(c)}\mathrm{div}\,\underline{\eta}+|\underline{\eta}|^2\big)-(1+s+k)\big(\,{}^{(c)}\mathrm{div}\,\eta+|\eta|^2\big)\Big)U
\end{aligned}$$

as stated. $\square$

The commutation formulas with the diez operators become particularly simple in the case of linear perturbations around Schwarzschild when $\widehat{\chi}, \underline{\widehat{\chi}}, \eta, \underline{\eta}, \xi, \underline{\xi}, \beta, \underline{\beta}$, ${}^*\rho$ are all linear quantities, i.e. they all vanish in Kerr.

Corollary 3.3.9. *For linearized perturbations near Schwarzschild we have, for any linear, horizontal, tensorfield U of signature s,*

$$[\,{}^{(c)}\nabla_3^{\#}, {}^{(c)}\nabla]U = [\,{}^{(c)}\nabla_4^{\#}, {}^{(c)}\nabla]U = 0.$$

and

$$[\,{}^{(c)}\nabla_4^{\#}, {}^{(c)}\nabla_3^{\#}]U \quad = \quad -s\big(4\rho - \frac{1}{2}tr\,\underline{\chi}\,tr\,\chi\big)U$$

Leibnitz rule for the diez operators

Lemma 3.3.10. *We have*

$$\begin{aligned}
{}^{(c)}\nabla_3^{\#}(\psi_1\cdot\psi_2) &= {}^{(c)}\nabla_3^{\#}\psi_1\cdot\psi_2 + \psi_1\cdot{}^{(c)}\nabla_3^{\#}\psi_2 - \frac{1}{2}tr\,\underline{\chi}\psi_1\cdot\psi_2\\
{}^{(c)}\nabla_4^{\#}(\psi_1\cdot\psi_2) &= {}^{(c)}\nabla_4^{\#}\psi_1\cdot\psi_2 + \psi_1\cdot{}^{(c)}\nabla_4^{\#}\psi_2 - \frac{1}{2}tr\,\chi\psi_1\cdot\psi_2
\end{aligned}\tag{3.3.5}$$

Also

$$\begin{aligned}
{}^{(c)}\nabla_3^{\#}(\psi_1\cdot\psi_2\cdot\psi_3) &= {}^{(c)}\nabla_3^{\#}\psi_1\cdot\psi_2\cdot\psi_3 + \psi_1\cdot{}^{(c)}\nabla_3^{\#}\psi_2\cdot\psi_3 + \psi_1\cdot\psi_2\cdot{}^{(c)}\nabla_3^{\#}\psi_3\\
&- tr\,\underline{\chi}\psi_1\cdot\psi_2\cdot\psi_3\\
{}^{(c)}\nabla_4^{\#}(\psi_1\cdot\psi_2\cdot\psi_3) &= {}^{(c)}\nabla_4^{\#}\psi_1\cdot\psi_2\cdot\psi_3 + \psi_1\cdot{}^{(c)}\nabla_4^{\#}\psi_2\cdot\psi_3 + \psi_1\cdot\psi_2\cdot{}^{(c)}\nabla_4^{\#}\psi_3\\
&- tr\,\chi\psi_1\cdot\psi_2\cdot\psi_3
\end{aligned}\tag{3.3.6}$$

Proof. Assume $\mathrm{sign}(\psi_i) = s_i$, $\mathrm{rank}(\psi_i) = k_i$. Note that $\mathrm{sign}(\psi_1\cdot\psi_2) = s_1 + s_2$ and $\mathrm{rank}(\psi_1\cdot\psi_2) = k_1 + k_2$. Therefore,

$$\begin{aligned}
&{}^{(c)}\nabla_3^{\#}(\psi_1\cdot\psi_2)\\
=\;& {}^{(c)}\nabla_3(\psi_1\cdot\psi_2) + \frac{1}{2}\big(1-(s_2+s_2)+(k_1+k_2)\big)\mathrm{tr}\,\underline{\chi}\psi_1\cdot\psi_2\\
=\;& {}^{(c)}\nabla_3\psi_1\cdot\psi_2 + \psi_1\cdot{}^{(c)}\nabla_3\psi_2 + \frac{1}{2}\big(1-(s_2+s_2)+(k_1+k_2)\big)\mathrm{tr}\,\underline{\chi}\psi_1\cdot\psi_2\\
=\;& \big({}^{(c)}\nabla_3\psi_1 + \frac{1}{2}(1-s_1+k_1)\mathrm{tr}\,\underline{\chi}\psi_1\big)\cdot\psi_2 + \psi_1\cdot\big({}^{(c)}\nabla_3\psi_2 + \frac{1}{2}(1-s_2+k_2)\mathrm{tr}\,\underline{\chi}\psi_2\big)\\
&-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\psi_1\cdot\psi_2\\
=\;& {}^{(c)}\nabla_3^{\#}\psi_1\cdot\psi_2 + \psi_1\cdot{}^{(c)}\nabla_3^{\#}\psi_2 - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\psi_1\cdot\psi_2
\end{aligned}$$

The second identity in (3.3.5) follows in the same manner. Similarly, $\mathrm{sign}(\psi_1\cdot\psi_2\cdot\psi_3) = s_1+s_2+s_3$ and $\mathrm{rank}(\psi_1\cdot\psi_2\cdot\psi_3) = k_1+k_2+k_3$. Hence

$$^{(c)}\nabla_3^{\#}\psi_i = \, ^{(c)}\nabla_3\psi_1 + \frac{1}{2}(1-s_i+k_i)\mathrm{tr}\,\underline{\chi}\psi_1$$

and

$$^{(c)}\nabla_3^{\#}(\psi_1\cdot\psi_2\cdot\psi_3) \;\; = \;\; ^{(c)}\nabla_3\big(\psi_1\cdot\psi_2\cdot\psi\big) + \frac{1}{2}(1-\sum s+\sum k)\mathrm{tr}\,\underline{\chi}\psi_1\cdot\psi_2\cdot\psi_3$$

from which the result easily follows.

□

3.3.3 Double null and geodesic foliations

Definition 3.3.11 (Double null)**.** *An optical function u is a regular solution (i.e. $du\neq 0$), of the Eikonal equation*

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0.$$

In that case $L = -\mathbf{g}^{\alpha\beta}\partial_\beta u\partial_\alpha$ is null and geodesic, i.e. $\mathbf{D}_L L = 0$ and is called the null geodesic generators of the null hypersurfaces generated by the level surfaces of u.

Definition 3.3.12. *Consider a region $\mathcal{D} = \mathcal{D}(u_*, \underline{u}_*)$ of a vacuum spacetime $(\mathcal{M}, \mathbf{g})$ spanned by a double null foliation generated by the optical functions $(u, \underline{u})$ increasing towards the future, $0\le u\le u_*$ and $0\le \underline{u}\le \underline{u}_*$. We denote by H_u the outgoing null hypersurfaces generated by the level surfaces of u and by $\underline{H}_{\underline{u}}$ the incoming null hypersurfaces generated level hypersurfaces of $\underline{u}$. We write $S_{u,\underline{u}} = H_u\cap\underline{H}_{\underline{u}}$ Let L, $\underline{L}$ be the geodesic vectorfields associated to the two foliations and define[19],*

$$\frac{1}{2}\Omega^2 = -\mathbf{g}(L,\, \underline{L})^{-1} \tag{3.3.7}$$

The normalized symmetric null pair is defined by,

$$e_3 = \Omega\,\underline{L}, \quad e_4 = \Omega L, \qquad \mathbf{g}(e_3, e_4) = -2$$

Given a 2-surfaces $S(u,\underline{u})$ and $(e_a)_{a=1,2}$ an arbitrary frame tangent to it we recall the Ricci coefficients,

$$\Gamma_{(\lambda)(\mu)(\nu)} = g(e_{(\lambda)}, \mathbf{D}_{e_{(\nu)}}e_{(\mu)}), \quad \lambda,\mu,\nu = 1,2,3,4 \tag{3.3.8}$$

[19]Observe that the flat value of Ω is 1.

These coefficients are completely determined by the following components,

$$
\begin{aligned}
\chi_{ab} &= \mathbf{g}(\mathbf{D}_a e_4, e_b), & \underline{\chi}_{ab} &= \mathbf{g}(\mathbf{D}_a e_3, e_b),\\
\eta_a &= -\frac{1}{2}\, g(\mathbf{D}_3 e_a, e_4), & \underline{\eta}_a &= -\frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_a, e_3)\\
\omega &= -\frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_3, e_4), & \underline{\omega} &= -\frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_4, e_3),\\
\zeta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_a e_4, e_3)
\end{aligned}
\tag{3.3.9}
$$

where $\mathbf{D}_a = \mathbf{D}_{e_{(a)}}$, $\mathbf{D}_3 = \mathbf{D}_{e_3}, \mathbf{D}_4 = \mathbf{D}_{e_4}$.

Lemma 3.3.13. *For a double null foliation we have,*

$$
\begin{aligned}
\omega &= -\frac{1}{2}\nabla_4(\log\Omega), & \underline{\omega} &= -\frac{1}{2}\nabla_3(\log\Omega),\\
\eta_a &= \zeta_a + \nabla_a(\log\Omega), & \underline{\eta}_a &= -\zeta_a + \nabla_a(\log\Omega)
\end{aligned}
\tag{3.3.10}
$$

Proof. Straightforward verification. Compare also with the proof of Lemma 3.3.15 below. □

For a more detailed exposition of double null foliations see[20] [Kl-Ni1] and in [Chr-BH] in the context of Christodolou's famous result[21] on formation of trapped surfaces.

Definition 3.3.14 (Geodesic). *A geodesic foliations are given by the level surfaces of function (u, s) where u is an outgoing (or incoming) optical function u,*

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = g^{ab}\partial_a u \partial_b u = 0$$

and s verifies,

$$L(s) = 1, \qquad L = -g^{ab}\partial_b u \partial_a.$$

We denote $S_{u,s}$ the 2-surfaces of intersection between the level surfaces of u and s. We then choose $e_4 = L$ and e_3 the unique null vectorfield orthogonal to $S_{u,s}$ and such that $g(e_3, e_4) = -2$. We also introduce

$$\underline{\Omega} := e_3(s). \tag{3.3.11}$$

[20] [Kl-Ni1] contains a proof of the stability of the Minkowski space in the exterior of the domain of influence of a compact region. A modern version of the result can be found in [Shen:Mink-ext]

[21] See also [Kl-Rod2] and [An-Luk] for more recent versions of the result.

Lemma 3.3.15. *We have*

$$\omega = \xi = 0, \quad \eta = \zeta, \quad \underline{\eta} = -\zeta, \quad e_\theta(\underline{\Omega}) = -\underline{\xi}, \quad e_4(\underline{\Omega}) = -2\underline{\omega}. \tag{3.3.12}$$

Proof. Since e_4 is geodesic, we have $\omega = \xi = 0$. Next, note that

$$e_\theta(u) = e_\theta(s) = e_4(u) = 0$$

and

$$e_3(u) = g(e_3, -L) = -g(e_3, e_4) = 2, \quad e_4(s) = 1.$$

Applying the vectorfield

$$[e_3, e_a] = \underline{\xi}_a e_4 + (\eta - \zeta)_a e_3 - \underline{\chi}_{ab} e_b$$

to u we deduce $\eta = \zeta$. Applying then

$$[e_3, e_a] = \underline{\xi}_a e_4 - \underline{\chi}_{ab} e_b$$

to s we deduce $e_a(\underline{\Omega}) = -\underline{\xi}_a$. Applying

$$[e_4, e_a] = (\underline{\eta} + \zeta)_a e_4 - \chi_{ab} e_b$$

to s, we deduce that

$$0 = -e_s(e_4(s)) = (\underline{\eta} + \zeta) e_4(s) = \underline{\eta} + \zeta$$

and hence $\underline{\eta} + \zeta = 0$. Finally applying

$$[e_4, e_3] = -4\zeta_a e_a - 2\underline{\omega} e_4$$

to s we infer $e_4(e_3(s)) = -2\underline{\omega}$, i.e. $e_4(\underline{\Omega}) = -2\underline{\omega}$ as desired. □

3.3.4 Teukolski and Regge-Wheeler equations in the integrable case

3.4 Main equations in complex notations

In this section we introduce complex notations for the Ricci coefficients and the curvature components with the objective of simplifying the main equations. From the real scalars, 1-tensors and symmetric traceless 2-tensors already introduced, we define their complexified version which results in anti-self dual tensors.

3.4.1 Complex notations

Recall Definition 3.0.8 of the set of real horizontal k-tensors $\mathfrak{s}_k = \mathfrak{s}_k(\mathcal{M}, \mathbb{R})$ on $\mathcal{M}$. For instance,

- $(a, b) \in \mathfrak{s}_0$ is a pair of real scalar function on $\mathcal{M}$,
- $f \in \mathfrak{s}_1$ is a real horizontal 1-tensor on $\mathcal{M}$,
- $u \in \mathfrak{s}_2$ is a real horizontal symmetric traceless 2-tensor on $\mathcal{M}$.

By Definition 3.0.9, the duals of real horizontal tensors are real horizontal tensors of the same type, i.e. ${}^*f \in \mathfrak{s}_1$ and ${}^*u \in \mathfrak{s}_2$.

We define the complexified version of horizontal tensors on $\mathcal{M}$.

Definition 3.4.1. *We denote by $\mathfrak{s}_k(\mathbb{C}) = \mathfrak{s}_k(\mathcal{M}, \mathbb{C})$ the set of complex anti-self dual k-tensors on $\mathcal{M}$. More precisely,*

- $a + ib \in \mathfrak{s}_0(\mathbb{C})$ *is a complex scalar function on $\mathcal{M}$ if $(a, b) \in \mathfrak{s}_0$,*
- $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ *is a complex anti-self dual 1-tensor on $\mathcal{M}$ if $f \in \mathfrak{s}_1$,*
- $U = u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$ *is a complex anti-self dual symmetric traceless 2-tensor on $\mathcal{M}$ if $u \in \mathfrak{s}_2$.*

Observe that $F \in \mathfrak{s}_1(\mathbb{C})$ and $U \in \mathfrak{s}_2(\mathbb{C})$ are indeed anti-self dual tensors, i.e.:

$$ {}^*F = -iF, \qquad {}^*U = -iU. $$

More precisely

$$ U_{12} = U_{21} = i\,{}^*U_{12} = i \in_{12} U_{22} = -iU_{11}, \qquad U_{11} = iU_{12}. $$

Recall that the derivatives ∇_3, ∇_4 and ∇_a are real derivatives. We can use the dual operators to define the complexified version of the ∇_a derivative, which allows to simplify the notations in the main equations.

Definition 3.4.2. *We define the complexified version of the horizontal derivative as*

$$ \mathcal{D} = \nabla + i\,{}^*\nabla, \qquad \overline{\mathcal{D}} = \nabla - i\,{}^*\nabla. $$

More precisely, we have

- *for* $a+ib \in \mathfrak{s}_0(\mathbb{C})$,

$$\mathcal{D}(a+ib) := (\nabla + i\,{}^*\nabla)(a+ib), \qquad \overline{\mathcal{D}}(a+ib) := (\nabla - i\,{}^*\nabla)(a+ib).$$

- *For* $f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$,

$$\begin{aligned} \mathcal{D}\cdot(f+i\,{}^*f) &:= (\nabla + i\,{}^*\nabla)\cdot(f+i\,{}^*f) = 0, \\ \overline{\mathcal{D}}\cdot(f+i\,{}^*f) &:= (\nabla - i\,{}^*\nabla)\cdot(f+i\,{}^*f), \\ \mathcal{D}\widehat{\otimes}(f+i\,{}^*f) &:= (\nabla + i\,{}^*\nabla)\widehat{\otimes}(f+i\,{}^*f). \end{aligned}$$

- *For* $u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$,

$$\begin{aligned} \mathcal{D}\cdot(u+i\,{}^*u) &:= (\nabla + i\,{}^*\nabla)\cdot(u+i\,{}^*u) = 0 \\ \overline{\mathcal{D}}\cdot(u+i\,{}^*u) &:= (\nabla - i\,{}^*\nabla)\cdot(u+i\,{}^*u). \end{aligned}$$

Note that

$${}^*\mathcal{D} = -i\mathcal{D}.$$

For $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ the operator $-\frac{1}{2}\mathcal{D}\widehat{\otimes}$ is formally adjoint to the operator $\overline{\mathcal{D}}\cdot U$ applied to $U \in \mathfrak{s}_2(\mathbb{C})$. For $h = a+ib \in \mathfrak{s}_0(\mathbb{C})$ the operator $-\mathcal{D}h$ is formally adjoint to the operator $\overline{\mathcal{D}}\cdot F$ applied to $F \in \mathfrak{s}_1(\mathbb{C})$. These notions makes sense literally only if the horizontal structure is integrable.

Lemma 3.4.3. *For* $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ *and* $U = u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$, *we have*

$$(\mathcal{D}\widehat{\otimes}F)\cdot\overline{U} = -2F\cdot(\mathcal{D}\cdot\overline{U}) - ((H+\underline{H})\widehat{\otimes}F)\cdot\overline{U} + 2\mathcal{D}\cdot(F\cdot\overline{U}). \tag{3.4.1}$$

Proof. We look at the real parts. Then

$$(\nabla\widehat{\otimes}f)\cdot u = \big(\nabla_a f_b + \nabla_b f_a - \delta_{ab}\mathrm{d}iv\, f\big)u_{ab} = 2(\nabla_a f_b)u_{ab} = 2\nabla_a(u_{ab}f_b) - 2(\mathrm{d}iv\, u)\cdot f$$

Using Lemma 3.0.41 applied to $\xi = u\cdot f$ we obtain

$$\begin{aligned} (\nabla\widehat{\otimes}f)\cdot u &= 2\nabla^a(u_{ab}f_b) - 2(\eta+\underline{\eta})\cdot(u\cdot f) - 2(\mathrm{d}iv\, u)\cdot f \\ &= -2(\mathrm{d}iv\, u)\cdot f - ((\eta+\underline{\eta})\widehat{\otimes}f)\cdot u + 2\mathrm{d}iv\,(u\cdot f) \end{aligned}$$

By complexifying, we obtain the stated identity. □

Lemma 3.4.4. *The following holds.*

- *If* $\xi, \eta \in \mathfrak{s}_1$

$$\begin{aligned}\xi \cdot \eta + i\, {}^*\xi \cdot \eta &= \frac{1}{2}\Big((\xi + i\, {}^*\xi) \cdot (\overline{\eta + i\, {}^*\eta})\Big),\\ \xi \widehat{\otimes} \eta + i\, {}^*(\xi \widehat{\otimes} \eta) &= \frac{1}{2}\Big((\xi + i\, {}^*\xi) \widehat{\otimes} (\eta + i\, {}^*\eta)\Big).\end{aligned}$$

- *If* $\eta \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$

$$\begin{aligned}u \cdot \eta + i\, {}^*u \cdot \eta &= \frac{1}{2}(u + i\, {}^*u) \cdot (\overline{\eta + i\, {}^*\eta}),\\ u \cdot \eta + i\, {}^*(u \cdot \eta) &= \frac{1}{2}(u + i\, {}^*u) \cdot (\overline{\eta + i\, {}^*\eta}).\end{aligned}$$

- *If* $u, v \in \mathfrak{s}_2$

$$u \cdot v + i\, {}^*u \cdot v = \frac{1}{2}(u + i\, {}^*u) \cdot (\overline{v + i\, {}^*v}).$$

- *If* $(a, b) \in \mathfrak{s}_0$

$$\nabla a - {}^*\nabla b + i({}^*\nabla a + \nabla b) = \mathcal{D}(a + ib).$$

- *If* $\xi \in \mathfrak{s}_1$

$$\begin{aligned}div\, \xi + i curl\, \xi &= \frac{1}{2}\overline{\mathcal{D}} \cdot (\xi + i\, {}^*\xi)\\ \nabla \widehat{\otimes} \xi + i\, {}^*(\nabla \widehat{\otimes} \xi) &= \frac{1}{2}\mathcal{D} \widehat{\otimes} (\xi + i\, {}^*\xi).\end{aligned}$$

- *If* $u \in \mathfrak{s}_2$

$$div\, u + i\, {}^*(div\, u) = \frac{1}{2}\overline{\mathcal{D}} \cdot (u + i\, {}^*u).$$

Proof. Straightforward verification. □

Lemma 3.4.5. *Let* $E, F \in \mathfrak{s}_1(\mathbb{C})$ *and* $U \in \mathfrak{s}_2(\mathbb{C})$. *Then*

$$E \widehat{\otimes} (\overline{F} \cdot U) + F \widehat{\otimes} (\overline{E} \cdot U) = 2(E \cdot \overline{F} + \overline{E} \cdot F)\, U. \tag{3.4.2}$$

Proof. See proof of Lemma 2.4.5 in [GKS-2022]. □

Leibniz formulas

We collect here Leibniz formulas involving the derivative operators defined above.

Lemma 3.4.6. *Let h be a scalar function, $F \in \mathfrak{s}_1(\mathbb{C})$, $U \in \mathfrak{s}_2(\mathbb{C})$. Then*

$$\begin{aligned}\overline{\mathcal{D}} \cdot (hF) &= h\overline{\mathcal{D}} \cdot F + \overline{\mathcal{D}}(h) \cdot F,\\ \mathcal{D}\widehat{\otimes}(hF) &= h\mathcal{D}\widehat{\otimes}F + \mathcal{D}(h)\widehat{\otimes}F,\\ \overline{\mathcal{D}} \cdot (hU) &= \overline{\mathcal{D}}(h) \cdot U + h(\overline{\mathcal{D}} \cdot U),\\ \mathcal{D}\widehat{\otimes}(\overline{F} \cdot U) &= 2(\mathcal{D} \cdot \overline{F})U + 2(\overline{F} \cdot \mathcal{D})U,\\ U \cdot \overline{\mathcal{D}}F &= U(\overline{\mathcal{D}} \cdot F).\end{aligned} \tag{3.4.3}$$

Also,

$$\begin{aligned}F\widehat{\otimes}(\overline{\mathcal{D}} \cdot U) &= 2(F \cdot \overline{\mathcal{D}})U = 4F \cdot \nabla U,\\ (F \cdot \overline{\mathcal{D}})U + (\overline{F} \cdot \mathcal{D})U &= 4f \cdot \nabla U = 2(F + \overline{F}) \cdot \nabla U.\end{aligned} \tag{3.4.4}$$

Proof. Straightforward verifications, see section **??**. □

Lemma 3.4.7. *As a corollary of* (3.4.4) *we derive the following formula for $U \in \mathfrak{s}_2(\mathbb{C})$*

$$\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}} \cdot U) \;=\; 2\triangle_2 U - 4\,{}^{(h)}K U - i\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)U \tag{3.4.5}$$

where

$${}^{(h)}K = -\frac{1}{4}tr\,\chi tr\underline{\chi} - \frac{1}{4}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}} - \frac{1}{4}\rho.$$

Proof. See proof of Lemma 2.4.7 in [GKS-2022]. □

3.4.2 Main equations in complex form

We now extend the definitions for the Ricci coefficients and curvature components given in Sections 3.1.1 and 3.1.2, to the complex case by using the anti-self dual tensors defined above.

Definition 3.4.8. *We define the following complex anti-self dual tensors:*

$$A := \alpha + i\,{}^*\alpha, \quad B := \beta + i\,{}^*\beta, \quad P := \rho + i\,{}^*\rho, \quad \underline{B} := \underline{\beta} + i\,{}^*\underline{\beta}, \quad \underline{A} := \underline{\alpha} + i\,{}^*\underline{\alpha},$$

and

$$X = \chi + i\,{}^*\chi, \quad \underline{X} = \underline{\chi} + i\,{}^*\underline{\chi}, \quad H = \eta + i\,{}^*\eta, \quad \underline{H} = \underline{\eta} + i\,{}^*\underline{\eta}, \quad Z = \zeta + i\,{}^*\zeta,$$
$$\Xi = \xi + i\,{}^*\xi, \quad \underline{\Xi} = \underline{\xi} + i\,{}^*\underline{\xi}.$$

In particular, note that

$$trX = tr\,\chi - i\,{}^{(a)}tr\chi, \quad \widehat{X} = \widehat{\chi} + i\,{}^*\widehat{\chi}, \quad tr\underline{X} = tr\,\underline{\chi} - i\,{}^{(a)}tr\underline{\chi}, \quad \widehat{\underline{X}} = \widehat{\underline{\chi}} + i\,{}^*\widehat{\underline{\chi}}.$$

Remark 3.4.9. *The pairing relations described in section 3.1.3 imply the following transformation rules with respect to the interchange of $L = e_4$, $\underline{L} = e_3$*

$$A \leftrightarrow \overline{\underline{A}}, \quad B \leftrightarrow -\overline{\underline{B}}, \quad P \leftrightarrow P, \quad trX \leftrightarrow \overline{tr\underline{X}}, \quad \widehat{X} \leftrightarrow \overline{\widehat{\underline{X}}}, \quad H \leftrightarrow \overline{\underline{H}}, \quad \Xi \leftrightarrow \overline{\underline{\Xi}},$$
$$Z \leftrightarrow -\overline{Z}, \quad \omega \leftrightarrow \underline{\omega}, \quad \mathcal{D} \to \overline{\mathcal{D}}.$$

Note the anomaly $P \leftrightarrow P$ rather than $P \leftrightarrow \overline{P}$. This is consistent however to setting $\underline{P} = \rho - i\,{}^*\rho$ and then $\overline{\underline{P}} = P$.

The complex notations allow us to rewrite the Ricci equations in a more compact form.

Proposition 3.4.10.

$$\begin{aligned}
\nabla_3 tr\underline{X} + \frac{1}{2}(tr\underline{X})^2 + 2\underline{\omega}\, tr\underline{X} &= \mathcal{D}\cdot\overline{\underline{\Xi}} + \underline{\Xi}\cdot\overline{\underline{H}} + \overline{\underline{\Xi}}\cdot(H - 2Z) - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_3\widehat{\underline{X}} + \Re(tr\underline{X})\widehat{\underline{X}} + 2\underline{\omega}\,\widehat{\underline{X}} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \frac{1}{2}\underline{\Xi}\widehat{\otimes}(H + \underline{H} - 2Z) - \underline{A},
\end{aligned}$$

$$\begin{aligned}
\nabla_3 trX + \frac{1}{2}tr\underline{X}\,trX - 2\underline{\omega}trX &= \mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P + \underline{\Xi}\cdot\overline{\Xi} - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}},\\
\nabla_3\widehat{X} + \frac{1}{2}tr\underline{X}\,\widehat{X} - 2\underline{\omega}\widehat{X} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}H + \frac{1}{2}H\widehat{\otimes}H - \frac{1}{2}\overline{trX}\widehat{\underline{X}} + \frac{1}{4}\underline{\Xi}\widehat{\otimes}\Xi,
\end{aligned}$$

$$\begin{aligned}
\nabla_4 tr\underline{X} + \frac{1}{2}trX\,tr\underline{X} - 2\omega tr\underline{X} &= \mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}} + 2\overline{P} + \Xi\cdot\overline{\underline{\Xi}} - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}},\\
\nabla_4\widehat{\underline{X}} + \frac{1}{2}trX\,\widehat{\underline{X}} - 2\omega\widehat{\underline{X}} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{H} + \frac{1}{2}\underline{H}\widehat{\otimes}\underline{H} - \frac{1}{2}\overline{tr\underline{X}}\widehat{X} + \frac{1}{4}\Xi\widehat{\otimes}\underline{\Xi},
\end{aligned}$$

$$\begin{aligned}
\nabla_4 trX + \frac{1}{2}(trX)^2 + 2\omega trX &= \mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot(H + 2Z) - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},\\
\nabla_4\widehat{X} + \Re(trX)\widehat{X} + 2\omega\widehat{X} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\Xi + \frac{1}{2}\Xi\widehat{\otimes}(\underline{H} + H + 2Z) - A.
\end{aligned}$$

Also,

$$
\begin{aligned}
\nabla_3 Z + \frac{1}{2} tr\underline{X}(Z+H) - 2\underline{\omega}(Z-H) &= -2\mathcal{D}\underline{\omega} - \frac{1}{2}\widehat{\underline{X}} \cdot (\overline{Z} + \overline{H}) \\
&\quad + \frac{1}{2} trX\underline{\Xi} + 2\omega\underline{\Xi} - \underline{B} + \frac{1}{2}\overline{\underline{\Xi}} \cdot \widehat{X}, \\
\nabla_4 Z + \frac{1}{2} trX(Z - \underline{H}) - 2\omega(Z + \underline{H}) &= 2\mathcal{D}\omega + \frac{1}{2}\widehat{X} \cdot (-\overline{Z} + \overline{\underline{H}}) \\
&\quad - \frac{1}{2} tr\underline{X}\Xi - 2\underline{\omega}\Xi - B - \frac{1}{2}\overline{\Xi} \cdot \widehat{\underline{X}}, \\
\nabla_3 \underline{H} - \nabla_4 \underline{\Xi} &= -\frac{1}{2}\overline{tr\underline{X}}(\underline{H} - H) - \frac{1}{2}\widehat{\underline{X}} \cdot (\overline{\underline{H}} - \overline{H}) - 4\omega\underline{\Xi} + \underline{B}, \\
\nabla_4 H - \nabla_3 \Xi &= -\frac{1}{2}\overline{trX}(H - \underline{H}) - \frac{1}{2}\widehat{X} \cdot (\overline{H} - \overline{\underline{H}}) - 4\underline{\omega}\Xi - B,
\end{aligned}
$$

and

$$
\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega} - \xi \cdot \underline{\xi} - (\eta - \underline{\eta}) \cdot \zeta + \eta \cdot \underline{\eta} = \rho.
$$

Also,

$$
\begin{aligned}
\frac{1}{2}\overline{\mathcal{D}} \cdot \widehat{X} + \frac{1}{2}\widehat{X} \cdot \overline{Z} &= \frac{1}{2}\mathcal{D}\overline{trX} + \frac{1}{2}\overline{trX}Z - i\Im(trX)H - i\Im(tr\underline{X})\Xi - B, \\
\frac{1}{2}\overline{\mathcal{D}} \cdot \widehat{\underline{X}} - \frac{1}{2}\widehat{\underline{X}} \cdot \overline{Z} &= \frac{1}{2}\mathcal{D}\overline{tr\underline{X}} - \frac{1}{2}\overline{tr\underline{X}}Z - i\Im(tr\underline{X})\,\underline{H} - i\Im(trX)\underline{\Xi} + \underline{B},
\end{aligned}
$$

and,

$$
curl\,\zeta = -\frac{1}{2}\widehat{\chi} \wedge \widehat{\underline{\chi}} + \frac{1}{4}\Big(tr\,\chi^{(a)}tr\underline{\chi} - tr\,\underline{\chi}^{(a)}tr\chi\Big) + \omega^{(a)}tr\underline{\chi} - \underline{\omega}^{(a)}tr\chi + {}^*\rho.
$$

We rewrite the Gauss equation in Proposition 3.0.27 for complex tensors.

Proposition 3.4.11. *The following identity holds true for $\Psi \in \mathfrak{s}_k(\mathbb{C})$ for $k = 1, 2$:*

$$
[\nabla_a, \nabla_b]\Psi = \left(\frac{1}{2}({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4)\Psi - ik\,{}^{(h)}K\Psi\right) \in_{ab} \tag{3.4.6}
$$

where

$$
{}^{(h)}K = -\frac{1}{8}trX\overline{tr\underline{X}} - \frac{1}{8}tr\underline{X}\overline{trX} + \frac{1}{4}\widehat{X} \cdot \overline{\widehat{\underline{X}}} + \frac{1}{4}\overline{\widehat{X}} \cdot \widehat{\underline{X}} - \frac{1}{2}P - \frac{1}{2}\overline{P}.
$$

The complex notations allow us to rewrite the Bianchi identities as follows.

Proposition 3.4.12. *We have,*

$$
\begin{aligned}
\nabla_3 A - \frac{1}{2}\mathcal{D}\widehat{\otimes} B &= -\frac{1}{2}tr\underline{X}A + 4\underline{\omega}A + \frac{1}{2}(Z+4H)\widehat{\otimes}B - 3\overline{P}\widehat{X},\\
\nabla_4 B - \frac{1}{2}\overline{\mathcal{D}}\cdot A &= -2\overline{trX}B - 2\omega B + \frac{1}{2}A\cdot(\overline{2Z+\underline{H}}) + 3\overline{P}\,\Xi,\\
\nabla_3 B - \mathcal{D}\overline{P} &= -tr\underline{X}B + 2\underline{\omega}B + \overline{\underline{B}}\cdot\widehat{X} + 3\overline{P}H + \frac{1}{2}A\cdot\underline{\Xi},\\
\nabla_4 P - \frac{1}{2}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}trXP + \frac{1}{2}(2\,\underline{H}+Z)\cdot\overline{B} - \overline{\Xi}\cdot\underline{B} - \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A},\\
\nabla_3 P + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{B} &= -\frac{3}{2}\overline{tr\underline{X}}P - \frac{1}{2}(\overline{2H-Z})\cdot\underline{B} + \underline{\Xi}\cdot\overline{B} - \frac{1}{4}\overline{\widehat{X}}\cdot\underline{A},\\
\nabla_4\underline{B} + \mathcal{D}P &= -trX\underline{B} + 2\omega\underline{B} + \overline{B}\cdot\widehat{\underline{X}} - 3P\,\underline{H} - \frac{1}{2}\underline{A}\cdot\overline{\Xi},\\
\nabla_3\underline{B} + \frac{1}{2}\overline{\mathcal{D}}\cdot\mathcal{A} &= -2\overline{tr\underline{X}}\,\underline{B} - 2\underline{\omega}\,\underline{B} - \frac{1}{2}\underline{A}\cdot(\overline{-2Z+H}) - 3P\,\underline{\Xi},\\
\nabla_4\underline{A} + \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} &= -\frac{1}{2}\overline{trX}\underline{A} + 4\omega\underline{A} + \frac{1}{2}(Z-4\,\underline{H})\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}}.
\end{aligned}
$$

Proof. Straightforward verifications by complexifying the Bianchi identities of Proposition 3.1.6. □

Remark 3.4.13. *Note that both the complex null structure and null Bianchi equations are both invariant with respect to the pairing relations of Remark 3.4.9.*

Remark 3.4.14. *Note that the complex Bianchi identities can be also derived directly from the equations*

$$\mathbf{D}^\alpha \mathcal{R}_{\alpha\beta\gamma\delta} = 0, \qquad \mathcal{R} = \mathbf{R} + i\,{}^*\mathbf{R}.$$

In view of (3.1.6), $\underline{\alpha}({}^*R) = {}^*\underline{\alpha}(\mathbf{R})$, $\alpha({}^*\mathbf{R}) = -{}^*\alpha(\mathbf{R})$, $\underline{\beta}({}^*\mathbf{R}) = {}^*\underline{\beta}(\mathbf{R})$, $\beta({}^*\mathbf{R}) = -{}^*\beta(\mathbf{R})$, $\rho({}^*\mathbf{R}) = {}^*\rho$ *and therefore*

$$\mathcal{R}_{a4b4} = \overline{A}_{ab}, \quad \mathcal{R}_{a434} = 2\overline{B}_a, \quad \mathcal{R}_{3434} = 4P, \quad \mathcal{R}_{a334} = 2\underline{B}_a \quad \mathcal{R}_{a3b3} = \underline{A}. \tag{3.4.7}$$

The derivation is done in appendix

3.4.3 Main complex equations using conformal derivatives

Definition 3.4.15. *We define the following conformal angular derivatives in the complex notation:*

- *For* $a+ib \in \mathfrak{s}_0(\mathbb{C})$ *we define*

$$
{}^{(c)}\mathcal{D}(a+ib) \;:=\; \big({}^{(c)}\nabla + i\,{}^*{}^{(c)}\nabla\big)(a+ib).
$$

- *For* $f+i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ *we define*

$$
\begin{aligned}
{}^{(c)}\mathcal{D}(f+i\,{}^*f) &:= \big({}^{(c)}\nabla + i\,{}^*{}^{(c)}\nabla\big)\cdot(f+i\,{}^*f),\\
{}^{(c)}\mathcal{D}\widehat{\otimes}(f+i\,{}^*f) &:= ({}^{(c)}\nabla + i\,{}^*{}^{(c)}\nabla)\widehat{\otimes}(f+i\,{}^*f).
\end{aligned}
$$

- *For* $u+i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$ *we define*

$$
{}^{(c)}\mathcal{D}\cdot(u+i\,{}^*u) \;:=\; \big({}^{(c)}\nabla + i\,{}^*{}^{(c)}\nabla\big)\cdot(u+i\,{}^*u).
$$

- *In all the above cases we set*

$$
\overline{{}^{(c)}\mathcal{D}} \;:=\; {}^{(c)}\nabla - i\,{}^{(c)}\nabla.
$$

These complex notations allow us to rewrite the null structure equations as follows.

Proposition 3.4.16. *We have*

$$
\begin{aligned}
{}^{(c)}\nabla_3 tr\underline{X} + \frac{1}{2}(tr\underline{X})^2 &= {}^{(c)}\mathcal{D}\cdot\overline{\underline{\Xi}} + \underline{\Xi}\cdot\overline{\underline{H}} + \overline{\underline{\Xi}}\cdot H - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{\underline{\widehat{X}}},\\
{}^{(c)}\nabla_3\underline{\widehat{X}} + \Re(tr\underline{X})\underline{\widehat{X}} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \frac{1}{2}\underline{\Xi}\widehat{\otimes}(H+\underline{H}) - \underline{A},
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_3 trX + \frac{1}{2}tr\underline{X}\,trX &= {}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P + \underline{\Xi}\cdot\overline{\Xi} - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{\widehat{X}},\\
{}^{(c)}\nabla_3\widehat{X} + \frac{1}{2}tr\underline{X}\,\widehat{X} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}H + \frac{1}{2}H\widehat{\otimes}H - \frac{1}{2}\overline{trX}\,\underline{\widehat{X}} + \frac{1}{4}\underline{\Xi}\widehat{\otimes}\Xi,
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_4 tr\underline{X} + \frac{1}{2}trX\,tr\underline{X} &= {}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}} + 2\overline{P} + \Xi\cdot\overline{\underline{\Xi}} - \frac{1}{2}\widehat{X}\cdot\overline{\underline{\widehat{X}}},\\
{}^{(c)}\nabla_4\underline{\widehat{X}} + \frac{1}{2}trX\,\underline{\widehat{X}} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{H} + \frac{1}{2}\underline{H}\widehat{\otimes}\underline{H} - \frac{1}{2}\overline{tr\underline{X}}\,\widehat{X} + \frac{1}{4}\Xi\widehat{\otimes}\underline{\Xi},
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_4 trX + \frac{1}{2}(trX)^2 &= {}^{(c)}\mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot H - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},\\
{}^{(c)}\nabla_4\widehat{X} + \Re(trX)\widehat{X} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \frac{1}{2}\Xi\widehat{\otimes}(\underline{H}+H) - A,
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_3 \underline{H} - {}^{(c)}\nabla_4 \underline{\Xi} &= -\frac{1}{2}\overline{tr\underline{X}}(\underline{H} - H) - \frac{1}{2}\widehat{\underline{X}} \cdot (\overline{\underline{H}} - \overline{H}) + \underline{B},\\
{}^{(c)}\nabla_4 H - {}^{(c)}\nabla_3 \Xi &= -\frac{1}{2}\overline{trX}(H - \underline{H}) - \frac{1}{2}\widehat{X} \cdot (\overline{H} - \overline{\underline{H}}) - B.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot \widehat{X} &= \frac{1}{2}{}^{(c)}\mathcal{D}\overline{trX} - i\Im(trX)H - i\Im(tr\underline{X})\Xi - B,\\
\frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot \widehat{\underline{X}} &= \frac{1}{2}{}^{(c)}\mathcal{D}\overline{tr\underline{X}} - i\Im(tr\underline{X})\,\underline{H} - i\Im(trX)\underline{\Xi} + \underline{B}.
\end{aligned}
$$

The complex notations allow us to rewrite the Bianchi identities as follows.

Proposition 3.4.17. *We have*

$$
\begin{aligned}
{}^{(c)}\nabla_3 A - \frac{1}{2}{}^{(c)}\mathcal{D}\widehat{\otimes} B &= -\frac{1}{2}tr\underline{X}A + 2H\widehat{\otimes}B - 3\overline{P}\widehat{X},\\
{}^{(c)}\nabla_4 B - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot A &= -2\overline{trX}B + \frac{1}{2}A \cdot \overline{\underline{H}} + 3\overline{P}\,\Xi,\\
{}^{(c)}\nabla_3 B - {}^{(c)}\mathcal{D}\overline{P} &= -tr\underline{X}B + \overline{\underline{B}} \cdot \widehat{X} + 3\overline{P}H + \frac{1}{2}A \cdot \overline{\underline{\Xi}},\\
{}^{(c)}\nabla_4 P - \frac{1}{2}{}^{(c)}\mathcal{D} \cdot \overline{B} &= -\frac{3}{2}trXP + \underline{H} \cdot \overline{B} - \overline{\Xi} \cdot \underline{B} - \frac{1}{4}\widehat{\underline{X}} \cdot \overline{A},\\
{}^{(c)}\nabla_3 P + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot \underline{B} &= -\frac{3}{2}\overline{tr\underline{X}}P - \overline{H} \cdot \underline{B} + \underline{\Xi} \cdot \overline{B} - \frac{1}{4}\overline{\widehat{X}} \cdot \underline{A},\\
{}^{(c)}\nabla_4 \underline{B} + {}^{(c)}\mathcal{D}P &= -trX\underline{B} + \overline{B} \cdot \widehat{\underline{X}} - 3P\,\underline{H} - \frac{1}{2}\underline{A} \cdot \overline{\Xi},\\
{}^{(c)}\nabla_3 \underline{B} + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot \mathcal{A} &= -2\overline{tr\underline{X}}\,\underline{B} - \frac{1}{2}\underline{A} \cdot \overline{H} - 3P\,\underline{\Xi},\\
{}^{(c)}\nabla_4 \underline{A} + \frac{1}{2}{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} &= -\frac{1}{2}trX\underline{A} - 2\,\underline{H}\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}}.
\end{aligned}
$$

Remark 3.4.18. *The complex Bianchi equations can also be derived directly from the equation* $\mathbf{D}^{\alpha}\mathcal{R}_{\alpha\beta\gamma\delta} = 0$*, see Remark 3.4.2.*

3.4.4 Connection to the Newman-Penrose formalism

In the Newman-Penrose NP formalism, one chooses a specific orthonormal basis of horizontal vectors (e_1, e_2) and defines all connection coefficients relative to the complexified frame $(n, l, m, \overline{m})$ where $n = \frac{1}{2}e_3$, $l = e_4$, $m = e_1 + ie_2$, $\overline{m} = e_1 - ie_2$. Thus, all quantities of interest are complex scalars instead of our horizontal tensors such as $\mathfrak{s}_1, \mathfrak{s}_2$. The NP formalism works well for deriving the basic equations, but has the disadvantage of substantially increasing the number of variables. Moreover, the calculations become far more cumbersome when deriving equations involving higher derivatives of the main quantities, in perturbations of Kerr. Another advantage of the formalism used here is that all important equations look similar to the ones in [Ch-Kl]. We refer to [NP] for the original form of the NP formalism.

The formalism used here is also related to the so-called Geroch-Held-Penrose formalism GHP formalism, which also introduced derivatives with boost weights, which are the scalar equivalent of the conformal derivatives used here, see Lemma 3.1.13. Nevertheless, the GHP formalism still involves complex scalars instead of horizontal tensors. We refer to [GHP] for the original form of the GHP formalism.

3.5 The wave operator using complex derivatives-See section 4.7.3 in [?]

We now express the laplacian in terms of complex derivatives. We summarize the result in the following.

Lemma 3.5.1. *We have for $\psi \in \mathfrak{s}_2(\mathbb{C})$,*

$$\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) = 4\triangle_2\psi - 2i\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\psi - 8\,{}^{(h)}K\psi \tag{3.5.1}$$

where ${}^{(h)}K$ is defined in (3.1.9)*. In particular, in perturbations of Kerr we have*

$$\begin{aligned}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) &= 4\triangle_2\psi - 2i\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\psi \\ &\quad +2\left(tr\,\chi tr\,\underline{\chi} + {}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + 4\rho\right)\psi + (\Gamma_g\cdot\Gamma_b)\cdot\psi, \end{aligned}\tag{3.5.2}$$

$$\begin{aligned}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) &= 4\triangle_2\psi - 2i\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\psi \\ &\quad +2\left(\frac{1}{2}trX\overline{tr\underline{X}} + \frac{1}{2}tr\underline{X}\overline{trX} + 2P + 2\overline{P}\right)\psi + (\Gamma_g\cdot\Gamma_b)\cdot\psi. \end{aligned}\tag{3.5.3}$$

Proof. See section 4.7.3 in [**?**]. □

We rewrite the above using the conformal derivatives introduced in Lemma 3.1.13.

Lemma 3.5.2. *We have for $\psi \in \mathfrak{s}_2(\mathbb{C})$ s-conformally invariant,*

$$\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\psi) = {}& 4\,{}^{(c)}\triangle_2\psi - 2i\left({}^{(a)}tr\chi\,{}^{(c)}\nabla_3 + {}^{(a)}tr\underline{\chi}\,{}^{(c)}\nabla_4\right)\psi \\
&+ 2\Big[\left(tr\,\chi tr\underline{\chi} + {}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + 4\rho\right) \\
&- is\left(\frac{1}{2}\left(tr\,\chi\,{}^{(a)}tr\underline{\chi} - tr\,\underline{\chi}\,{}^{(a)}tr\chi\right) + 2\,{}^*\rho\right)\Big]\psi \\
&+ (\Gamma_g\cdot\Gamma_b)\cdot\psi
\end{aligned} \tag{3.5.4}$$

where ${}^{(c)}\triangle_2 := \gamma^{ab}\,{}^{(c)}\nabla_a\,{}^{(c)}\nabla_b$ is the conformal Laplacian operator for horizontal 2-tensors.

Proof. See section 4.7.3 in [?]. □

By putting together the canonical expression for the wave operator given in Lemma 3.2.10 and the expression for the Laplacian given in Lemma 3.5.1, we obtain the following.

Corollary 3.5.3. *We have, for $\psi \in \mathfrak{s}_2(\mathbb{C})$,*

$$\begin{aligned}
\dot{\square}_2\psi = {}& -\nabla_4\nabla_3\psi + \frac{1}{4}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) + \left(2\omega - \frac{1}{2}trX\right)\nabla_3\psi - \frac{1}{2}tr\underline{X}\nabla_4\psi + 2\underline{\eta}\cdot\nabla\psi \\
&+ \left(-\frac{1}{2}tr\,\chi tr\underline{\chi} - \frac{1}{2}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} - 2\rho\right)\psi + 2i\left({}^*\rho - \eta\wedge\underline{\eta}\right)\psi + (\Gamma_b\cdot\Gamma_g)\cdot\psi,
\end{aligned} \tag{3.5.5}$$

which can be rewritten as

$$\begin{aligned}
\dot{\square}_2\psi = {}& -\nabla_4\nabla_3\psi + \frac{1}{4}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) + \left(2\omega - \frac{1}{2}trX\right)\nabla_3\psi - \frac{1}{2}tr\underline{X}\nabla_4\psi + 2\underline{\eta}\cdot\nabla\psi \\
&+ \left(-\frac{1}{4}trX\overline{tr\underline{X}} - \frac{1}{4}tr\underline{X}\overline{trX} - 2\overline{P}\right)\psi - 2i\left(\eta\wedge\underline{\eta}\right)\psi + (\Gamma_b\cdot\Gamma_g)\cdot\psi.
\end{aligned} \tag{3.5.6}$$

To do from the Bianchi equations using the hodge operators

Chapter 4

Derivation of the main equations

4.1 Teukolsky equation for A

It is known that the curvature components A and $\underline{A}$ satisfy wave equations which decouple from all other components at the linear level, the celebrated Teukolsky equations. In this section we derive, using our formalism, the corresponding Teukolsky equation for A while keeping track of the error terms generated by the perturbation from Kerr expressed in terms of (Γ_b, Γ_g).

4.1.1 The Teukolsky equation for A

Proposition 4.1.1. *The complex tensor $A \in \mathfrak{s}_2(\mathbb{C})$ satisfies the following equation:*

$$\mathcal{L}(A) \quad = \quad Err[\mathcal{L}(A)] \tag{4.1.1}$$

where

$$\begin{aligned}\mathcal{L}(A) = -\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + \frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) + \left(-\frac{1}{2}trX - 2\overline{trX}\right)\,{}^{(c)}\nabla_3 A\\ -\frac{1}{2}tr\underline{X}\,{}^{(c)}\nabla_4 A + \big(4H + \underline{H} + \overline{\underline{H}}\big)\cdot\,{}^{(c)}\nabla A + \big(-\overline{trX}\,tr\underline{X} + 2\overline{P}\big)\,A + H\widehat{\otimes}(\overline{\underline{H}}\cdot A),\end{aligned} \tag{4.1.2}$$

with error term expressed schematically

$$Err[\mathcal{L}(A)] \quad = \quad r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big) + {}^{(c)}\nabla_3\Xi\cdot B + \Gamma_b\cdot\Gamma_g\cdot A. \tag{4.1.3}$$

Proof. See proof of proposition 5.1.1. ◻

4.2 Generalized Regge-Wheeler equation for $\mathfrak{q}$

In this section we derive the generalized Regge-Wheeler-type equation.

4.2.1 The invariant quantities Q and $\mathfrak{q}$

We start with the following lemma.

Lemma 4.2.1. *Let C_1 and C_2 be scalar functions. The expression*

$$Q(A) \quad = \quad {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A + C_1\, {}^{(c)}\nabla_3 A + C_2 A \in \mathfrak{s}_2(\mathbb{C}) \tag{4.2.1}$$

is 0-conformally invariant provided C_1 is -1-conformally invariant and C_2 is -2-conformally invariant.

Proof. Direct verification in view of the definition of the conformal derivative ${}^{(c)}\nabla_3$. □

Definition 4.2.2. *Given a fixed null pair (e_3, e_4) and scalar functions r and θ as in Section 4.1 of [GKS-2022] we define our main quantity $\mathfrak{q} \in \mathfrak{s}_2(\mathbb{C})$ as*

$$\mathfrak{q} \quad = \quad q\overline{q}^3 Q(A) = q\overline{q}^3 \left({}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A + C_1\, {}^{(c)}\nabla_3 A + C_2 A \right) \tag{4.2.2}$$

where $q = r + ia\cos\theta$, and the scalar function C_1, C_2 are given by

$$\begin{aligned} C_1 &= 2tr\underline{\chi} - 2\frac{{}^{(a)}tr\underline{\chi}^2}{tr\underline{\chi}} - 4i\, {}^{(a)}tr\underline{\chi}, \\ C_2 &= \frac{1}{2}tr\underline{\chi}^2 - 4\, {}^{(a)}tr\underline{\chi}^2 + \frac{3}{2}\frac{{}^{(a)}tr\underline{\chi}^4}{tr\underline{\chi}^2} + i\left(-2tr\underline{\chi}\, {}^{(a)}tr\underline{\chi} + 4\frac{{}^{(a)}tr\underline{\chi}^3}{tr\underline{\chi}}\right). \end{aligned} \tag{4.2.3}$$

Remark 4.2.3. *Note that $\mathfrak{q}$ is independent of the particular normalization. More precisely if $e_3' = \lambda^{-1}e_3, e_4' = \lambda e_4$ and $A' = \lambda^2 A$ then $\mathfrak{q}' = \mathfrak{q}$.*

4.2.2 The derivation of the gRW equation for $\mathfrak{q}$

We now state the first main result of Part I concerning the wave equation satisfied by $\mathfrak{q}$.

Theorem 4.2.4. *The invariant symmetric traceless 2-tensor $\mathfrak{q} \in \mathfrak{s}_2(\mathbb{C})$ in Definition 4.2.2 satisfies the equation*

$$\dot{\square}_2\mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V\mathfrak{q} \quad = \quad L_{\mathfrak{q}}[A] + Err[\dot{\square}_2\mathfrak{q}], \tag{4.2.4}$$

where:

- $\mathbf{T}$ *is the vectorfield given by Definition 5.2.8, see also Remark* **??** *below.*

- *The potential V is the* ***real*** *scalar function given by*

$$V \quad = \quad \frac{4}{|q|^2}\frac{r^2-2mr+2a^2}{r^2} - \frac{4a^2\cos^2\theta}{|q|^6}(r^2+6mr+a^2\cos^2\theta), \tag{4.2.5}$$

  which for $a = 0$ coincides with the potential of the Regge-Wheeler equation in Schwarzschild, i.e. $V = -\mathrm{tr}\,\chi\mathrm{tr}\underline{\chi} + O(\frac{|a|}{r^4})$, see also Remark **??** *below.*

- $L_{\mathfrak{q}}[A]$ *is a linear second order operator in A, given in the outgoing frame by*

$$\begin{aligned} L_{\mathfrak{q}}[A] \quad &= \quad q\overline{q}^3\Bigg(-\frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\nabla_3 A - \frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\nabla_3 A \\ &\qquad + W_4\ \nabla_4 A + W_3\nabla_3 A + W\cdot\nabla A + W_0 A\Bigg), \end{aligned}$$

  where W_4, W_3, W_0 are complex functions of (r,θ) and W is the product of a complex function of (r,θ) with ${}^*\mathfrak{R}(\mathfrak{J})$, with the following fall-off in r

$$q\overline{q}^3 W_4 = q\overline{q}^3 W_3 = q\overline{q}^3 W = O(a), \qquad q\overline{q}^3 W_0 = O\left(\frac{a}{r}\right).$$

- $Err[\dot{\square}_2\mathfrak{q}]$ *is the nonlinear correction term, which under the additional condition*

$$\check{H} \in \Gamma_g$$

  is given schematically by the expression

$$\begin{aligned} Err[\dot{\square}_2\mathfrak{q}] &= r^2\mathfrak{d}^{\leq 3}(\Gamma_g\cdot(A,B)) + \nabla_3(r^2\mathfrak{d}^{\leq 2}(\Gamma_b\cdot(A,B))) \\ &\quad + \mathfrak{d}^{\leq 1}(\Gamma_g\cdot\mathfrak{q}) + r^3\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Gamma_g\cdot\Gamma_g\big). \end{aligned} \tag{4.2.6}$$

4.2.3 The real part of the gRW equation

Since $\mathfrak{q} \in \mathfrak{s}_2(\mathbb{C})$ is a complex anti-self dual tensor, we can decompose it as

$$\mathfrak{q} = \psi + i\, {}^*\psi \tag{4.2.7}$$

for some $\psi = \Re(\mathfrak{q}) \in \mathfrak{s}_2(\mathbb{R})$. Taking the real part of (4.2.4), since V is real, we then obtain an equation for ψ, which is given by

$$\dot{\square}_2\psi + \frac{4a\cos\theta}{|q|^2}\, {}^*\nabla_{\mathbf{T}}\psi - V\psi \;\; = \;\; \Re(L_{\mathfrak{q}}[A]) + \Re(\mathrm{Err}[\dot{\square}_2\mathfrak{q}]).$$

We summarize in the following.

Proposition 4.2.5. *The tensor $\psi \in \mathfrak{s}_2(\mathbb{R})$ satisfies*

$$\dot{\square}_2\psi - V_0\psi = -\frac{4a\cos\theta}{|q|^2}\, {}^*\nabla_T\psi + N, \qquad V_0 = \frac{4\Delta}{(r^2+a^2)|q|^2}, \tag{4.2.8}$$

with the right hand side N being given by

$$\begin{aligned} N &:= \big(V - V_0\big)\psi + \Re(L_{\mathfrak{q}}[A]) + \Re(Err[\dot{\square}_2\mathfrak{q}]) \\ &= N_0 + N_L + N_{Err} \end{aligned} \tag{4.2.9}$$

where:

- *N_0 denotes the zero-th order term in ψ, i.e.*

$$N_0 := \left(V - \frac{4\Delta}{(r^2+a^2)|q|^2}\right)\psi = O\left(\frac{a}{r^4}\right)\psi. \tag{4.2.10}$$

- *N_L denotes the lower order terms in ψ, i.e.*

$$\begin{aligned} N_L := \Re\Bigg(q\overline{q}^3\Bigg[&-\frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\nabla_3 A - \frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\nabla_3 A \\ &+ W_4\nabla_4 A + W_3\nabla_3 A + W\cdot\nabla A + W_0 A\Bigg]\Bigg) \end{aligned} \tag{4.2.11}$$

where W_4, W_3, W_0 are complex functions of (r,θ), and W is the product of a complex function of (r,θ) with ${}^*\Re(\mathfrak{J})$, having the following fall-off in r

$$q\overline{q}^3 W_4 = q\overline{q}^3 W_3 = q\overline{q}^3 W = O(a), \qquad q\overline{q}^3 W_0 = O\left(\frac{a}{r}\right).$$

- $N_{Err}[\psi]$ *denotes the error terms, i.e.*

$$N_{Err}[\psi] := \Re(Err[\dot{\square}_2\mathfrak{q}]) \tag{4.2.12}$$

which are schematically given by

$$\begin{aligned} N_{Err}[\psi] &= r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot(\alpha,\beta)) + \nabla_3(r^2\mathfrak{d}^{\leq 2}(\Gamma_b\cdot(\alpha,\beta))) \\ &+\mathfrak{d}^{\leq 1}(\Gamma_g\psi) + r^3\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Gamma_g\cdot\Gamma_g\big). \end{aligned}$$

Also, recall that ψ and A are related by the differential relation:

$$\psi = \Re\Big(q\overline{q}^3\big({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + C_1\,{}^{(c)}\nabla_3 A + C_2 A\big)\Big),$$

with

$$\begin{aligned} C_1 &= 2tr\underline{\chi} - 2\frac{{}^{(a)}tr\underline{\chi}^2}{tr\underline{\chi}} - 4i\,{}^{(a)}tr\underline{\chi}, \\ C_2 &= \frac{1}{2}tr\underline{\chi}^2 - 4\,{}^{(a)}tr\underline{\chi}^2 + \frac{3}{2}\frac{{}^{(a)}tr\underline{\chi}^4}{tr\underline{\chi}^2} + i\left(-2tr\underline{\chi}\,{}^{(a)}tr\underline{\chi} + 4\frac{{}^{(a)}tr\underline{\chi}^3}{tr\underline{\chi}}\right). \end{aligned}$$

4.3 Generalized Regge-Wheeler equation for $\underline{\mathfrak{q}}$

In this section, we derive the generalized Regge-Wheeler equation for $\underline{\mathfrak{q}}$.

4.3.1 The Teukolsky equation for $\underline{A}$

Here we derive the Teukolsky equation for $\underline{A}$. In order to capture correctly the non linear terms in the equation, we express the Bianchi identity for $\underline{A}$ in terms of

$$\underline{A}_4 = {}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A},$$

which has an improved decay rate as compared to ${}^{(c)}\nabla_4\underline{A}$. In the derivation of the Teukolsky equation below, we express explicitly the error terms which decay less than $r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)$.

Proposition 4.3.1. *We have*

$$\left({}^{(c)}\nabla_3 + 2\overline{tr\underline{X}} + \frac{1}{2}tr\underline{X}\right)\underline{A}_4 = \frac{1}{4}\big({}^{(c)}\mathcal{D} + H + 4\,\underline{H}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big) + 3P\underline{A} + Err_{TE} \tag{4.3.1}$$

where Err_{TE} is given schematically by

$$Err_{TE} = trX\underline{\Xi}\widehat{\otimes}\underline{B} + (\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}} + (\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B} + (\Gamma_b\cdot\Gamma_b)\cdot(A,B) + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).$$

Proof. See Proposition 5.3.1 in [GKS-2022]. □

4.3.2 The invariant quantities $\underline{Q}$ and $\underline{\mathfrak{q}}$

In this section we consider the analog $\underline{\mathfrak{q}}$ of $\mathfrak{q}$ and derive its corresponding gRW equation.

Definition 4.3.2. *Given a fixed null pair (e_3, e_4) and scalar functions r and θ as in Section **??**, we define our second main quantity $\underline{\mathfrak{q}} \in \mathfrak{s}_2(\mathbb{C})$ as*

$$\underline{\mathfrak{q}} = \overline{q}q^3\underline{Q}(\underline{A}) = \overline{q}q^3\left({}^{(c)}\nabla_4\,{}^{(c)}\nabla_4\underline{A} + \underline{C}_1\,{}^{(c)}\nabla_4\underline{A} + \underline{C}_2\underline{A}\right), \tag{4.3.2}$$

with complex scalars

$$\begin{aligned}\underline{C}_1 &= 2tr\,\chi - 2\frac{{}^{(a)}tr\chi^2}{tr\,\chi} - 4i\,{}^{(a)}tr\chi,\\ \underline{C}_2 &= \frac{1}{2}tr\,\chi^2 - 4\,{}^{(a)}tr\chi^2 + \frac{3}{2}\frac{{}^{(a)}tr\chi^4}{tr\,\chi^2} + i\left(-2tr\,\chi\,{}^{(a)}tr\chi + 4\frac{{}^{(a)}tr\chi^3}{tr\,\chi}\right).\end{aligned} \tag{4.3.3}$$

4.3.3 The derivation of the gRW equation for $\underline{\mathfrak{q}}$

We state below the gRW equation satisfied by $\underline{\mathfrak{q}}$.

Theorem 4.3.3. *The invariant symmetric traceless 2-tensor $\underline{\mathfrak{q}} \in \mathfrak{s}_2(\mathbb{C})$ in Definition 4.3.2 satisfies the equation*

$$\dot{\square}_2\underline{\mathfrak{q}} + i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\underline{\mathfrak{q}} - V\underline{\mathfrak{q}} = L_{\underline{\mathfrak{q}}}[\underline{A}] + Err[\dot{\square}_2\underline{\mathfrak{q}}] \tag{4.3.4}$$

where:

- *The potential V is the **real** scalar function given by*

$$V = \frac{4}{|q|^2}\frac{r^2-2mr+2a^2}{r^2} - \frac{4a^2\cos^2\theta}{|q|^6}(r^2+6mr+a^2\cos^2\theta), \tag{4.3.5}$$

which for $a=0$ coincides with the potential of the Regge-Wheeler equation in Schwarzschild, i.e. $V=-tr\,\chi tr\underline{\chi}+O(\frac{|a|}{r^4})$.

- *$L_{\underline{\mathfrak{q}}}[\underline{A}]$ is a linear second order operator in $\underline{A}$ given in the ingoing frame by*

$$L_{\underline{\mathfrak{q}}}[\underline{A}] = q\overline{q}^3\left(\frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\underline{A}_4 + \frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\underline{A}_4 + \underline{W}_4\underline{A}_4 + \underline{W}_3\nabla_3\underline{A} + \underline{W}\cdot\nabla\underline{A} + \underline{W}_0\underline{A}\right),$$

where $\underline{W}_4$, $\underline{W}_3$, $\underline{W}_0$ are complex functions of (r,θ) and $\underline{W}$ is the product of a complex function of (r,θ) with ${}^*\mathfrak{R}(\mathfrak{J})$, with the following fall-off in r

$$q\overline{q}^3\underline{W}_4, q\overline{q}^3\underline{W} = O\left(\frac{a^2}{r}\right), \qquad q\overline{q}^3\underline{W}_3, q\overline{q}^3\underline{W}_0 = O\left(\frac{a^2}{r^2}\right).$$

- *$Err[\dot{\square}_2\underline{\mathfrak{q}}]$ is the nonlinear correction term, which under the additional conditions*[1]

$$\Xi = 0, \qquad \widecheck{\underline{H}} = 0, \quad for \quad r \geq r_0,$$

is given schematically by the expression

$$Err[\square_2\underline{\mathfrak{q}}] = r^2\mathfrak{d}^{\leq 2}(\Gamma_b\cdot(A,B)) + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b).$$

4.4 Teukolsky-Starobinski identity

We state here, in the context of perturbations of Kerr, one of the Teukolsky-Starobinski identities, which relate the complex curvature components A and $\underline{A}$ through fourth-order differential operators.

Proposition 4.4.1. *Assume that $\Xi = 0$ in $r \leq r_0$. The complex tensors A, $\underline{A} \in \mathfrak{s}_2(\mathbb{C})$ satisfy the following relation in the region $r \leq r_0$*

$$\left({}^{(c)}\nabla_4 + 2trX\right)^4\underline{A} = r^{-4}\mathfrak{d}^{\leq 4}A + \mathfrak{d}^{\leq 3}\left(\Gamma_b\cdot\Gamma_g\right). \tag{4.4.1}$$

[1]In fact, it suffices to assume that $\Xi \in r^{-2}\Gamma_g$ and $\widecheck{\underline{H}} \in r^{-1}\Gamma_g$. These additional conditions make the structure of Err$[\dot{\square}_2\underline{\mathfrak{q}}]$ in (4.3.6) possible. This structure is essential in the control of the nonlinear terms.

Proof. See Proposition 5. 4.1 in[GKS-2022].. □

Remark 4.4.2. *Proposition 4.4.1 is stated without proof in Chapter 7 of [K-S:Kerr]. Both the assumption $\Xi = 0$ and the restriction to $r \le r_0$ are unnecessary and assumed only for convenience, as they hold when applying Proposition 4.4.1 in Chapter 7 of [K-S:Kerr]. In particular, the restriction to $r \le r_0$ allows us to avoid having to track the precise powers of r in the nonlinear terms.*

Remark 4.4.3. *Choosing a normalization such that $\omega \in \Gamma_g$, and hence $trX = \frac{2}{q} + \Gamma_g$, we infer from Proposition 4.4.1, for $\Xi = 0$ in the region $r \le r_0$,*

$$\frac{1}{q^7}\nabla_4\left(q^2\nabla_4\left(q^2\nabla_4\left(q^2\nabla_4(q\underline{A})\right)\right)\right) \quad = \quad r^{-4}\mathfrak{d}^{\le 4}A + \mathfrak{d}^{\le 3}\big(\Gamma_b \cdot \Gamma_g\big).$$

4.5 The wave equation for P

Here we derive the wave equation satisfied by the curvature component P.

Lemma 4.5.1. *The curvature component P satisfies the following scalar wave equation:*

$$\begin{aligned}\square_{\mathbf{g}}P &= trX\nabla_3 P + \overline{tr\underline{X}}\,\nabla_4 P - \overline{H}\cdot\mathcal{D}P - \underline{H}\cdot\overline{\mathcal{D}}P\\ &\quad + \frac{3}{2}\Big[\overline{tr\underline{X}}trX + 2P - 2\,\underline{H}\cdot\overline{H}\Big]P + Err[\square_{\mathbf{g}}P],\end{aligned} \tag{4.5.1}$$

with error terms given by

$$\begin{aligned}Err[\square_{\mathbf{g}}P] &= -\,{}^{(c)}\nabla_3(\overline{\Xi}\cdot\underline{B}) - \frac{1}{4}\,{}^{(c)}\nabla_3(\widehat{\underline{X}}\cdot\overline{A}) + \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\left(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\right)\\ &\quad + \underline{H}\cdot\left(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\right) + \frac{1}{2}\Big(- \overline{\underline{B}}\cdot B + \underline{\Xi}\cdot\nabla_4\overline{B} - \overline{\widehat{\underline{X}}}\cdot{}^{(c)}\mathcal{D}\overline{B} - \overline{H}\cdot\overline{\widehat{\underline{X}}}\cdot B\Big)\\ &\quad - \frac{1}{2}(2\overline{tr\underline{X}} + tr\underline{X})\left(\overline{\Xi}\cdot\underline{B} + \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A}\right) + \frac{1}{2}H\cdot\left(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\right)\\ &\quad + \left(-\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{\underline{X}} - i\Im(trX)\underline{\Xi} + \underline{B} + {}^{(c)}\nabla_4\underline{\Xi} - \frac{1}{2}\widehat{\underline{X}}\cdot(\overline{\underline{H}} - \overline{H}) + \underline{B}\right)\cdot\overline{B}\\ &\quad - \frac{3}{2}\big(\underline{\Xi}\cdot\overline{\Xi} - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}}\big)P.\end{aligned} \tag{4.5.2}$$

Proof. See Appendix **??**. □

Remark 4.5.2. *From the expression in (4.5.2) and using the null structure equation for ${}^{(c)}\nabla_3\widehat{\underline{X}}$, observe that the error terms $Err[\square_{\mathbf{g}}P]$ can be schematically written as*

$$Err[\square_{\mathbf{g}}P] \quad = \quad r^{-1}\mathfrak{d}^{\le 1}(\Gamma_g\cdot\underline{B}) + \nabla_3(\Xi\cdot\underline{B}) + r^{-1}\mathfrak{d}^{\le 1}(\Gamma_b\cdot(A,B)) + r^{-3}(\Gamma_g\cdot\Gamma_b) - \underline{A}\cdot\overline{A}.$$

4.6 Derivation of the main equations in linear perturbations of Schwarzschild

We make use of the results of section 3.3.1 to derive the Teukolsky and RW equations in linear perturbations of Schwarzschild, i.e. we assume that ${}^{(a)}\mathrm{tr}\chi = {}^{(a)}\mathrm{tr}\underline{\chi} = 0$, $\widehat{\chi}, \widehat{\underline{\chi}}, \eta, \underline{\eta}, \zeta, \xi, \underline{\xi}, \alpha, \underline{\alpha}, \beta, \underline{\beta}$ are linear quantities and that we neglect all quadratic and higher order expressions involving them.

4.7 Derivation of the Teukolsky equations

Proposition 4.7.1 (Bianchi-Schw#). *In linear perturbations of Schwarzschild we have*[2]

$$
\begin{aligned}
{}^{(c)}\nabla_3^{\#}\alpha &= {}^{(c)}\nabla\widehat{\otimes}\beta - 3\rho\widehat{\chi},\\
{}^{(c)}\nabla_4^{\#}\beta &= {}^{(c)}div\,\alpha + 3\xi\rho,\\
{}^{(c)}\nabla_3^{\#}\beta &= -{}^{(c)}div\,\varrho + 3\rho\eta,
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_4^{\#}\rho &= {}^{(c)}div\,\beta,\\
{}^{(c)}\nabla_3^{\#}\rho &= -{}^{(c)}div\,\underline{\beta},\\
{}^{(c)}\nabla_4^{\#}\ {}^*\rho &= -{}^{(c)}curl\,\beta - \frac{3}{2}(tr\,\chi\ {}^*\rho - {}^{(a)}tr\chi\rho)\\
{}^{(c)}\nabla_3^{\#}\ {}^*\rho &= -{}^{(c)}curl\,\underline{\beta} - \frac{3}{2}(tr\underline{\chi}\ {}^*\rho + {}^{(a)}tr\underline{\chi}\rho)
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_4^{\#}\underline{\beta} &= -{}^{(c)}div\,\check{\varrho} - 3\rho\underline{\eta}\\
{}^{(c)}\nabla_3^{\#}\underline{\beta} &= -{}^{(c)}div\,\underline{\alpha} - 3\underline{\xi}\rho\\
{}^{(c)}\nabla_4^{\#}\underline{\alpha} &= -{}^{(c)}\nabla\widehat{\otimes}\underline{\beta} - 3\rho\widehat{\underline{\chi}}.
\end{aligned}
$$

Proof. We only need to check the equations for ρ, ${}^*\rho$. Neglecting quadratic terms the

[2]Here $div\,\varrho = -(\nabla\rho + {}^*\nabla\ {}^*\rho)$, $\quad div\,\check{\varrho} = -(\nabla\rho - {}^*\nabla\ {}^*\rho)$.

Bianchi equations for ρ, ${}^*\rho$ take the form

$$\begin{aligned}
{}^{(c)}\nabla_4\rho - {}^{(c)}\mathrm{d}iv\,\beta &= -\frac{3}{2}\mathrm{tr}\,\chi\rho\\
{}^{(c)}\nabla_4\ {}^*\rho + {}^{(c)}curl\,\beta &= -\frac{3}{2}(\mathrm{tr}\,\chi\ {}^*\rho - {}^{(a)}\mathrm{tr}\chi\rho)\\
{}^{(c)}\nabla_3\rho + {}^{(c)}\mathrm{d}iv\,\underline{\beta} &= -\frac{3}{2}\mathrm{tr}\,\underline{\chi}\rho\\
{}^{(c)}\nabla_3\ {}^*\rho + {}^{(c)}curl\,\underline{\beta} &= -\frac{3}{2}(\mathrm{tr}\,\underline{\chi}\ {}^*\rho + {}^{(a)}\mathrm{tr}\underline{\chi}\rho)
\end{aligned}$$

$\square$

Remark 4.7.2. *In view of our definition for ${}^{(c)}\nabla^\#$ applied to Ricci coefficients*

$$\begin{aligned}
{}^{(c)}\nabla_3^\#\widehat{\chi} &= {}^{(c)}\nabla_3\widehat{\chi} + \frac{1}{2}(1-1+1)tr\,\underline{\chi}\widehat{\chi} = {}^{(c)}\nabla_3\widehat{\chi} + \frac{1}{2}tr\,\underline{\chi}\chi\\
{}^{(c)}\nabla_3^\#tr\,\chi &= {}^{(c)}\nabla_3 tr\,\chi + \frac{1}{2}tr\,\underline{\chi}tr\,\chi\\
{}^{(c)}\nabla_4^\#\widehat{\chi} &= {}^{(c)}\nabla_4\widehat{\chi} + \frac{1}{2}(1+1+1)tr\,\underline{\chi}\widehat{\chi} = {}^{(c)}\nabla_4\widehat{\chi} + \frac{3}{2}tr\,\chi\widehat{\chi}\\
{}^{(c)}\nabla_4^\#tr\,\chi &= {}^{(c)}\nabla_4 tr\,\chi + \frac{3}{2}tr\,\chi^2
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_3^\#\widehat{\underline{\chi}} &= {}^{(c)}\nabla_3\widehat{\underline{\chi}} + \frac{1}{2}(1+1+1)tr\,\underline{\chi}\,\widehat{\underline{\chi}} = {}^{(c)}\nabla_3\widehat{\underline{\chi}} + \frac{3}{2}tr\,\underline{\chi}\,\widehat{\underline{\chi}}\\
{}^{(c)}\nabla_3^\#tr\,\underline{\chi} &= {}^{(c)}\nabla_3 tr\,\underline{\chi} + \frac{3}{2}tr\,\underline{\chi}^2\\
{}^{(c)}\nabla_4^\#\widehat{\underline{\chi}} &= {}^{(c)}\nabla_4\widehat{\underline{\chi}} + \frac{1}{2}(1-1+1)tr\,\underline{\chi\widehat{\chi}} = {}^{(c)}\nabla_3\widehat{\underline{\chi}} + \frac{1}{2}tr\,\chi\widehat{\underline{\chi}}\\
{}^{(c)}\nabla_4^\#tr\,\underline{\chi} &= {}^{(c)}\nabla_4 tr\,\underline{\chi} + \frac{1}{2}tr\,\chi tr\,\underline{\chi}
\end{aligned}$$

Proposition 3.3.1 becomes

Proposition 4.7.3 (Ricci-Schwarzschild#)**.** *We have*

$$\begin{aligned}
{}^{(c)}\nabla_3^\#tr\,\underline{\chi} &= tr\,\underline{\chi}^2 + 2\,{}^{(c)}div\,\underline{\xi},\\
{}^{(c)}\nabla_3^\#\widehat{\underline{\chi}} &= \frac{1}{2}tr\,\underline{\chi}\,\widehat{\underline{\chi}} + {}^{(c)}\nabla\widehat{\otimes}\underline{\xi} - \underline{\alpha},
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_3^\#tr\,\chi &= 2\,{}^{(c)}div\,\eta + 2\rho,\\
{}^{(c)}\nabla_3^\#\widehat{\chi} &= -\frac{1}{2}tr\,\chi\widehat{\underline{\chi}} + {}^{(c)}\nabla\widehat{\otimes}\eta,
\end{aligned}$$

$$
\begin{aligned}
{}^{(c)}\nabla_4^\# tr\,\underline{\chi} &= 2\,{}^{(c)}div\,\underline{\eta} + 2\rho,\\
{}^{(c)}\nabla_4^\# \widehat{\underline{\chi}} &= -\frac{1}{2}tr\,\underline{\chi}\widehat{\chi} + {}^{(c)}\nabla\widehat{\otimes}\underline{\eta},
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_4^\# tr\,\chi &= tr\,\chi^2 + 2\,{}^{(c)}div\,\xi,\\
{}^{(c)}\nabla_4^\# \widehat{\chi} &= \frac{1}{2}tr\,\chi\,\widehat{\chi} + {}^{(c)}\nabla\widehat{\otimes}\xi - \alpha,
\end{aligned}
$$

Also,

$$
\begin{aligned}
{}^{(c)}div\,\widehat{\chi} &= \frac{1}{2}\,{}^{(c)}\nabla(tr\,\chi) - \frac{1}{2}\,{}^*\,{}^{(c)}\nabla(\,{}^{(a)}tr\chi) - \beta,\\
{}^{(c)}div\,\widehat{\underline{\chi}} &= \frac{1}{2}\,{}^{(c)}\nabla(tr\underline{\chi}) - \frac{1}{2}\,{}^*\,{}^{(c)}\nabla(\,{}^{(a)}tr\underline{\chi}) + \underline{\beta}.
\end{aligned}
$$

We will make use of the commutation formulas of Corollary 3.3.9.

For linearized perturbations near Schwarzschild we have, for any linear, horizontal, tensorfield U of signature s,

$$
\begin{aligned}
[\,{}^{(c)}\nabla_3^\#, {}^{(c)}\nabla]U &= [\,{}^{(c)}\nabla_4^\#, {}^{(c)}\nabla]U = 0.\\
[\,{}^{(c)}\nabla_4^\#, {}^{(c)}\nabla_3^\#]U &= -s\big(4\rho - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi\big)U.
\end{aligned}
$$

4.7.1 Useful calculations

Lemma 4.7.4. *We have, ignoring quadratic and higher order terms,*

$$
\begin{aligned}
{}^{(c)}\nabla_3^\#(tr\,\chi tr\,\underline{\chi}) &= tr\,\underline{\chi}\big(\frac{1}{2}tr\,\chi tr\,\underline{\chi} + 2\rho\big) + 2\big(tr\,\chi\,div\,\underline{\xi} + tr\,\underline{\chi}\,div\,\eta\big)\\
{}^{(c)}\nabla_4^\#(tr\,\chi tr\,\underline{\chi}) &= tr\,\chi\big(\frac{1}{2}tr\,\chi tr\,\underline{\chi} + 2\rho\big) + 2\big(tr\,\underline{\chi}\,div\,\xi + tr\,\chi\,div\,\underline{\eta}\big)
\end{aligned}
\tag{4.7.1}
$$

Also,

$$
{}^{(c)}\nabla_4^\#(\rho\widehat{\chi}) = \rho\big(\,{}^{(c)}\nabla\widehat{\otimes}\xi - \alpha\big).
$$

Proof. Using the equations of Proposition 4.7.3 we calculate,

$$
\begin{aligned}
{}^{(c)}\nabla_3^\#(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}) &= {}^{(c)}\nabla_3^\#\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + \mathrm{tr}\,\chi\,{}^{(c)}\nabla_3^\#\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}tr\,\chi\mathrm{tr}\,\underline{\chi}\\
&= (2\mathrm{d}iv\,\eta + 2\rho)\mathrm{tr}\,\underline{\chi} + \mathrm{tr}\,\chi(\mathrm{tr}\,\underline{\chi}^2 + 2\mathrm{d}iv\,\underline{\xi}) - \frac{1}{2}\mathrm{tr}\,\chi tr\,\chi^2\\
&= \mathrm{tr}\,\underline{\chi}\big(\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} + 2\rho\big) + 2\big(\mathrm{tr}\,\chi\mathrm{d}iv\,\underline{\xi} + \mathrm{tr}\,\underline{\chi}\mathrm{d}iv\,\eta\big)
\end{aligned}
$$

Similarly

$$
{}^{(c)}\nabla_4^{\#}(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}) \;=\; \mathrm{t}r\,\chi\big(\frac{1}{2}\mathrm{t}r\,\chi\mathrm{tr}\,\underline{\chi}+2\rho\big)+2\big(\mathrm{tr}\,\underline{\chi}\mathrm{d}iv\,\xi+\mathrm{t}r\,\chi\mathrm{d}iv\,\underline{\eta}\big).
$$

Now, using the Leibnitz rule for ${}^{(c)}\nabla^{\#}$ operators and the equations ${}^{(c)}\nabla_4^{\#}\rho=0$, ${}^{(c)}\nabla_4^{\#}(\widehat{\chi})=\frac{1}{2}\mathrm{t}r\,\chi\,\widehat{\chi}+{}^{(c)}\nabla\widehat{\otimes}\xi-\alpha$ of Proposition 4.7.3, we deduce

$$
\begin{aligned}
{}^{(c)}\nabla_4^{\#}(\rho\widehat{\chi}) &= {}^{(c)}\nabla_4^{\#}\rho\widehat{\chi}+\rho\,{}^{(c)}\nabla_4^{\#}\widehat{\chi}-\frac{1}{2}\rho\mathrm{t}r\,\chi\widehat{\chi}=\rho\big({}^{(c)}\nabla_4^{\#}\widehat{\chi}-\frac{1}{2}\mathrm{t}r\,\chi\widehat{\chi}\big)\\
&= \rho\big({}^{(c)}\nabla\widehat{\otimes}\xi-\alpha\big)
\end{aligned}
$$

as stated.

□

4.7.2 Teukolsky Equation for α

Proposition 4.7.5. *We have*

$$
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}\alpha \;=\; {}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}div\,\alpha+3\rho\alpha. \tag{4.7.2}
$$

Also,

$$
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}\alpha \;=\; \triangle\alpha+\big(5\rho+\frac{1}{2}tr\,\chi tr\,\underline{\chi}\big)\alpha. \tag{4.7.3}
$$

Proof. We write the first Bianchi pair

$$
\begin{aligned}
{}^{(c)}\nabla_3^{\#}\alpha &= {}^{(c)}\nabla\widehat{\otimes}\beta-3\rho\widehat{\chi},\\
{}^{(c)}\nabla_4^{\#}\beta &= {}^{(c)}\mathrm{d}iv\,\alpha+3\xi\rho
\end{aligned}
$$

in the form

$$
\begin{aligned}
{}^{(c)}\nabla_3^{\#}\alpha &= {}^{(c)}\nabla\widehat{\otimes}\beta+i\\
{}^{(c)}\nabla_4^{\#}\beta &= {}^{(c)}\mathrm{d}iv\,\alpha+j\\
i &= -3\rho\widehat{\chi}\\
j &= 3\xi\rho
\end{aligned}
$$

Lemma 4.7.6. *Given*

$$
i \;=\; -3\rho\widehat{\chi}, \qquad j=3\xi\rho
$$

we have

$$
{}^{(c)}\nabla_4^{\#}i+{}^{(c)}\nabla\widehat{\otimes}j \;=\; 3\rho\alpha \tag{4.7.4}
$$

Taking ${}^{(c)}\nabla_4^\#$ and of the first equation, ${}^{(c)}\nabla\widehat{\otimes}$ of the second equation and using Lemma 3.3.8 we derive

$$\begin{aligned} {}^{(c)}\nabla_4^\#\, {}^{(c)}\nabla_3^\#\alpha &= {}^{(c)}\nabla\widehat{\otimes}\, {}^{(c)}\nabla_4^\#\beta + {}^{(c)}\nabla_4^\# i \\ {}^{(c)}\nabla\widehat{\otimes}\, {}^{(c)}\nabla_4^\#\beta &= {}^{(c)}\nabla\widehat{\otimes}\, {}^{(c)}\mathrm{d}iv\,\alpha + {}^{(c)}\nabla\widehat{\otimes} j \end{aligned}$$

Combining we deduce,

$$ {}^{(c)}\nabla_4^\#\, {}^{(c)}\nabla_3^\#\alpha = {}^{(c)}\nabla\widehat{\otimes}\, {}^{(c)}\mathrm{d}iv\,\alpha + {}^{(c)}\nabla_4^\# i + {}^{(c)}\nabla\widehat{\otimes} j. $$

Now, making use of the formula ${}^{(c)}\nabla_4^\#(\rho\widehat{\chi}) = \rho\big({}^{(c)}\nabla\widehat{\otimes}\xi - \alpha\big)$ in Lemma 4.7.4

$$ {}^{(c)}\nabla_4^\# i + {}^{(c)}\nabla\widehat{\otimes} j = -3\rho\big({}^{(c)}\nabla\widehat{\otimes}\xi - \alpha\big) + 3\,{}^{(c)}\nabla\widehat{\otimes}(\rho\xi) = 3\rho\alpha \tag{4.7.5} $$

We deduce

$$ {}^{(c)}\nabla_4^\#\, {}^{(c)}\nabla_3^\#\alpha = {}^{(c)}\nabla\widehat{\otimes}\, {}^{(c)}\mathrm{d}iv\,\alpha + 3\rho\alpha. $$

as stated. To prove the second part of the proposition we make use of the Lemma

Lemma 4.7.7. *We have*

$$ {}^{(c)}\nabla\widehat{\otimes} div = \triangle - 2K \tag{4.7.6} $$

where $K = -\rho - \frac{1}{4} tr\,\chi tr\underline{\chi}$ is the Gauss curvature

Proof. The Lemma follows from the formula $\not{\mathcal{D}}_2^*\not{\mathcal{D}}_2 = -\frac{1}{2}\triangle + K$, $\not{\mathcal{D}}_2 = \mathrm{d}iv$, $\not{\mathcal{D}}_2^* = -\frac{1}{2}\nabla\widehat{\otimes}$. □

Therefore,

$$\begin{aligned} {}^{(c)}\nabla_4^\#\, {}^{(c)}\nabla_3^\#\alpha &= \triangle\alpha - 2K\alpha + 3\rho\alpha = -2(-\rho - \frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi})\alpha + 3\rho\alpha \\ &= \triangle\alpha + \big(5\rho + \frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}\big)\alpha \end{aligned}$$

as stated. □

Lemma 4.7.8. *Teukolsky equation takes the form*

$$ {}^{(c)}\nabla_4\, {}^{(c)}\nabla_3\alpha + \frac{5}{2} tr\,\underline{\chi}\, {}^{(c)}\nabla_3\alpha + \frac{1}{2} tr\,\underline{\chi}\, {}^{(c)}\nabla_4\alpha + (-2\rho + \frac{1}{2} tr\,\chi tr\,\underline{\chi})\alpha = {}^{(c)}\nabla\widehat{\otimes}\, {}^{(c)} div\,\alpha. $$

Proof.

$$\begin{aligned}
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}\alpha &= {}^{(c)}\nabla_4({}^{(c)}\nabla_3^{\#}\alpha)+\frac{5}{2}\mathrm{tr}\,\underline{\chi}({}^{(c)}\nabla_3^{\#}\alpha)\\
&= {}^{(c)}\nabla_4({}^{(c)}\nabla_3\alpha+\frac{1}{2}\mathrm{tr}\,\underline{\chi}\alpha)+\frac{5}{2}\mathrm{tr}\,\underline{\chi}({}^{(c)}\nabla_3\alpha+\frac{1}{2}\mathrm{tr}\,\underline{\chi}\alpha)\\
&= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\alpha+\frac{5}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3\alpha+\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4\alpha+\big(\frac{1}{2}\,{}^{(c)}\nabla_4\mathrm{tr}\,\underline{\chi}+\frac{5}{4}\mathrm{tr}\,\underline{\chi}^2\big)\alpha.
\end{aligned}$$

Recall that

$$ {}^{(c)}\nabla_4\mathrm{tr}\,\underline{\chi} = -\frac{1}{2}\mathrm{t}r\,\chi\mathrm{tr}\,\underline{\chi}+2\,{}^{(c)}\mathrm{d}iv\,\underline{\eta}+2\rho $$

Hence

$$ {}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}\alpha = {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\alpha+\frac{5}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3\alpha+\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4\alpha+\big(\rho+\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2\big)\alpha $$

Therefore equation ${}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}={}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\mathrm{d}iv\,\alpha+3\rho\alpha$ takes the form

$$ {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\alpha+\frac{5}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3\alpha+\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4\alpha+\big(\rho+\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2\big)\alpha={}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\mathrm{d}iv\,\alpha+3\rho\alpha $$

or,

$$ {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\alpha+\frac{5}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3\alpha+\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4\alpha+(-2\rho+\frac{1}{2}\mathrm{t}r\,\chi\mathrm{tr}\,\underline{\chi})\alpha = {}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\mathrm{d}iv\,\alpha. $$

□

4.8 Derivation of the RW equation

4.8.1 The commuted First Bianchi system

We commute the first Bianchi system with ${}^{(c)}\nabla_3^{\#}$ and $Q={}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}$

$$\begin{aligned}
{}^{(c)}\nabla_3^{\#}\alpha &= {}^{(c)}\nabla\widehat{\otimes}\beta-3\rho\widehat{\chi}\\
{}^{(c)}\nabla_4^{\#}\beta &= {}^{(c)}\mathrm{d}iv\,\alpha+3\rho\xi
\end{aligned}$$

Proposition 4.8.1. *Ignoring quadratic terms we have*

$$\begin{aligned}
{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}\alpha &= {}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^{\#}\beta-3\,{}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi}),\\
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}\beta &= {}^{(c)}div\ {}^{(c)}\nabla_3^{\#}\alpha+\left(-4\rho+\frac{1}{2}tr\underline{\chi}tr\chi\right)\beta+3\,{}^{(c)}\nabla_3^{\#}(\rho\xi).
\end{aligned} \tag{4.8.1}$$

$$
\begin{aligned}
{}^{(c)}\nabla_3^{\#} Q(\alpha) &= {}^{(c)}\nabla\widehat{\otimes} Q(\beta) - 3\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi})\\
{}^{(c)}\nabla_4^{\#} Q(\beta) &= {}^{(c)}div\, Q(\alpha) + {}^{(c)}\nabla_3^{\#}\left(\left(-4\rho+\frac{1}{2}tr\underline{\chi}tr\chi\right)\beta\right) + 3\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\xi)
\end{aligned}
\tag{4.8.2}
$$

Proof. We start with

$$
{}^{(c)}\nabla_3^{\#}\alpha = {}^{(c)}\nabla\widehat{\otimes}\beta - 3\rho\widehat{\chi}.
$$

Apply ${}^{(c)}\nabla_3^{\#}$ (note that ${}^{(c)}\nabla_3^{\#}\alpha$ has signature 1 and rank 3) and apply the commutator lemma to obtain

$$
{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}\alpha = {}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^{\#}\beta - 3\,{}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi}).
$$

which is the first equation in (4.8.1).

Starting with

$$
{}^{(c)}\nabla_4^{\#}\beta = {}^{(c)}\mathrm{d}iv\,\alpha + 3\rho\xi
$$

we apply ${}^{(c)}\nabla_3^{\#}$ to both sides

$$
{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_4^{\#}\beta = {}^{(c)}\mathrm{d}iv\,{}^{(c)}\nabla_3^{\#}\alpha + 3\,{}^{(c)}\nabla_3^{\#}(\rho\xi).
$$

or,

$$
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}\beta + [\,{}^{(c)}\nabla_3^{\#},\,{}^{(c)}\nabla_4^{\#}]\beta = {}^{(c)}\mathrm{d}iv\,{}^{(c)}\nabla_3^{\#}\alpha + 3\,{}^{(c)}\nabla_3^{\#}(\rho\xi).
$$

Using the commutator identity of Lemma 3.3.8, where $s = \mathrm{sign}(\psi) = 1$,

$$
[\,{}^{(c)}\nabla_3^{\#},\,{}^{(c)}\nabla_4^{\#}]\beta = \left(4\rho - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\right)\beta =
$$

we deduce

$$
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}\beta \;=\; {}^{(c)}\mathrm{d}iv\,{}^{(c)}\nabla_3^{\#}\alpha - \left(4\rho - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\right)\beta + 3\,{}^{(c)}\nabla_3^{\#}(\rho\xi).
$$

Hence ${}^{(c)}\nabla_3^{\#}\alpha$, ${}^{(c)}\nabla_3^{\#}\beta$ verify the system

$$
\begin{aligned}
{}^{(c)}\nabla_3^{\#}(\,{}^{(c)}\nabla_3^{\#}\alpha) &= {}^{(c)}\nabla\widehat{\otimes}(\,{}^{(c)}\nabla_3^{\#}\beta) - 3\,{}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi})\\
{}^{(c)}\nabla_4^{\#}(\,{}^{(c)}\nabla_3^{\#}\beta) &= {}^{(c)}\mathrm{d}iv\,(\,{}^{(c)}\nabla_3^{\#}\alpha) - \left(4\rho - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\right)\beta + 3\,{}^{(c)}\nabla_3^{\#}(\rho\xi).
\end{aligned}
\tag{4.8.3}
$$

as stated in (4.8.1).

Next we apply apply ${}^{(c)}\nabla_3^{\#}$ to the first equation in (4.8.1), commute, and derive

$$\begin{aligned}{}^{(c)}\nabla_3^{\#} Q(\alpha) &= {}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}\beta - 3\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi})\\ &= {}^{(c)}\nabla\widehat{\otimes} Q(\beta) - 3\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi}).\end{aligned}$$

which is the first equation in (4.8.2).

Applying ${}^{(c)}\nabla_3^{\#}$ to the second equation in (4.8.3) and commuting as before we derive

$$\begin{aligned}&{}^{(c)}\nabla_4^{\#}(\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}\beta) + [\,{}^{(c)}\nabla_3^{\#},\,{}^{(c)}\nabla_4^{\#}]\,{}^{(c)}\nabla_3^{\#}\beta\\ =\;& {}^{(c)}\mathrm{d}iv\,(\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}\alpha) - {}^{(c)}\nabla_3^{\#}\Big(\big(4\rho - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\beta\Big) + 3\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\xi).\end{aligned}$$

Since ${}^{(c)}\nabla_3^{\#}\beta$ has signature zero $[\,{}^{(c)}\nabla_3^{\#},\,{}^{(c)}\nabla_4^{\#}]\,{}^{(c)}\nabla_3^{\#}\beta = 0$. We deduce,

$${}^{(c)}\nabla_4^{\#} Q(\beta) = {}^{(c)}\mathrm{d}iv\, Q(\beta) + {}^{(c)}\nabla_3^{\#}\Big(\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\beta\Big) + 3\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\xi).$$

which is the second equation in (4.8.2). □

4.8.2 Teuklosky equation for $Q(\alpha)$

We rewrite the result of the second part of Proposition 4.8.1 in the form

$$\begin{aligned}{}^{(c)}\nabla_3^{\#} Q(\alpha) &= {}^{(c)}\nabla\widehat{\otimes} Q(\beta) + I\\ {}^{(c)}\nabla_4^{\#} Q(\beta) &= \mathrm{d}iv\, Q(\alpha) + J\end{aligned} \tag{4.8.4}$$

where

$$\begin{aligned}I &= -3\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi})\\ J &= {}^{(c)}\nabla_3^{\#}\left(\left(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\right)\beta\right) + 3\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\xi)\end{aligned} \tag{4.8.5}$$

Proposition 4.8.2. *The following identity holds true*

$${}^{(c)}\nabla_4^{\#} I + {}^{(c)}\nabla\widehat{\otimes} J = \big(-\rho + \frac{1}{2} tr\,\chi\, tr\,\underline{\chi}\big) Q(\alpha). \tag{4.8.6}$$

The proof is an immediate consequence of the following calculations.

Lemma 4.8.3. *The following identities hold true.*

$$\begin{aligned} {}^{(c)}\nabla_4^\# I = \;& 3\rho Q(\alpha) - 3tr\underline{\chi}\rho\,{}^{(c)}\nabla_3^\#\alpha - 3\,{}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#\big(\rho\xi\big)\\ & -3\,{}^{(c)}\nabla_3^\#\Big(\big(4\rho - \frac{1}{2}tr\,\chi tr\underline{\chi}\big)(\rho\widehat{\chi})\Big). \end{aligned} \tag{4.8.7}$$

$$\begin{aligned} {}^{(c)}\nabla\widehat{\otimes}J = & \big(-4\rho + \frac{1}{2}tr\underline{\chi}tr\chi\big)Q(\alpha) + 3\rho tr\underline{\chi}\rho\,{}^{(c)}\nabla_3^\#\alpha\\ & + 3\,{}^{(c)}\nabla_3^\#\Big(\big(-4\rho + \frac{1}{2}tr\underline{\chi}tr\chi\big)\rho\widehat{\chi}\Big) + 3\,{}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#(\rho\nabla\widehat{\otimes}\xi) \end{aligned} \tag{4.8.8}$$

Proof. We start with checking (4.8.8). Making use of

$$ {}^{(c)}\nabla\widehat{\otimes}\beta \;=\; {}^{(c)}\nabla_3^\#\alpha + 3\widehat{\chi}\rho$$

we deduce

$$\begin{aligned} &{}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^\#\Big(\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\beta\Big) = {}^{(c)}\nabla_3^\#\Big(\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\,{}^{(c)}\nabla\widehat{\otimes}\beta\Big)\\ = \;& {}^{(c)}\nabla_3^\#\Big(\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\big({}^{(c)}\nabla_3^\#\alpha + 3\rho\widehat{\chi}\big)\Big)\\ = \;& {}^{(c)}\nabla_3^\#\Big(\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\,{}^{(c)}\nabla_3^\#\alpha\Big) + 3\,{}^{(c)}\nabla_3^\#\Big(\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\rho\widehat{\chi}\Big)\\ = \;& {}^{(c)}\nabla_3^\#\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\,{}^{(c)}\nabla_3^\#\alpha + \big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)Q(\alpha)\\ & -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\,{}^{(c)}\nabla_3^\#\alpha + 3\,{}^{(c)}\nabla_3^\#\Big(\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\rho\widehat{\chi}\Big) \end{aligned}$$

Hence

$$\begin{aligned} &{}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^\#\Big(\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\beta\Big)\\ = \;& \big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)Q(\alpha) + 3\,{}^{(c)}\nabla_3^\#\Big(\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\rho\widehat{\chi}\Big)\\ & +\Big({}^{(c)}\nabla_3^\#\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\Big)\,{}^{(c)}\nabla_3^\#\alpha \end{aligned}$$

In view of Lemma 4.7.4 we have, modulo linear terms,

$$ {}^{(c)}\nabla_3^\#(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}) \;= \mathrm{tr}\,\underline{\chi}\big(\tfrac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} + 2\rho\big).$$

Thus, since also ${}^{(c)}\nabla_3^\#(\rho) = 0$,

$$ {}^{(c)}\nabla_3^\#\big(-4\rho + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big) = \frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 2\rho\big)$$

and therefore,

$$ {}^{(c)}\nabla_3^\#\big(-4\rho+\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(-4\rho+\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big) \;=\; 3\mathrm{tr}\,\underline{\chi}\rho. $$

Henceforth

$$ \begin{aligned} {}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^\#\Big(\big(-4\rho+\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\beta\Big) \;&=\; \big(-4\rho+\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)Q(\alpha)+3\rho\mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^\#\alpha\\ &+3\,{}^{(c)}\nabla_3^\#\Big(\big(-4\rho+\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\big)\rho\widehat{\chi}\Big) \end{aligned} $$

from which (4.8.8) easily follows.

To check (4.8.7) we write, commuting ${}^{(c)}\nabla_4^\#$ with ${}^{(c)}\nabla_3^\#$ twice,

$$ \begin{aligned} {}^{(c)}\nabla_4^\#\,{}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#(\rho\widehat{\chi}) \;&=\; {}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_4^\#\,{}^{(c)}\nabla_3^\#(\rho\widehat{\chi})+[\,{}^{(c)}\nabla_4^\#,\,{}^{(c)}\nabla_3^\#]\,{}^{(c)}\nabla_3^\#(\rho\widehat{\chi})\\ &=\; {}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_4^\#(\rho\widehat{\chi})+\,{}^{(c)}\nabla_3^\#\Big([\,{}^{(c)}\nabla_4^\#,\,{}^{(c)}\nabla_3^\#](\rho\widehat{\chi})\Big)\\ &\quad+[\,{}^{(c)}\nabla_4^\#,\,{}^{(c)}\nabla_3^\#]\,{}^{(c)}\nabla_3^\#(\rho\widehat{\chi})\\ &=\; I_1+I_2+I_3 \end{aligned} $$

Recall, see 4.7.4,

$$ {}^{(c)}\nabla_4^\#(\rho\widehat{\chi}) \;=\; \rho\big(\,{}^{(c)}\nabla\widehat{\otimes}\xi-\alpha\big). $$

Thus,

$$ \begin{aligned} I_1 \;&=\; {}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_4^\#(\rho\widehat{\chi})={}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#\Big(\rho\big(\,{}^{(c)}\nabla\widehat{\otimes}\xi-\alpha\big)\Big)\\ &=\; -\,{}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#(\rho\alpha)+\,{}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#(\rho\,{}^{(c)}\nabla\widehat{\otimes}\xi)\\ &=\; -\,{}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#(\rho\alpha)+\,{}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^\#\,{}^{(c)}\nabla_3^\#\big(\rho\xi\big) \end{aligned} $$

Also, since ${}^{(c)}\nabla_3^\#\rho=0$,

$$ {}^{(c)}\nabla_3^\#(\rho\alpha) \;=\; {}^{(c)}\nabla_3^\#\rho\alpha+\rho\,{}^{(c)}\nabla_3^\#\alpha-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\rho\alpha=\rho\,{}^{(c)}\nabla_3^\#\alpha-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\rho\alpha $$

$$
\begin{aligned}
{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\alpha) &= {}^{(c)}\nabla_3^{\#}\Big(\rho\,{}^{(c)}\nabla_3^{\#}\alpha - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\rho\alpha\Big)\\
&= \Big({}^{(c)}\nabla_3^{\#}\rho\,{}^{(c)}\nabla_3^{\#}\alpha + \rho\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}\alpha - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^{\#}\alpha\Big) - \frac{1}{2}\,{}^{(c)}\nabla_3^{\#}\big(\mathrm{tr}\,\underline{\chi}\rho\alpha\big)\\
&= \rho Q(\alpha) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^{\#}\alpha - \frac{1}{2}\,{}^{(c)}\nabla_3^{\#}\big(\mathrm{tr}\,\underline{\chi}\rho\alpha\big)\\
&= \rho Q(\alpha) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^{\#}\alpha\\
&\quad -\frac{1}{2}\Big({}^{(c)}\nabla_3^{\#}\alpha(\mathrm{tr}\,\underline{\chi}\rho) + \alpha\,{}^{(c)}\nabla_3^{\#}(\mathrm{tr}\,\underline{\chi}\rho) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(\mathrm{tr}\,\underline{\chi}\rho\alpha\big)\Big)\\
&= \rho Q(\alpha) - \mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^{\#}\alpha - \frac{1}{2}\alpha\,{}^{(c)}\nabla_3^{\#}(\mathrm{tr}\,\underline{\chi}\rho) + \frac{1}{4}\mathrm{tr}\,\underline{\chi}^2\rho\alpha
\end{aligned}
$$

We deduce,

$$
\begin{aligned}
{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\alpha) &= \rho Q(\alpha) - \mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^{\#}\alpha + \frac{1}{4}\mathrm{tr}\,\underline{\chi}^2\rho\alpha\\
&\quad -\frac{1}{2}\alpha\big({}^{(c)}\nabla_3^{\#}\mathrm{tr}\,\underline{\chi}\rho + \mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3^{\#}\rho - \frac{1}{2}\mathrm{tr}\,\underline{\chi}^2\rho\big)\\
&= \rho Q(\alpha) - \mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^{\#}\alpha + \frac{1}{4}\mathrm{tr}\,\underline{\chi}^2\rho\alpha - \frac{1}{2}\alpha\big(\mathrm{tr}\,\underline{\chi}^2\rho - \frac{1}{2}\mathrm{tr}\,\underline{\chi}^2\rho\big)\\
&= \rho Q(\alpha) - \mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^{\#}\alpha
\end{aligned}
$$

and therefore,

$$
I_1 = -\rho Q(\alpha) + \mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^{\#}\alpha + {}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}\big(\rho\xi\big). \tag{4.8.9}
$$

Note that the signature of ${}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi})$ is zero and therefore, in view of the commutator Lemma 3.3.8

$$
I_3 = [\,{}^{(c)}\nabla_4^{\#},\,{}^{(c)}\nabla_3^{\#}]\,{}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi}) = 0. \tag{4.8.10}
$$

It remains to calculate I_2.

$$
I_2 = {}^{(c)}\nabla_3^{\#}\Big([\,{}^{(c)}\nabla_4^{\#},\,{}^{(c)}\nabla_3^{\#}](\rho\widehat{\chi})\Big)
$$

In view of the commutator Lemma 3.3.8 and

$$
[\,{}^{(c)}\nabla_4^{\#},\,{}^{(c)}\nabla_3^{\#}](\rho\widehat{\chi}) = \big(4\rho - \frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi}\big)(\rho\widehat{\chi})
$$

Thus

$$
I_2 = {}^{(c)}\nabla_3^{\#}\Big(\big(4\rho - \frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi}\big)(\rho\widehat{\chi})\Big)
$$

We deduce

$$
\begin{aligned}
& {}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}(\rho\widehat{\chi}) = I_1 + I_2 + I_3 \\
= \;& -\rho Q(\alpha) + \mathrm{tr}\,\underline{\chi}\rho\,{}^{(c)}\nabla_3^{\#}\alpha + {}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_3^{\#}\,{}^{(c)}\nabla_3^{\#}\big(\rho\xi\big) + {}^{(c)}\nabla_3^{\#}\Big(\big(4\rho - \frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}\big)(\rho\widehat{\chi})\Big)
\end{aligned}
$$

from which (4.8.7) follows.

□

As a corollary of (4.8.4) and Proposition 4.8.2 we derive the Teukolski equation for $Q(\alpha)$.

Proposition 4.8.4. *$Q(\alpha)$ verifies,*

$$
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}Q(\alpha) \;=\; {}^{(c)}\nabla\widehat{\otimes}div\,Q(\alpha) + \big(-\rho + \frac{1}{2}tr\,\chi tr\,\underline{\chi}\big)Q(\alpha) \tag{4.8.11}
$$

Proof.

$$
\begin{aligned}
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}Q(\alpha) &= {}^{(c)}\nabla\widehat{\otimes}\,{}^{(c)}\nabla_4 Q(\beta) + {}^{(c)}\nabla_4^{\#}I \\
&= {}^{(c)}\nabla\widehat{\otimes}\big(\mathrm{d}iv\,Q(\alpha) + J\big) + {}^{(c)}\nabla_4^{\#}I \\
&= {}^{(c)}\nabla\widehat{\otimes}\mathrm{d}iv\,Q(\alpha) + {}^{(c)}\nabla\widehat{\otimes}J + {}^{(c)}\nabla_4^{\#}I \\
&= {}^{(c)}\nabla\widehat{\otimes}\mathrm{d}iv\,Q(\alpha) + \big(-\rho + \frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}\big)Q(\alpha)
\end{aligned}
$$

as stated.

□

Proposition 4.8.5. *We have*

$$
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}Q(\alpha) \;=\; \triangle Q(\alpha) + \big(\rho + tr\,\chi tr\,\underline{\chi}\big)Q(\alpha) \tag{4.8.12}
$$

Proof. Note that

$$
{}^{(c)}\nabla\widehat{\otimes}\mathrm{d}iv \;=\; \triangle - 2K \tag{4.8.13}
$$

where $K = -\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} - \rho + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}} = -\rho - \frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}$ is the Gauss curvature This follows from the formula $\not{d}_2^*\not{d}_2 = -\frac{1}{2}\triangle + K$ where $\not{d}_2 = \mathrm{d}iv$, $\not{d}_2^* = -\frac{1}{2}\nabla\widehat{\otimes}$. □

Remark 4.8.6. *We have*

$$
{}^{(c)}\nabla_4^{\#}\,{}^{(c)}\nabla_3^{\#}Q(\alpha) \;=\; {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 Q(\alpha) + \frac{5}{2}\Big(tr\,\chi\,{}^{(c)}\nabla_3 Q(\alpha) + tr\underline{\chi}\,{}^{(c)}\nabla_4 Q(\alpha)\Big) + 5(\rho + tr\,\chi tr\underline{\chi}).
$$

Proof. We have $\mathrm{sign}(Q(\alpha))=0$, $\mathrm{rank}(Q(\alpha))=4$ and $\mathrm{sign}({}^{(c)}\nabla_3 Q(\alpha))=-1$, $\mathrm{rank}({}^{(c)}\nabla_3(Q(\alpha))=5$,

$$\begin{aligned}
{}^{(c)}\nabla_3^\# Q(\alpha) &= {}^{(c)}\nabla_3 Q(\alpha)+\frac{5}{2}\mathrm{tr}\,\underline{\chi} Q(\alpha)\\
{}^{(c)}\nabla_4^\#\,{}^{(c)}\nabla_3^\# Q(\alpha) &= {}^{(c)}\nabla_4^\#\big({}^{(c)}\nabla_3 Q(\alpha)+\frac{5}{2}\mathrm{tr}\,\underline{\chi} Q(\alpha)\big)\\
&= {}^{(c)}\nabla_4\big({}^{(c)}\nabla_3 Q(\alpha)+\frac{5}{2}\mathrm{tr}\,\underline{\chi} Q(\alpha)\big)+\frac{5}{2}\mathrm{tr}\,\chi\big({}^{(c)}\nabla_3 Q(\alpha)+\frac{5}{2}\mathrm{tr}\,\underline{\chi}\big)Q(\alpha)\\
&= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 Q(\alpha)+\frac{5}{2}\mathrm{tr}\,\chi\,{}^{(c)}\nabla_3 Q(\alpha)+\frac{5}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4 Q(\alpha)\\
&\quad +\frac{5}{2}\big({}^{(c)}\nabla_4(\mathrm{tr}\,\underline{\chi})+\frac{5}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}\big)Q(\alpha)
\end{aligned}$$

The result then follows by a simple computation using ${}^{(c)}\nabla_4(\mathrm{tr}\,\underline{\chi})=-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+2\rho$. ☐

4.8.3 Function r and normalizations

Outgoing frame for which e_4 is gedoesic

In this case $e_4=\Upsilon^{-1}\partial_t+\partial_r$, $e_3=\partial_t-\Upsilon\partial_r$, $\xi=\omega=0$, $\underline{\omega}=\frac{m}{r^2}$, $\mathrm{tr}\,\chi=\frac{2}{r}$, $\mathrm{tr}\,\underline{\chi}=-\frac{2}{r}\Upsilon$ We have

$$ {}^{(c)}\nabla_3(r)=\frac{r}{2}\mathrm{tr}\,\underline{\chi},\qquad {}^{(c)}\nabla_4(r)=\frac{r}{2}\mathrm{tr}\,\chi$$

Lemma 4.8.7. *For a tensor ψ of signature s and rank k we have*

$$\begin{aligned}
{}^{(c)}\nabla_3^\#(\psi) &= r^{-1+s-k}\,{}^{(c)}\nabla_3(r^{1-s+k}\psi)\\
{}^{(c)}\nabla_4^\#(\psi) &= r^{-1-s-k}\,{}^{(c)}\nabla_4(r^{1+s+k}\psi).
\end{aligned}\tag{4.8.14}$$

Ingoing frame for which e_3 is gedoesic

In this case $e_3=\Upsilon^{-1}\partial_t+\partial_r$, $e_3=\partial_t+\Upsilon\partial_r$, $\underline{\xi}=\underline{\omega}=0, \omega=-\frac{m}{r^2}$, $\mathrm{tr}\,\chi=\frac{2\Upsilon}{r}$, $\mathrm{tr}\,\underline{\chi}=-\frac{1}{r}$. In this case also

$$ {}^{(c)}\nabla_3(r)=\frac{r}{2}\mathrm{tr}\,\underline{\chi},\qquad {}^{(c)}\nabla_4(r)=\frac{r}{2}\mathrm{tr}\,\chi$$

Thus in both cases

Lemma 4.8.8. *For a tensor $\psi$of signature s and rank k we have*

$$
\begin{aligned}
{}^{(c)}\nabla_3^\#(\psi) &= r^{-1+s-k}\,{}^{(c)}\nabla_3(r^{1-s+k}\psi)\\
{}^{(c)}\nabla_4^\#(\psi) &= r^{-1-s-k}\,{}^{(c)}\nabla_4(r^{1+s+k}\psi).
\end{aligned}
\tag{4.8.15}
$$

Proof. Indeed

$$
\begin{aligned}
{}^{(c)}\nabla_3^\#(\psi) &= {}^{(c)}\nabla_3 + \frac{1-s+k}{2}\mathrm{tr}\,\underline{\chi}\psi\\
{}^{(c)}\nabla_4^\#(\psi) &= {}^{(c)}\nabla_4 + \frac{1+s+k}{2}\mathrm{tr}\,\chi\psi
\end{aligned}
$$

Hence

$$
\begin{aligned}
r^{-1+s-k}\,{}^{(c)}\nabla_3(r^{1-s+k}\psi) &= {}^{(c)}\nabla_3\psi + (1-s+k)r^{-1+s-k}r^{-s+k}\frac{r}{2}\mathrm{tr}\,\underline{\chi}\psi\\
&= {}^{(c)}\nabla_3\psi + \frac{1-s+k}{2}\mathrm{tr}\,\underline{\chi}\psi = {}^{(c)}\nabla_3^\#\psi
\end{aligned}
$$

and similarly for the second equation in (4.8.15).

□

4.8.4 Reggee-Wheeler equation

Lemma 4.8.9. *We have*

$$
{}^{(c)}\nabla_4^\#\,{}^{(c)}\nabla_3^\# Q(\alpha) = r^{-5}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\big(r^5 Q(\alpha)\big)
\tag{4.8.16}
$$

Proof. To check (4.8.16) we write (note that $Q(\alpha)$ has signature 0 and rank 4.) Thus, since ${}^{(c)}\nabla_3^\# Q(\alpha)$ has signature -1 and rank 5.

$$
\begin{aligned}
{}^{(c)}\nabla_4^\#\,{}^{(c)}\nabla_3^\# Q(\alpha) &= r^{-1-(-1)-5}\,{}^{(c)}\nabla_4\big(r^{1+(-1)+5}\,{}^{(c)}\nabla_3^\# Q(\alpha)\big) = r^{-5}\,{}^{(c)}\nabla_4\big(r^5\,{}^{(c)}\nabla_3^\# Q(\alpha)\big)\\
&= r^{-5}\,{}^{(c)}\nabla_4\big(r^5 r^{-1+0-4}\,{}^{(c)}\nabla_3\big(r^{1+0+4}Q(\alpha)\big)\\
&= r^{-5}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\big(r^5 Q(\alpha)\big)
\end{aligned}
$$

$$
{}^{(c)}\nabla_4^\#\,{}^{(c)}\nabla_3^\# Q(\alpha) = r^{-5}\,{}^{(c)}\nabla_4\big(r^5\,{}^{(c)}\nabla_3^\# Q(\alpha)\big) = r^{-5}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\big(r^5 Q(\alpha)\big)
$$

□

Theorem 4.8.10. *The quantity* $\mathfrak{q} = r^4 Q(\alpha)$ *verifies,*

$$\dot{\square}\mathfrak{q} + tr\,\chi tr\,\underline{\chi}\mathfrak{q} = 0. \tag{4.8.17}$$

Proof. Recall that the wave operator for $\psi \in \mathfrak{s}_2(\mathbb{C})$ is given by

$$\dot{\square} \quad = \quad -\nabla_4\nabla_3\psi - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\nabla_4\psi + \left(2\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\psi + \triangle_2\psi$$

Equation (4.8.12) takes the form, with $\mathfrak{q} = r^4 Q(\alpha)$

$$^{(c)}\nabla_4\,^{(c)}\nabla_3(r\mathfrak{q}) \quad = \quad \triangle r\mathfrak{q} + \big(\rho + \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}\big)r\mathfrak{q} \tag{4.8.18}$$

Now, using the equation $\nabla_4\mathrm{tr}\,\underline{\chi} = -\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 2\rho + 2\omega\mathrm{tr}\,\underline{\chi}$,

$$\begin{aligned}
^{(c)}\nabla_3(r\mathfrak{q}) &= \nabla_3(r\mathfrak{q}) = \frac{r}{2}\mathrm{tr}\,\underline{\chi}\mathfrak{q} + r\nabla_3\mathfrak{q}\\
^{(c)}\nabla_4\,^{(c)}\nabla_3(r\mathfrak{q}) &= \nabla_4\big(\frac{r}{2}\mathrm{tr}\,\underline{\chi}\mathfrak{q} + r\nabla_3\mathfrak{q}\big) - 2\omega\big(\frac{r}{2}\mathrm{tr}\,\underline{\chi}\mathfrak{q} + r\nabla_3\mathfrak{q}\big)\\
&= \frac{r}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}\mathfrak{q} + \frac{r}{2}\nabla_4\mathrm{tr}\,\underline{\chi}\mathfrak{q} + \frac{r}{2}\mathrm{tr}\,\underline{\chi}\nabla_4\mathfrak{q} + \frac{r}{2}\mathrm{tr}\,\chi\nabla_3\mathfrak{q} + r\nabla_4\nabla_3\mathfrak{q}\\
&\quad -2\omega\big(\frac{r}{2}\mathrm{tr}\,\underline{\chi}\mathfrak{q} + r\nabla_3\mathfrak{q}\big)\\
&= r\nabla_4\nabla_3\mathfrak{q} + r\big(\frac{1}{2}\mathrm{tr}\,\chi - 2\omega\big)\nabla_3\mathfrak{q} + \frac{r}{2}\mathrm{tr}\,\underline{\chi}\nabla_4\mathfrak{q}\\
&\quad +\frac{1}{2}r\mathfrak{q}\big(\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + \nabla_4\mathrm{tr}\,\underline{\chi} - 2\omega\mathrm{tr}\,\underline{\chi}\big).\\
&= r\nabla_4\nabla_3\mathfrak{q} + r\big(\frac{1}{2}\mathrm{tr}\,\chi - 2\omega\big)\nabla_3\mathfrak{q} + \frac{r}{2}\mathrm{tr}\,\underline{\chi}\nabla_4\mathfrak{q} + r\rho\mathfrak{q}
\end{aligned}$$

Using (4.8.18)

$$r\nabla_4\nabla_3\mathfrak{q} + r\big(\frac{1}{2}\mathrm{tr}\,\chi - 2\omega\big)\nabla_3\mathfrak{q} + \frac{r}{2}\mathrm{tr}\,\underline{\chi}\nabla_4\mathfrak{q} + r\rho\mathfrak{q} \quad = \quad r\triangle\mathfrak{q} + \big(\rho + \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}\big)r\mathfrak{q}$$

Therefore

$$-\nabla_4\nabla_3\mathfrak{q} + \triangle\mathfrak{q} - \big(\frac{1}{2}\mathrm{tr}\,\chi - 2\omega\big)\nabla_3\mathfrak{q} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\nabla_4 + \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}\mathfrak{q} = 0.$$

Therefore,

$$\dot{\square}\mathfrak{q} + \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}\mathfrak{q} = 0.$$

as stated.

□

Chapter 5

The Kerr spacetime

5.1 Boyer-Lindquist coordinates

We consider the Kerr metric in standard Boyer-Lindquist coordinates (t, r, θ, ϕ),

$$\mathbf{g}_{a,m} = -\frac{|q|^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{|q|^2}{\Delta}(dr)^2 + |q|^2(d\theta)^2,$$

where

$$q = r + ia\cos\theta, \tag{5.1.1}$$

and

$$\begin{cases} \Delta & = & r^2 - 2mr + a^2, \\ |q|^2 & = & r^2 + a^2(\cos\theta)^2, \\ \Sigma^2 & = & (r^2+a^2)|q|^2 + 2mra^2(\sin\theta)^2 = (r^2+a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

Observe that

$$(2mr - |q|^2)\Sigma^2 = -|q|^4\Delta + 4a^2m^2r^2(\sin\theta)^2.$$

The metric $\mathbf{g} = \mathbf{g}_{a,m}$ can also be written in the form

$$\mathbf{g} \quad = \quad -\frac{\left(\Delta - a^2\sin^2\theta\right)}{|q|^2}dt^2 - \frac{4amr}{|q|^2}\sin^2\theta dt d\phi + \frac{|q|^2}{\Delta}dr^2 + |q|^2 d\theta^2 + \frac{\Sigma^2}{|q|^2}\sin^2\theta d\phi^2.$$

Note that $\mathbf{g}_{tt}\mathbf{g}_{\phi\phi} - \mathbf{g}_{t\phi}^2 = -\Delta\sin^2\theta$ and that the non-vanishing components of the inverse metric are given by

$$\begin{aligned}
&\mathbf{g}^{00} = -\frac{\Sigma^2}{|q|^2\Delta}, \qquad \mathbf{g}^{0\phi} = -\frac{2amr}{|q|^2\Delta}, \qquad \mathbf{g}^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{|q|^2\Delta\sin^2\theta}, \\
&\mathbf{g}^{rr} = \frac{\Delta}{|q|^2}, \qquad \mathbf{g}^{\theta\theta} = \frac{1}{|q|^2}.
\end{aligned} \tag{5.1.2}$$

The volume element $d\mu$ of $\mathbf{g}$ is given by

$$d\mu \;=\; |q|^2\sin\theta dt dr d\theta d\phi, \qquad \sqrt{|g|} = |q|^2\sin\theta.$$

We also note that

$$\mathbf{T} = \partial_t, \qquad \mathbf{Z} = \partial_\phi, \tag{5.1.3}$$

are both Killing. Also

$$\mathbf{g}(\mathbf{T},\mathbf{T}) = -\frac{\Delta - a^2\sin^2\theta}{|q|^2} = -\frac{|q|^2 - 2mr}{|q|^2}.$$

Thus and $\mathbf{T}$ is only time-like in the complement of the ergoregion, i.e. the region

$$\Delta > a^2\sin^2\theta.$$

The domain of outer communication of the Kerr metric is given by,

$$\mathcal{R} = \{(\theta, r, t, \phi) \in (0,\pi)\times(r_+,\infty)\times\mathbb{R}\times\mathbb{S}^1\},$$

where $r_+ := m + \sqrt{m^2 - a^2}$, the larger root of Δ, corresponds to the event horizon.

Definition 5.1.1. *We introduce the vectorfields $\widehat{T}, \widehat{R}$ as follows:*

$$\widehat{T} : \;=\; \partial_t + \frac{a}{r^2+a^2}\partial_\phi, \qquad \widehat{R} := \frac{\Delta}{r^2+a^2}\partial_r. \tag{5.1.4}$$

Lemma 5.1.2.

$$\mathbf{g}(\widehat{T},\widehat{T}) \;=\; -\Delta\frac{|q|^2}{(r^2+a^2)^2}, \qquad \mathbf{g}(\widehat{R},\widehat{R}) = \Delta\frac{|q|^2}{(r^2+a^2)^2}, \qquad \mathbf{g}(\widehat{T},\widehat{R}) = 0.$$

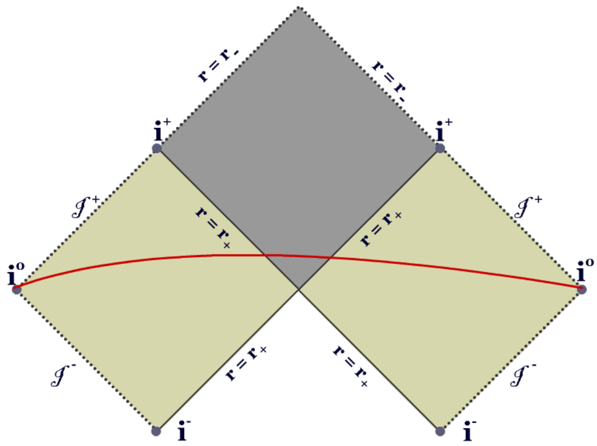


Figure 5.1: Penrose diagram of Kerr for $0 < |a| < m$. The surface $r = r_+$, the larger root of $\Delta = 0$, is the event horizon of the black hole, $r > r_+$ the domain of outer communication, $\mathcal{I}^+$ is the future null infinity, corresponding to $r = +\infty$.

5.2 Principal null frames

The Kerr metric is a spacetime of Petrov Type D, i.e. its Weyl curvature can be diagonalized with two linearly independent eigenvectors, the so-called principal null (PN) directions. There are two basic normalizations for these directions, the ingoing normalization, defined for all $r > 0$,

$$\begin{aligned} e_4^{(in)} &= \frac{r^2+a^2}{|q|^2}\partial_t + \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi, \\ e_3^{(in)} &= \frac{r^2+a^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\phi, \end{aligned} \tag{5.2.1}$$

for which $e_3 = e_3^{(in)}$ is geodesic, i.e. $\mathbf{D}_{e_3}e_3 = 0$, and the outgoing one

$$\begin{aligned} e_4^{(out)} &= \frac{|q|^2}{\Delta}e_4^{(in)} = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi, \\ e_3^{(out)} &= \frac{\Delta}{|q|^2}e_3^{(in)} = \frac{r^2+a^2}{|q|^2}\partial_t - \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi, \end{aligned} \tag{5.2.2}$$

for which $e_4 = e_4^{(iout)}$ is geodesic $\mathbf{D}_{e_4}e_4 = 0$. The horizontal structure $\mathcal{H}$ is spanned by the orthonormal vectors

$$e_1 = \frac{1}{|q|}\partial_\theta, \qquad e_2 = \frac{a\sin\theta}{|q|}\partial_t + \frac{1}{|q|\sin\theta}\partial_\phi. \tag{5.2.3}$$

We refer to (5.2.3) as the *canonical horizontal basis* of Kerr.

Remark 5.2.1. *Note that $\mathcal{H}(r)=0$ and, relative to the r,θ BL-coordinates,*

$$\begin{aligned} e_4^{(in)}(r) &= \frac{\Delta}{|q|^2}, \qquad e_3^{(in)}(r)=-1, \qquad e_3(\theta)=0\\ e_3^{(out)}(r) &= -\frac{\Delta}{|q|^2}, \qquad e_4^{(out)}(r)=1, \qquad e_4(\theta)=0. \end{aligned} \tag{5.2.4}$$

Lemma 5.2.2. *In both cases we have, consistent with the definition of principal null frames and the Penrose-Saks theorem,*

$$\widehat{X}=\underline{\widehat{X}}=\Xi=\underline{\Xi}, \qquad A=B=\underline{A}=\underline{B}=0. \tag{5.2.5}$$

Proof. Straightforward calculation. □

The null structure and null Bianchi equations, Propositions 3.4.16 and 3.4.17, take the form (in both the incoming and outgoing frames)

Proposition 5.2.3. *1. We have*

$$\begin{aligned}
{}^{(c)}\nabla_3 tr\underline{X}+\frac{1}{2}(tr\underline{X})^2 &= 0\\
{}^{(c)}\mathcal{D}\widehat{\otimes}H+H\widehat{\otimes}H &= 0\\
{}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H}+\underline{H}\widehat{\otimes}\,\underline{H} &= 0\\
{}^{(c)}\nabla_4 trX+\frac{1}{2}(trX)^2 &= 0\\
{}^{(c)}\nabla_3\,\underline{H}+\frac{1}{2}\overline{tr\underline{X}}(\,\underline{H}-H) &= 0,\\
{}^{(c)}\nabla_4 H+\frac{1}{2}\overline{trX}(H-\,\underline{H}) &= 0\\
\frac{1}{2}\,{}^{(c)}\mathcal{D}\overline{trX}-i\Im(trX)H &= 0\\
\frac{1}{2}\,{}^{(c)}\mathcal{D}\overline{tr\underline{X}}-i\Im(tr\underline{X})\,\underline{H} &= 0.
\end{aligned}$$

2. The complex scalar P verifies:

$$\begin{aligned}
-\,{}^{(c)}\mathcal{D}\overline{P} &= 3\overline{P}H\\
{}^{(c)}\mathcal{D}P &= -3P\,\underline{H},\\
{}^{(c)}\nabla_4 P &= -\frac{3}{2}trXP\\
{}^{(c)}\nabla_3 P &= -\frac{3}{2}\overline{tr\underline{X}}P.
\end{aligned}$$

3. *The components of* $\mathbf{B}$ *are given by, see Proposition 3.1.3,*

$$\begin{aligned}
\mathbf{B}_{abc3} &= -\mathbf{B}_{ab3c} = -tr\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - {}^{(a)}tr\underline{\chi}\big(\in_{ca}\eta_b - \in_{cb}\eta_a\big),\\
\mathbf{B}_{abc4} &= -\mathbf{B}_{ab4c} = -tr\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - {}^{(a)}tr\chi\big(\in_{ca}\underline{\eta}_b - \in_{cb}\underline{\eta}_a\big),\\
\mathbf{B}_{ab34} &= -\mathbf{B}_{ab43} = -4\big(\eta_a\underline{\eta}_b - \underline{\eta}_a\eta_b\big),\\
\mathbf{B}_{1212} &= -\mathbf{B}_{1221} = \mathbf{B}_{2121} = \frac{1}{2}tr\,\chi tr\,\underline{\chi} + \frac{1}{2}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi}.
\end{aligned} \tag{5.2.6}$$

Definition 5.2.4. *We define the following complex horizontal* 1*-tensor in Kerr, given in components relative to* e_1, e_2 *by*

$$\mathfrak{J}_1 = \frac{i\sin\theta}{|q|}, \qquad \mathfrak{J}_2 = \frac{\sin\theta}{|q|}.$$

Note that $\mathfrak{J}$ *is regular*[1] *(even at the axis), and anti-selfadjoint, i.e.* ${}^*\mathfrak{J} = -i\mathfrak{J}$ *and* $|\mathfrak{J}|^2 = 2\frac{\sin^2\theta}{|q|^2}$.

Remark 5.2.5. *Note that* $\mathfrak{J} = j + i\,{}^*j$ *can also be written as with* $j_1 = -\,{}^*j_2 = 0$ *and* $j_2 = {}^*j_1 = \frac{\sin\theta}{|q|}$.

Lemma 5.2.6. *In addition to* (5.2.5) *the remaining null Ricci and curvature coefficients are given by the following:*

1. *Relative to the ingoing PN frame* $\underline{\omega} = 0$ *and* $H = Z$. *Moreover* $P = -\frac{2m}{q^3}$ *and*

$$trX = \frac{2\Delta\overline{q}}{|q|^4}, \quad tr\underline{X} = -\frac{2}{\overline{q}}, \quad \omega = -\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right) = -\frac{a^2\cos^2\theta(r-m) + mr^2 - a^2r}{|q|^4}.$$

2. *Relative to the outgoing PN frame* $\omega = 0$, $\underline{H} = -Z$. *Moreover* $P = -\frac{2m}{q^3}$ *and*

$$trX = \frac{2}{q}, \quad tr\underline{X} = -\frac{2\Delta q}{|q|^4}, \quad \underline{\omega} = \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right) = \frac{a^2\cos^2\theta(r-m) + mr^2 - a^2r}{|q|^4}.$$

3. *In both cases, in view of the definition of* $\mathfrak{J}$,

$$\underline{H} \;=\; -\frac{a}{q}\mathfrak{J} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad H = \frac{a}{\overline{q}}\mathfrak{J} = \frac{aq}{|q|^2}\mathfrak{J}.$$

In particular $q\,\underline{H} = -\overline{q}H$ *and* $|H|^2 = |\,\underline{H}|^2$.

[1] While the basis (e_1, e_2) is not.

4. *The scalar q satisfies, for both the outgoing and incoming PN frames,*

$$\nabla_4 q = \frac{1}{2} trX q, \quad \nabla_3 q = \frac{1}{2} \overline{tr\underline{X}} q, \quad \mathcal{D} q = q\, \underline{H}, \quad \overline{\mathcal{D}} q = q\overline{H}. \tag{5.2.7}$$

5. *Moreover, in both frames,*

$$\begin{aligned} tr\,\chi\,^{(a)}tr\underline{\chi} + tr\,\underline{\chi}\,^{(a)}tr\chi &= 0, \\ |\eta|^2 - |\underline{\eta}|^2 &= 0, \\ div\,(\eta - \underline{\eta}) &= 0, \\ div\,(\,^*\eta + \,^*\underline{\eta}) &= 0, \end{aligned}$$

Proof. Straightforward verification. □

Lemma 5.2.7. *The complex 1-form $\mathfrak{J}$ verifies the following properties*

1. *We have*

$$^*\mathfrak{J} = -i\mathfrak{J}, \qquad \mathfrak{J} \cdot \overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}, \quad |\Re(\mathfrak{J})|^2 = \frac{(\sin\theta)^2}{|q|^2}.$$

2. *We have*

$$\overline{\mathcal{D}} \cdot \mathfrak{J} = \frac{4i(r^2+a^2)\cos\theta}{|q|^4}, \qquad \mathcal{D}\widehat{\otimes}\mathfrak{J} = 0,$$

3. *We have*

$$\mathcal{D}(q) = -a\mathfrak{J}, \qquad \mathcal{D}(\overline{q}) = a\mathfrak{J}.$$

4. *Relative to $e_3 = e_3^{(in)}, e_4 = e_4^{(in)}$*

$$\nabla_4 \mathfrak{J} + \frac{1}{2} trX\mathfrak{J} = 0, \qquad \nabla_3 \mathfrak{J} + \frac{1}{2} tr\underline{X}\mathfrak{J} = 0, \tag{5.2.8}$$

5. *We have*

$$\nabla_4(q\mathfrak{J}) = \nabla_3(\overline{q}\mathfrak{J}) = 0. \tag{5.2.9}$$

Proof. Straightforward verification. □

Lemma 5.2.8. *The vectorfields $\mathbf{T}, \mathbf{Z}, \widehat{T}, \widehat{R}$ can be expressed as follows*

- *If the normalization of* (e_3, e_4) *is ingoing, we have*

$$\begin{aligned}\mathbf{T} &:= \frac{1}{2}\left(e_4 + \frac{\Delta}{|q|^2}e_3 - 2a\Re(\mathfrak{J})^b e_b\right),\\ \mathbf{Z} &:= \frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_4 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_3\right).\end{aligned}$$

and

$$\widehat{T} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4^{(in)} + \frac{\Delta}{r^2+a^2}e_3^{(in)}\right), \quad \widehat{R} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4^{(in)} - \frac{\Delta}{r^2+a^2}e_3^{(in)}\right).$$

- *If the normalization of* (e_3, e_4) *is outgoing, we have*

$$\begin{aligned}\mathbf{T} &:= \frac{1}{2}\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right),\\ \mathbf{Z} &:= \frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_3 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_4\right).\end{aligned}$$

and

$$\widehat{T} = \frac{1}{2}\left(\frac{\Delta}{r^2+a^2}e_4^{(out)} + \frac{|q|^2}{r^2+a^2}e_3^{(out)}\right), \quad \widehat{R} = \frac{1}{2}\left(\frac{\Delta}{r^2+a^2}e_4^{(out)} - \frac{|q|^2}{r^2+a^2}e_3^{(out)}\right).$$

- *We also have the following relations, valid in any frame:*

$$\begin{aligned}{}^{(a)}tr\chi e_3 + {}^{(a)}tr\underline{\chi} e_4 &= \frac{4a\cos\theta(r^2+a^2)}{|q|^4}\widehat{T},\\ {}^{(a)}tr\chi e_3 + {}^{(a)}tr\underline{\chi} e_4 + 2(\eta+\underline{\eta})\cdot {}^*\nabla &= \frac{4a\cos\theta}{|q|^2}\mathbf{T},\\ {}^{(a)}tr\chi e_3 + {}^{(a)}tr\underline{\chi} e_4 - 4\frac{r^2+a^2}{|q|^3}\cot\theta e_2 &= -\frac{4\cos\theta}{|q|^2\sin^2\theta}\mathbf{Z}.\end{aligned} \tag{5.2.10}$$

5.2.1 Eddington-Finkelstein (EF) coordinates

Ingoing EF-coordinates

Let r_0 be a constant $r_0 > r_+$. We introduce the adapted ingoing Eddington-Finkelstein function $\underline{u}$ defined by[2]

$$\underline{u} \;=\; t + \int_{r_0}^{r}\frac{r'^2+a^2}{\Delta(r')}dr'.$$

[2]Note that the choice of u and $\underline{u}$ is such that we have $u = \underline{u} = t$ on the timelike hypersurface $r = r_0$.

Note that

$$e_4^{(in)}(\underline{u}) = \frac{2(r^2+a^2)}{|q|^2}, \qquad e_3^{(in)}(\underline{u}) = 0, \qquad e_1(\underline{u}) = 0, \qquad e_2(\underline{u}) = \frac{a\sin\theta}{|q|}.$$

Remark 5.2.9. *Note that the non-vanishing of $e_2(\underline{u})$ in Kerr is connected with the lack of integrability of the null pair $(e_3^{(in)}, e_4^{(in)})$.*

Definition 5.2.10. *The principal null pair $(e_3^{(in)}, e_4^{(in)})$ together with the BL function r, such that $e_3^{(in)}(r) = 1$, is called the canonical, ingoing, principal geodesic structure (PG) of Kerr. The associated, ingoing, Eddington-Finkelstein coordinates $(\underline{u}, r, \theta, \varphi_+)$ are given by*

$$\underline{u} := t + f(r), \quad f'(r) = \frac{r^2+a^2}{\Delta}, \qquad \varphi_+ := \phi + h(r), \quad h'(r) = \frac{a}{\Delta},$$

such that,

$$e_3^{(in)}(r) = 1, \qquad e_3^{(in)}(\underline{u}) = e_3^{(in)}(\theta) = e_3^{(in)}(\varphi_+) = 0.$$

In particular

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\underline{u}\partial_\beta\underline{u} = \frac{a^2\sin^2\theta}{|q|^2}$$

Outgoing EF coordinates

Let r_0 be a constant $r_0 > r_+$. We introduce the adapted outgoing Eddington-Finkelstein function u defined by

$$u \;\; := \;\; t - \int_{r_0}^{r} \frac{r'^2+a^2}{\Delta(r')}dr'.$$

Note that

$$e_4^{(out)}(u) = 0, \quad e_3^{(out)}(u) = \frac{2(r^2+a^2)}{|q|^2}, \quad e_1(u) = 0, \quad e_2(u) = \frac{a\sin\theta}{|q|},$$

and

$$\mathbf{g}(\mathbf{D}u, \mathbf{D}u) = \frac{a^2\sin^2\theta}{|q|^2}.$$

Definition 5.2.11. *The principal null pair* $(e_3^{(out)}, e_4^{(out)})$ *together with the BL function* r, *such that* $e_4^{(out)}(r) = 1$, *is called the canonical, outgoing, PG structure of Kerr. The associated, outgoing, Eddington-Finkelstein coordinates* $(u, r, \theta, \varphi_-)$ *are given by*

$$u := t - f(r), \quad f'(r) = \frac{r^2 + a^2}{\Delta}, \qquad \varphi_- := \phi - h(r), \quad h'(r) = \frac{a}{\Delta},$$

such that,

$$e_4^{(out)}(r) = 1, \qquad e_4^{(out)}(u) = e_4^{(out)}(\theta) = e_4^{(out)}(\varphi_-) = 0.$$

Lemma 5.2.12. *Relative to the outgoing Eddington-Finkelstein coordinates* $(u, r, \theta, \varphi_-)$ *we have:*

1. *The action of the outgoing PG frame on the coordinates* $(u, r, \theta, \varphi_-)$ *is given by*

$$\begin{array}{llll} e_4(r) = 1, & e_4(u) = 0, & e_4(\theta) = 0, & e_4(\varphi_-) = 0, \\ e_3(r) = -\frac{\Delta}{|q|^2}, & e_3(u) = \frac{2(r^2+a^2)}{|q|^2}, & e_3(\theta) = 0, & e_3(\varphi_-) = \frac{2a}{|q|^2}, \\ e_1(r) = 0, & e_1(u) = 0, & e_1(\theta) = \frac{1}{|q|}, & e_1(\varphi_-) = 0, \\ e_2(r) = 0, & e_2(u) = \frac{a\sin\theta}{|q|}, & e_2(\theta) = 0, & e_2(\varphi_-) = \frac{1}{|q|\sin\theta}. \end{array} \tag{5.2.11}$$

2. *In particular*

$$\begin{pmatrix} e_4 \\ e_3 \\ e_2 \\ e_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\Delta}{|q|^2} & \frac{2(r^2+a^2)}{|q|^2} & 0 & \frac{2a}{|q|^2} \\ 0 & \frac{a\sin\theta}{|q|} & 0 & \frac{1}{|q|\sin\theta} \\ 0 & 0 & \frac{1}{|q|} & 0 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_u \\ \partial_\theta \\ \partial_{\varphi_-} \end{pmatrix}.$$

3. *In the outgoing EF coordinates, the metric takes the form*

$$\begin{aligned} \mathbf{g} \;=\; & -\left(1 - \frac{2mr}{|q|^2}\right)(du)^2 - 2drdu + 2a(\sin\theta)^2 drd\varphi_- \\ & -\frac{4mra(\sin\theta)^2}{|q|^2} dud\varphi_- + |q|^2(d\theta)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}(d\varphi_-)^2. \end{aligned}$$

Proof. Straightforward verification. $\square$

5.2.2 Aymptotically null optical time function

Lemma 5.2.13. *There exists a function τ, defined for $r \geq r_+ - \delta_{\mathcal{H}}$ which verifies the following properties:*

$$\mathbf{g}(N_\Sigma, N_\Sigma) \leq -\frac{m^2}{8r^2}, \qquad e_4(\tau) > 0, \qquad e_3(\tau) > 0, \qquad |\nabla\tau|^2 \leq \frac{8}{9} e_4(\tau) e_3(\tau).$$

with N_Σ the future unit normal to the level surfaces $\Sigma - \Sigma(\tau)$ of τ. In addition, for r large,

$$\frac{m^2}{r^2} \lesssim e_4(\tau) \lesssim \frac{m^2}{r^2}, \qquad 1 \lesssim e_3(\tau) \lesssim 1.$$

Moreover we can choose τ s.t.

$$\mathbf{T}(\tau) = 1, \qquad \nabla(\tau) = a\Re(\mathfrak{J}).$$

Proof. See Section D.3.1 in [K-S:Kerr]. □

5.3 Inverse Kerr metric and Killing tensors

Lemma 5.3.1. *The inverse Kerr metric can be written in the form*

$$|q|^2 \mathbf{g}^{\alpha\beta} = \Delta \partial_r^\alpha \partial_r^\beta + \frac{1}{\Delta} \mathcal{R}^{\alpha\beta} \tag{5.3.1}$$

with

$$\begin{aligned} \mathcal{R}^{\alpha\beta} &= -(r^2+a^2)^2 \partial_t^\alpha \partial_t^\beta - 2a(r^2+a^2) \partial_t^{(\alpha} \partial_\phi^{\beta)} - a^2 \partial_\phi^\alpha \partial_\phi^\beta + \Delta O^{\alpha\beta}, \\ O^{\alpha\beta} &= \partial_\theta^\alpha \partial_\theta^\beta + \frac{1}{\sin^2\theta} \partial_\phi^\alpha \partial_\phi^\beta + 2a \partial_t^{(\alpha} \partial_\phi^{\beta)} + a^2 \sin^2\theta \partial_t^\alpha \partial_t^\beta. \end{aligned} \tag{5.3.2}$$

Note that

$$\mathcal{R}^{\alpha\beta} = -(r^2+a^2)^2 \widehat{T}^\alpha \widehat{T}^\beta + \Delta O^{\alpha\beta}, \qquad O^{\alpha\beta} = |q|^2 \big(e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta \big), \tag{5.3.3}$$

thus the inverse metric can also be written in the form

$$|q|^2 \mathbf{g}^{\alpha\beta} \;\; = \;\; \frac{(r^2+a^2)^2}{\Delta} \big(-\widehat{T}^\alpha \widehat{T}^\beta + \widehat{R}^\alpha \widehat{R}^\beta \big) + O^{\alpha\beta}. \tag{5.3.4}$$

Proof. From the expression of the Kerr metric, the inverse metric can be written in the form

$$|q|^2\mathbf{g}^{\alpha\beta} = \Delta\partial_r^\alpha\partial_r^\beta + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}$$

with

$$\mathcal{R}^{\alpha\beta} \quad = \quad -\Sigma^2\partial_t^\alpha\partial_t^\beta - 2amr\partial_t^\alpha\partial_\phi^\beta - 2amr\partial_\phi^\alpha\partial_t^\beta + \Delta\partial_\theta^\alpha\partial_\theta^\beta + \frac{\Delta - a^2\sin^2\theta}{\sin^2\theta}\partial_\phi^\alpha\partial_\phi^\beta$$

which establishes (5.3.2). According to the definition (5.1.4) of $\widehat{T}$, we can write

$$\begin{aligned}\mathcal{R}^{\alpha\beta} &= -(r^2+a^2)^2\left(\partial_t^\alpha\partial_t^\beta + \frac{2a}{r^2+a^2}\partial_t^{(\alpha}\partial_\phi^{\beta)} + \frac{a^2}{(r^2+a^2)^2}\partial_\phi^\alpha\partial_\phi^\beta\right) + \Delta O^{\alpha\beta}\\ &= -(r^2+a^2)^2\widehat{T}^\alpha\widehat{T}^\beta + \Delta O^{\alpha\beta}\end{aligned}$$

which establishes the first expression in (5.3.3). Finally the second expression (5.3.3) can be easily checked from the expressions of e_1, e_2 in terms of the BL coordinates in (5.2.2). □

The relevance of the decomposition of the metric in (5.3.1) is in the fact that the operator $\mathcal{R}^{\alpha\beta}$ can be written in terms of $\partial_t^\alpha\partial_t^\beta$, $a\partial_t^{(\alpha}\partial_\phi^{\beta)}$, $a^2\partial_\phi^\alpha\partial_\phi^\beta$ and $O^{\alpha\beta}$ where $\partial_t, \partial_\varphi$ are Killing vectors and O is intimately related to the Carter operator as discussed in the next section.

Definition 5.3.2. *We define the following symmetric spacetime 2-tensors*

$$\begin{aligned}S_1^{\alpha\beta} &:= \mathbf{T}^\alpha\mathbf{T}^\beta = \partial_t^\alpha\partial_t^\beta,\\ S_2^{\alpha\beta} &:= a\mathbf{T}^{(\alpha}\mathbf{Z}^{\beta)} = a\partial_t^{(\alpha}\partial_\phi^{\beta)},\\ S_3^{\alpha\beta} &:= a^2\mathbf{Z}^\alpha\mathbf{Z}^\beta = a^2\partial_\phi^\alpha\partial_\phi^\beta,\\ S_4^{\alpha\beta} &:= O^{\alpha\beta} = |q|^2\big(e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta\big).\end{aligned}$$

We denote the set of the above tensors as $S_{\underline{a}}$, *for* $\underline{a} = 1, 2, 3, 4$.

With the above definition, from (5.3.2) we write

$$\mathcal{R}^{\alpha\beta} = -(r^2+a^2)^2 S_1^{\alpha\beta} - 2(r^2+a^2)S_2^{\alpha\beta} - S_3^{\alpha\beta} + \Delta S_4^{\alpha\beta}. \tag{5.3.5}$$

More compactly, using the repetition in $\underline{a}$ to signify summation over $\underline{a} = 1, 2, 3, 4$, we denote

$$\mathcal{R}^{\alpha\beta} = \mathcal{R}^{\underline{a}} S_{\underline{a}}^{\alpha\beta}, \tag{5.3.6}$$

with $\mathcal{R}^{\underline{a}}$, $\underline{a} = 1, 2, 3, 4$, given by

$$\mathcal{R}^1 = -(r^2+a^2)^2, \quad \mathcal{R}^2 = -2(r^2+a^2), \quad \mathcal{R}^3 = -1, \quad \mathcal{R}^4 = \Delta. \tag{5.3.7}$$

5.4 Carter tensor and Carter operator

One of the fundamental properties of the Kerr metric is that it admits a Killing tensor[3] K which cannot be written in terms of the Killing vectorfields $\mathbf{T}$ and $\mathbf{Z}$. This 2-tensor is called the Carter tensor [Carter].

Definition 5.4.1 (Carter tensor)**.** *The Carter tensor is defined by*

$$K^{\alpha\beta} \;=\; -(a^2\cos^2\theta)\mathbf{g}^{\alpha\beta} + O^{\alpha\beta} \tag{5.4.1}$$

where the tensor O is defined in (5.3.2)*.*

Note that the only non-vanishing components of K are:

$$K_{ab} \;=\; r^2\delta_{ab}, \qquad K_{34} = 2a^2\cos^2\theta.$$

Proposition 5.4.2. *The Carter tensor defined in* (5.4.1) *is a Killing tensor of the Kerr metric, i.e.* $\mathbf{D}_{(\mu}K_{\nu\rho)} = 0$.

Proof. See Proposition 3.7.2 in [GKS-2022]. □

Given $K^{\alpha\beta}$ we associate to it the second order operator $\mathcal{K}$

$$\mathcal{K}(\psi) := \dot{\mathbf{D}}_\beta(K^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi),$$

which has the fundamental property of commuting with the D'Alembertian operator for scalars in a vacuum spacetime, see Proposition 2.3.7 in [GKS-2022]. Its explicit expression in Kerr for K as in (5.4.1) is given in the following.

Proposition 5.4.3. *In Kerr spacetime, the Carter operator $\mathcal{K}$ for $\psi \in \mathfrak{s}_k$ is given by*

$$\mathcal{K} \;=\; -(a^2\cos^2\theta)\dot{\square}_k + \mathcal{O} \tag{5.4.2}$$

where $\mathcal{O}$ is the following second order angular operator:

$$\mathcal{O}(\psi) \;:=\; |q|^2\big(\triangle_k\psi + (\eta + \underline{\eta})\cdot\nabla\psi\big). \tag{5.4.3}$$

Proof. See Section **??**. □

[3] A Killing 2-tensor K is a symmetric 2-tensor satisfying $\mathbf{D}_{(\mu}K_{\alpha\beta)} = 0$.

Lemma 5.4.4. *In Kerr, the second order operator $\mathcal{O}$ defined in* (5.4.3) *is equivalent to*

$$\mathcal{O}(\psi) \quad = \quad |q|^2\left(\triangle_k\psi - \frac{2a^2\cos\theta}{|q|^2}\;{}^*\Re(\mathfrak{J})^b\nabla_b\psi\right). \tag{5.4.4}$$

Also, the operator $\mathcal{O}$ can be written in the following ways, all equivalent to (5.4.3)*:*

$$\begin{aligned}
&\mathcal{O}(\psi) = |q|^2\left(\triangle_k\psi + \frac{2\nabla(|q|)}{|q|}\cdot\nabla\psi\right),\\
&\mathcal{O}(\psi) = \nabla\cdot\left(|q|^2\nabla\psi\right),\\
&\mathcal{O}(\psi) = \dot{\mathbf{D}}_\beta(O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi) - \nabla(a^2\cos^2\theta)\cdot\nabla\psi,\\
&\mathcal{O}(\psi) = |q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi).
\end{aligned} \tag{5.4.5}$$

Proof. See Lemma 3.7.4 in [GKS-2022] □

Proposition 5.4.5. *In Kerr, for scalars $\psi\in\mathfrak{s}_0$, the operator $\mathcal{O}$ commutes with $|q|^2\dot{\square}_2$. This is no longer true for tensors $\psi\in\mathfrak{s}_2$. We have however, for $\psi\in\mathfrak{s}_2$,*

$$[\mathcal{O}, |q|^2\dot{\square}_2]\psi = |q|^2\left[\nabla\left(\frac{8a(r^2+a^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\widehat{T}}\;{}^*\psi + O(ar^{-2})\nabla_{\widehat{R}}^{\leq 1}\mathfrak{d}^{\leq 1}\psi\right], \tag{5.4.6}$$

where $\mathfrak{d}=(\nabla_3, r\nabla_4, r\nabla)$ denotes weighted derivatives as in [K-S:Schw] and [K-S:Kerr].

Proof. See Proposition 4.5.3 in [GKS-2022]. This should be redone. □

5.5 Null geodesics in Kerr

5.5.1 The constants of motion for geodesics

Let $\gamma(\lambda)$ be a null geodesic in Kerr. Using the expression for the inverse of the metric given by (5.3.1), along $\gamma(\lambda)$, since $\mathbf{g}(\dot\gamma,\dot\gamma)=0$ we have, with $\dot\gamma_r=\partial_r^\alpha\dot\gamma_\alpha$, $\dot\gamma_t=\partial_t^\alpha\dot\gamma_\alpha$, $\dot\gamma_\varphi=\partial_\varphi^\alpha\dot\gamma_\alpha$

$$0 = |q|^2\mathbf{g}^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta = \left(\Delta\partial_r^\alpha\partial_r^\beta + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot\gamma_\alpha\dot\gamma_\beta = \Delta\dot\gamma_r\dot\gamma_r + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta$$

with

$$\mathcal{R}^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta \quad = \quad -(r^2+a^2)^2\dot\gamma_t\dot\gamma_t - 2a(r^2+a^2)\dot\gamma_t\dot\gamma_\varphi - a^2\dot\gamma_\varphi\dot\gamma_\varphi + \Delta O^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta.$$

Since $\partial_t = T$ and $\partial_\varphi = Z$ are Killing vectorfields, we deduce that $\dot\gamma_t = \mathbf{g}(\dot\gamma, T)$ and $\dot\gamma_\varphi = \mathbf{g}(\dot\gamma, Z)$ are constants of the motion, i.e. constants along γ, and respectively called the energy and the azimuthal angular momentum. We write,

$$\mathbf{e} := -\mathbf{g}(\dot\gamma, T), \qquad \ell_{\mathbf{z}} := -\mathbf{g}(\dot\gamma, Z).$$

We also define[4]

$$\mathbf{k}^2 \;\; := \;\; K^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta$$

for the Carter tensor K in Kerr. Since K is Killing, $\mathbf{k}^2$ is also a constant of motion. Indeed, we have

$$\frac{d}{d\lambda}\mathbf{k}^2\big(\gamma(\lambda)\big) \;\; = \;\; \mathbf{D}_\lambda K_{\alpha\beta}\,\dot\gamma^\lambda\dot\gamma^\alpha\dot\gamma^\beta = 0.$$

Since from (5.4.1), $K = -(a^2\cos^2\theta)\mathbf{g} + O$ and γ is null we deduce, with $\dot\gamma_a = \mathbf{g}(\dot\gamma, e_a)$

$$\mathbf{k}^2 \;\; = \;\; O^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta = |q|^2\big(e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta\big)\dot\gamma_\alpha\dot\gamma_\beta = |q|^2\big(|\dot\gamma_1|^2 + |\dot\gamma_2|^2\big).$$

We summarize the result in the following.

Proposition 5.5.1. *The quantities*

$$\mathbf{e} = -\mathbf{g}(\dot\gamma, T), \qquad \ell_{\mathbf{z}} = -\mathbf{g}(\dot\gamma, Z), \qquad \mathbf{k}^2 = K^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta,$$

are constant along null geodesics. Moreover, relative to the null frame

$$\mathbf{k}^2 \;\; = |q|^2\big(|\dot\gamma_1|^2 + |\dot\gamma_2|^2\big).$$

With these constants we have

$$\begin{aligned}\mathcal{R}^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta &= -(r^2+a^2)^2\dot\gamma_t\dot\gamma_t - 2a(r^2+a^2)\dot\gamma_t\dot\gamma_\varphi - a^2\dot\gamma_\varphi\dot\gamma_\varphi + \Delta O^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta\\ &= -(r^2+a^2)^2\mathbf{e}^2 - 2a(r^2+a^2)\mathbf{e}\,\ell_{\mathbf{z}} - a^2{\ell_{\mathbf{z}}}^2 + \Delta\mathbf{k}^2\end{aligned}$$

which is only a function of r along any fixed null geodesic. We introduce the notation

$$\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) := -(r^2+a^2)^2\mathbf{e}^2 - 2a(r^2+a^2)\mathbf{e}\,\ell_{\mathbf{z}} - a^2{\ell_{\mathbf{z}}}^2 + \Delta\mathbf{k}^2.$$

Note that we have the identity

$$-\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) \;\; = \big((r^2+a^2)\mathbf{e} + a\ell_{\mathbf{z}}\big)^2 - \Delta\mathbf{k}^2. \tag{5.5.1}$$

[4]Observe that $\mathbf{k}^2$ is a positive constant of motion.

In view of the above, we infer that

$$0 = \Delta\dot{\gamma}_r\dot{\gamma}_r + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta = \Delta\dot{\gamma}_r\dot{\gamma}_r + \frac{1}{\Delta}\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}).$$

Since

$$\frac{dr}{d\lambda} = \dot{\gamma}^\alpha\frac{\partial r}{\partial\alpha} = \dot{\gamma}^r = \mathbf{g}^{rr}\dot{\gamma}_r = \frac{\Delta}{|q|^2}\dot{\gamma}_r,$$

we finally obtain

$$|q|^4\left(\frac{dr}{d\lambda}\right)^2 = -\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k})$$

which is the equation for a null geodesic with constants of motion $\mathbf{e}$, $\ell_{\mathbf{z}}$, $\mathbf{k}$.

5.5.2 Trapped null geodesics

There exist null geodesics along which $\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{q}) = 0$ i.e. r remains constant. These are called orbital null geodesics.

Remark 5.5.2. *Trapped null geodesics correspond to null geodesics that stay in a region $[r_1, r_2]$ of r with $r_+ < r_1 < r_2 < +\infty$ for all values of λ, and are thus a priori more general than orbital null geodesics. As it turns out, see for example Proposition 2 in [**?**], all trapped null geodesics in Kerr are in fact orbital null geodesics. Thus, from now on, we do not distinguish between trapped and orbital null geodesics.*

If r is constant we also have

$$-\partial_r\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) = \partial_r(|q|^4)\left(\frac{dr}{d\lambda}\right)^2 + 2|q|^4\frac{dr}{d\lambda}\partial_r\left(\frac{dr}{d\lambda}\right) = 0.$$

The r values for which such solutions are possible must then verify the equations

$$\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) = \partial_r\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) = 0.$$

Thus, introducing

$$\Pi := (r^2 + a^2)\mathbf{e} + a\ell_{\mathbf{z}}$$

we write from (5.5.1)

$$\begin{aligned} -\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) &= \Pi^2 - \Delta\mathbf{k}^2 = 0, \\ -\partial_r\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) &= 2\Pi(\partial_r\Pi) - (\partial_r\Delta)\mathbf{k}^2 = 0. \end{aligned}$$

From the second equation, we deduce

$$\mathbf{k}^2 = 2\Pi \frac{\partial_r \Pi}{\partial_r \Delta}. \tag{5.5.2}$$

Thus, substituting in the first equation,

$$\Pi^2 - 2\Pi \frac{\Delta \partial_r \Pi}{\partial_r \Delta} = 0$$

or, if $\Pi \neq 0$,

$$\Pi(\partial_r \Delta) - 2(\partial_r \Pi)\Delta = 0. \tag{5.5.3}$$

We make use of the following calculation.

Lemma 5.5.3. *We have the identity*

$$\Pi(\partial_r \Delta) - 2(\partial_r \Pi)\Delta = -2\mathcal{T}_{\mathbf{e},\ell_{\mathbf{z}}} \tag{5.5.4}$$

where

$$\mathcal{T}_{\mathbf{e},\ell_{\mathbf{z}}} := \big(r^3 - 3mr^2 + ra^2 + ma^2\big)\mathbf{e} - (r-m)a\ell_{\mathbf{z}}.$$

Proof. We have

$$\begin{aligned}
(\partial_r \Delta)\Pi - 2\Delta(\partial_r \Pi) &= 2(r-m)\big((r^2+a^2)\mathbf{e} + a\ell_{\mathbf{z}}\big) - 4r\big(r^2+a^2-2rm\big)\mathbf{e} \\
&= 2\Big((r-m)\big((r^2+a^2)\mathbf{e} + a\ell_{\mathbf{z}}\big) - 2r\big(r^2+a^2-2rm\big)\mathbf{e}\Big) \\
&= 2\Big(\big(-r^3 + 3mr^2 - ra^2 - ma^2\big)\mathbf{e} + (r-m)a\ell_{\mathbf{z}}\Big) \\
&= -2\mathcal{T}_{\mathbf{e},\ell_{\mathbf{z}}}
\end{aligned}$$

as stated. □

As a consequence of the Lemma we deduce that all orbital null geodesics are given by the equation

$$\mathcal{T}_{\mathbf{e},\ell_{\mathbf{z}}} = \big(r^3 - 3mr^2 + ra^2 + ma^2\big)\mathbf{e} - (r-m)a\ell_{\mathbf{z}} = 0.$$

Remark 5.5.4. *The following hold true.*

1. *There are no trapped null geodesics perpendicular to $T = \partial_t$ in the exterior of a non-extremal Kerr.*

2. *The values of r for which trapped null geodesics exist depends on the ratio $\ell_{\mathbf{z}}/\mathbf{e}$. More precisely, at trapped null geodesics, we have*

$$\frac{r^3 - 3mr^2 + ra^2 + ma^2}{r - m} = \frac{a\ell_{\mathbf{z}}}{\mathbf{e}}.$$

In particular, for $\ell_{\mathbf{z}} = 0$, the trapped null geodesics are given by the equation

$$\mathcal{T} := r^3 - 3mr^2 + a^2r + a^2m = 0. \tag{5.5.5}$$

Remark 5.5.5. *Note that one may specify possible values of r for which trapped null geodesics exists. Indeed, let*

$$\begin{aligned}\hat{r}_1 &:= 2m\left(1 + \cos\left(\frac{2}{3}\arccos\left(-\frac{|a|}{m}\right)\right)\right),\\ \hat{r}_2 &:= 2m\left(1 + \cos\left(\frac{2}{3}\arccos\left(\frac{|a|}{m}\right)\right)\right).\end{aligned} \tag{5.5.6}$$

Then, a trapped null geodesics satisfies $r \in [\hat{r}_1, \hat{r}_2]$, see for example [Teo].

Lemma 5.5.6. *The trapped null geodesics are unstable, i.e.*

$$\partial_r^2 \mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) < 0.$$

Proof. We have

$$\Pi = (r^2 + a^2)\mathbf{e} + a\ell_{\mathbf{z}}, \quad \partial_r\Pi = 2r\mathbf{e}, \quad \partial_r^2\Pi = 2\mathbf{e},$$

and using (5.5.2) to write $\mathbf{k}^2 = 4r\frac{\Pi}{\partial_r\Delta}\mathbf{e}$, we have

$$\begin{aligned}-\partial_r^2\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) &= 2(\partial_r\Pi)^2 + 2\Pi(\partial_r^2\Pi) - 2\mathbf{k}^2 = 8r^2\mathbf{e}^2 + 4\Pi\mathbf{e} - 8r\frac{\Pi}{\partial_r\Delta}\mathbf{e}\\ &= \frac{4}{\partial_r\Delta}\left(2r^2\partial_r\Delta\mathbf{e}^2 + (\partial_r\Delta)\Pi\mathbf{e} - 2r\Pi\mathbf{e}\right)\\ &= \frac{4\mathbf{e}}{\partial_r\Delta}\left(4r^2(r - m)\mathbf{e} - 2m\Pi\right).\end{aligned}$$

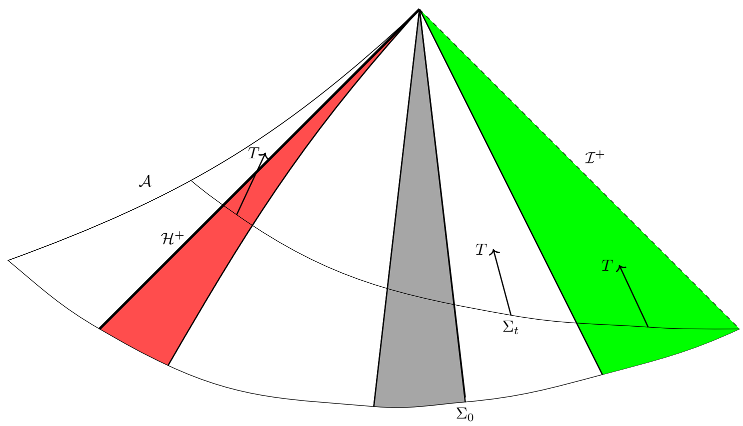
Using (5.5.3) to write $\Pi = \frac{4r\Delta\mathbf{e}}{\partial_r\Delta}$ we deduce

$$\begin{aligned}-\partial_r^2\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) &= \frac{4\mathbf{e}}{\partial_r\Delta}\left(4r^2(r - m)\mathbf{e} - 2m\frac{4r\Delta\mathbf{e}}{\partial_r\Delta}\right) = \frac{32r\mathbf{e}^2}{(\partial_r\Delta)^2}\left(r(r - m)^2 - m\Delta\right)\\ &= \frac{8r}{(r - m)^2}\mathbf{e}^2\Big(r(r - m)^2 - m(r^2 + a^2 - 2mr)\Big)\\ &= \frac{8r}{(r - m)^2}\mathbf{e}^2\Big((r - m)^3 + m(m^2 - a^2)\Big),\end{aligned}$$

which is positive since $r \geq m$ and $|a| \leq m$. □

Remark 5.5.7. *The trapped region is disjoint from the ergoregion if $\frac{|a|}{m} < 0.7$.*

Figure 5.2: Penrose diagram for the future of a spacelike hypersurface in the exterior of $\mathcal{K}(a, m)$ when $|a|/m$ is sufficiently small so that the trapped region is separated from the ergoregion. $\mathcal{H}^+$ is the future event $r = r_+$ and $\mathcal{A}$ a spacelike hypersurface inside the black hole.



.1 Direct proof of the complex Bianchi identities

We give a direct proof of Proposition 3.4.17 using the complex for of the Bianchi identities

$$\mathbf{D}^\alpha \mathcal{R}_{\alpha\beta\gamma\delta} = 0.$$

and the relations,

Lemma .1.1. *Let $\mathcal{R}_{\mu\nu\gamma\delta} = R_{\mu\nu\gamma\delta} + i\ {}^*R_{\mu\nu\gamma\delta}$. We have*

$$\begin{cases} \mathcal{R}_{4a4b} = \overline{A}_{ab} \\ \mathcal{R}_{3a3b} = \underline{A}_{ab} \\ \mathcal{R}_{a434} = 2\overline{B}_a \\ \mathcal{R}_{a334} = 2\underline{B}_a \\ \mathcal{R}_{4343} = 4P \\ \mathcal{R}_{a3b4} = -P\delta_{ab} + \ {}^*P \in_{ab} \\ \mathcal{R}_{ab34} = 2 \in_{ab} \ {}^*P \\ \mathcal{R}_{abc3} = -i \in_{ab} \underline{B}_c = \in_{ab} \ {}^*\underline{B}_c \\ \mathcal{R}_{abc4} = -i \in_{ab} \overline{B}_c = - \in_{ab} \ {}^*\overline{B}_c \end{cases}$$

Proof. Direct verification form the definition of $\mathcal{R} = \mathbf{R} + i\ {}^*\mathbf{R}$. □

According to Proposition 3.4.17 we have to check the following:

$$\begin{aligned} \nabla_3 A - \frac{1}{2}\mathcal{D}\widehat{\otimes} B &= -\frac{1}{2}\mathrm{tr}\underline{X} A + 4\underline{\omega} A + \frac{1}{2}(Z + 4H)\widehat{\otimes} B - 3\overline{P}\widehat{X}, \\ \nabla_4 B - \frac{1}{2}\overline{\mathcal{D}} \cdot A &= -2\overline{\mathrm{tr}X} B - 2\omega B + \frac{1}{2}A \cdot (\overline{2Z + \underline{H}}) + 3\overline{P}\,\Xi, \\ \nabla_3 B - \mathcal{D}\overline{P} &= -\mathrm{tr}\underline{X} B + 2\underline{\omega} B + \overline{\underline{B}} \cdot \widehat{X} + 3\overline{P} H + \frac{1}{2}A \cdot \overline{\underline{\Xi}}, \\ \nabla_4 P - \frac{1}{2}\mathcal{D} \cdot \overline{B} &= -\frac{3}{2}\mathrm{tr}X P + \frac{1}{2}(2\,\underline{H} + Z) \cdot \overline{B} - \overline{\Xi} \cdot \underline{B} - \frac{1}{4}\underline{\widehat{X}} \cdot \overline{A}, \\ \\ \nabla_3 P + \frac{1}{2}\overline{\mathcal{D}} \cdot \underline{B} &= -\frac{3}{2}\overline{\mathrm{tr}\underline{X}} P - \frac{1}{2}(\overline{2H - Z}) \cdot \underline{B} + \underline{\Xi} \cdot \overline{B} - \frac{1}{4}\overline{\widehat{X}} \cdot \underline{A}, \\ \nabla_4 \underline{B} + \mathcal{D} P &= -\mathrm{tr}X \underline{B} + 2\omega \underline{B} + \overline{B} \cdot \underline{\widehat{X}} - 3P\,\underline{H} - \frac{1}{2}\underline{A} \cdot \overline{\Xi}, \\ \nabla_3 \underline{B} + \frac{1}{2}\overline{\mathcal{D}} \cdot \underline{A} &= -2\overline{\mathrm{tr}\underline{X}}\,\underline{B} - 2\underline{\omega}\,\underline{B} - \frac{1}{2}\underline{A} \cdot (\overline{-2Z + H}) - 3P\,\underline{\Xi}, \\ \nabla_4 \underline{A} + \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} &= -\frac{1}{2}\mathrm{tr}X \underline{A} + 4\omega \underline{A} + \frac{1}{2}(Z - 4\,\underline{H})\widehat{\otimes}\underline{B} - 3P\underline{\widehat{X}}. \end{aligned}$$

Proof. We start with deriving the $\nabla_4 P$ equation, because the $\nabla_4 P$ is needed in order to do later Bianchi calculations in this proof.

Derivation of $\nabla_4 P$ Equation: We start with $0 = \mathbf{D}^\alpha \mathcal{R}_{\alpha 434}$ or $\frac{1}{2}\mathbf{D}_4 \mathcal{R}_{3434} = \mathbf{D}^b \mathcal{R}_{b434}$.. For the left-hand side, we compute

$$\begin{aligned}
\mathbf{D}_4 \mathcal{R}_{3434} =& e_4(\mathcal{R}(e_3, e_4, e_3, e_4)) - \mathcal{R}(\mathbf{D}_4 e_3, e_4, e_3, e_4) - \mathcal{R}(e_3, \mathbf{D}_4 e_4, e_3, e_4) \\
& - \mathcal{R}(e_3, e_4, \mathbf{D}_4 e_3, e_4) - \mathcal{R}(e_3, e_4, e_3, \mathbf{D}_4 e_4) \\
=& e_4(\mathcal{R}(e_3, e_4, e_3, e_4)) - 2\mathcal{R}(\mathbf{D}_4 e_3, e_4, e_3, e_4) - 2\mathcal{R}(e_3, \mathbf{D}_4 e_4, e_3, e_4) \\
=& e_4(4P) - 2\mathcal{R}(2\omega e_3 + 2\underline{\eta}_a e_a, e_4, e_3, e_4) - 2\mathcal{R}(e_3, -2\omega e_4 + 2\xi_a e_a, e_3, e_4) \\
=& 4\nabla_4 P - 4\omega \mathcal{R}(e_3, e_4, e_3, e_4) - 4\underline{\eta}_a \mathcal{R}(e_a, e_4, e_3, e_4) \\
& + 4\omega \mathcal{R}(e_3, e_4, e_3, e_4) - 4\xi_a \mathcal{R}(e_3, e_a, e_3, e_4) \\
=& 4\nabla_4 P - 4\underline{\eta}_a \mathcal{R}(e_a, e_4, e_3, e_4) - 4\xi_a \mathcal{R}(e_3, e_a, e_3, e_4) \\
=& 4\nabla_4 P - 4\underline{\eta}_a (2\overline{B_a}) - 4\xi_a(-2\underline{B}_a) \\
=& 4\nabla_4 P - 8\underline{\eta}_a \overline{B_a} + 8\xi_a \underline{B}_a \\
=& 4\nabla_4 P - 8\left(\frac{1}{2}\,\underline{H} \cdot \overline{B}\right) + 8\left(\frac{1}{2}\overline{\Xi} \cdot \underline{B}\right) \qquad \text{(by Lemma 2.4.4)} \\
=& 4\nabla_4 P - 4\,\underline{H} \cdot \overline{B} + 4\overline{\Xi} \cdot \underline{B}
\end{aligned}$$

and for the right-hand side we compute

$$\begin{aligned}
&\mathbf{D}_a \mathcal{R}_{b434} \\
&= e_a(\mathcal{R}(e_b, e_4, e_3, e_4)) - \mathcal{R}(\mathbf{D}_a e_b, e_4, e_3, e_4) - \mathcal{R}(e_b, \mathbf{D}_a e_4, e_3, e_4) \\
&\quad - \mathcal{R}(e_b, e_4, \mathbf{D}_a e_3, e_4) - \mathcal{R}(e_b, e_4, e_3, \mathbf{D}_a e_4) \\
&= e_a(\mathcal{R}(e_b, e_4, e_3, e_4)) - \mathcal{R}(\nabla_a e_b + \frac{1}{2}\chi_{ab} e_3 + \frac{1}{2}\underline{\chi}_{ab} e_4, e_4, e_3, e_4) - \mathcal{R}(e_b, \chi_{ac} e_c - \zeta_a e_4, e_3, e_4) \\
&\quad - \mathcal{R}(e_b, e_4, \underline{\chi}_{ac} e_c + \zeta_a e_3, e_4) - \mathcal{R}(e_b, e_4, e_3, \chi_{ac} e_c - \zeta_a e_4) \\
&= \nabla_a(2\overline{B_b}) - \frac{1}{2}\chi_{ab}\mathcal{R}_{3434} - \chi_{ac}\mathcal{R}_{bc34} + \zeta_a \mathcal{R}_{b434} - \underline{\chi}_{ac}\mathcal{R}_{b4c4} - \zeta_a \mathcal{R}_{b434} - \chi_{ac}\mathcal{R}_{b43c} + \zeta_a \mathcal{R}_{b434} \\
&= 2\nabla_a \overline{B_b} - \frac{1}{2}\chi_{ab}\mathcal{R}_{3434} - \chi_{ac}\mathcal{R}_{bc34} + \zeta_a \mathcal{R}_{b434} - \underline{\chi}_{ac}\mathcal{R}_{b4c4} + \chi_{ac}\mathcal{R}_{c3b4} \\
&= 2\nabla_a \overline{B_b} - \frac{1}{2}\chi_{ab}(4P) - \chi_{ac}(2 \in_{bc} {}^*P) + \zeta_a(2\overline{B_b}) - \underline{\chi}_{ac}\overline{A}_{bc} + \chi_{ac}(-P\delta_{cb} + {}^*P \in_{cb}) \\
&= 2\nabla_a \overline{B_b} - 2\chi_{ab} P - 2\chi_{ac} \in_{bc} {}^*P + 2\zeta_a \overline{B_b} - \underline{\chi}_{ac}\overline{A}_{bc} - \chi_{ab} P + {}^*P \in_{cb} \chi_{ac} \\
&= 2\nabla_a \overline{B_b} - 3\chi_{ab} P + 3\, {}^*P \in_{cb} \chi_{ac} + 2\zeta_a \overline{B_b} - \underline{\chi}_{ac}\overline{A}_{bc} \\
&= 2\nabla_a \overline{B_b} - 3\chi_{ab} P - \underline{\chi}_{ac}\overline{A}_{bc} + 3\, {}^*P \chi_{ab}{}^* + 2\zeta_a \overline{B_b}
\end{aligned}$$

We deduce

$$
\begin{aligned}
\mathbf{D}^b\mathcal{R}_{b434} &= 2\mathrm{d}iv\,\overline{B} - 3\mathrm{tr}\chi P - \underline{\chi}\cdot\overline{A} - 3\,{}^*P\,{}^{(a)}\mathrm{tr}\chi + 2\zeta\cdot\overline{B}\\
&= 2\frac{1}{2}\mathcal{D}\cdot\overline{B} - 3\mathrm{tr}\chi P - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{A} + 3iP\,{}^{(a)}\mathrm{tr}\chi + Z\cdot\overline{B}\\
&= \mathcal{D}\cdot\overline{B} - 3(\mathrm{tr}\chi - i\,{}^{(a)}\mathrm{tr}\chi)P - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{A} + Z\cdot\overline{B}\\
&= \mathcal{D}\cdot\overline{B} - 3\mathrm{tr}XP - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{A} + Z\cdot\overline{B}.
\end{aligned}
$$

Putting it all together gives

$$
\frac{1}{2}\left(4\nabla_4 P - 4\,\underline{H}\cdot\overline{B} + 4\Xi\cdot\underline{B}\right) = \mathcal{D}\cdot\overline{B} - 3\mathrm{tr}XP - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{A} + Z\cdot\overline{B}
$$

or

$$
\nabla_4 P - \frac{1}{2}\mathcal{D}\cdot\overline{B} = -\frac{3}{2}\mathrm{tr}XP + \frac{1}{2}(2\,\underline{H} + Z)\cdot\overline{B} - \overline{\Xi}\cdot\underline{B} - \frac{1}{4}\underline{\widehat{X}}\cdot\overline{A}
$$

<u>**Derivation of $\nabla_4 B$ Equation:**</u> We start with $\mathbf{D}^\alpha\mathcal{R}_{\alpha\beta\gamma\delta} = 0$, i.e. $0 = \mathbf{D}^\delta\mathcal{R}_{\delta 4a4} = \mathbf{D}^b\mathcal{R}_{b4a4} - \frac{1}{2}\mathbf{D}_4\mathcal{R}_{34a4}$. For the left-hand side

$$
\begin{aligned}
\mathbf{D}_4\mathcal{R}_{34a4} =& e_4(\mathcal{R}(e_3,e_4,e_a,e_4)) - \mathcal{R}(\mathbf{D}_4e_3,e_4,e_a,e_4) - \mathcal{R}(e_3,\mathbf{D}_4e_4,e_a,e_4)\\
&- \mathcal{R}(e_3,e_4,\mathbf{D}_4e_a,e_4) - \mathcal{R}(e_3,e_4,e_a,\mathbf{D}_4e_4)\\
=& e_4(\mathcal{R}(e_3,e_4,e_a,e_4)) - \mathcal{R}(2\omega e_3 + 2\underline{\eta}_b e_b,e_4,e_a,e_4) - \mathcal{R}(e_3,-2\omega e_4 + 2\xi_b e_b,e_a,e_4)\\
&- \mathcal{R}(e_3,e_4,\nabla_4 e_a + \underline{\eta}_a e_4 + \xi_a e_3,e_4) - \mathcal{R}(e_3,e_4,e_a,-2\omega e_4 + 2\xi_b e_b)\\
=& e_4(\mathcal{R}(e_3,e_4,e_a,e_4)) - 2\omega\mathcal{R}(e_3,e_4,e_a,e_4) - 2\underline{\eta}_b\mathcal{R}(e_b,e_4,e_a,e_4)\\
&+ 2\omega\mathcal{R}(e_3,e_4,e_a,e_4) - 2\xi_b\mathcal{R}(e_3,e_b,e_a,e_4)\\
&- \mathcal{R}(e_3,e_4,\nabla_4e_a,e_4) - \xi_a\mathcal{R}(e_3,e_4,e_3,e_4) + 2\omega\mathcal{R}(e_3,e_4,e_a,e_4) - 2\xi_b\mathcal{R}(e_3,e_4,e_a,e_b)\\
=& e_4(\mathcal{R}(e_3,e_4,e_a,e_4)) - 2\underline{\eta}_b\mathcal{R}(e_b,e_4,e_a,e_4) - 2\xi_b\mathcal{R}(e_3,e_b,e_a,e_4)\\
&- \mathcal{R}(e_3,e_4,\nabla_4e_a,e_4) - \xi_a\mathcal{R}(e_3,e_4,e_3,e_4) + 2\omega\mathcal{R}(e_3,e_4,e_a,e_4) - 2\xi_b\mathcal{R}(e_3,e_4,e_a,e_b)\\
=& \nabla_4(2\overline{B}_a) - 2\underline{\eta}_b\overline{A}_{ba} - 2\xi_b(P\delta_{ba} - \ {}^*P\in_{ba}) - \xi_a(4P) + 2\omega(2\overline{B}_a) - 2\xi_b(2\in_{ab}\ {}^*P)\\
=& 2\nabla_4\overline{B}_a - 2\underline{\eta}_b\overline{A}_{ba} - 2\xi_aP - 2\,{}^*\xi_a\,{}^*P - 4\xi_aP + 4\omega\overline{B}_a - 4\,{}^*\xi_a\,{}^*P\\
=& 2\nabla_4\overline{B}_a - 2\underline{\eta}_b\overline{A}_{ba} - 6\xi_aP - 6\,{}^*\xi_a\,{}^*P + 4\omega\overline{B}_a\\
=& 2\nabla_4\overline{B}_a - 2\underline{\eta}_b\overline{A}_{ba} - 6\xi_aP - 6\,{}^*\xi_a(-iP) + 4\omega\overline{B}_a\\
=& 2\nabla_4\overline{B}_a - 2\underline{\eta}_b\overline{A}_{ba} - 6(\xi_a - i\,{}^*\xi_a)P + 4\omega\overline{B}_a\\
=& 2\nabla_4\overline{B}_a - 2\underline{\eta}_b\overline{A}_{ba} - 6\overline{\Xi}_aP + 4\omega\overline{B}_a\\
=& 2\nabla_4\overline{B}_a - \ \underline{H}_b\overline{A}_{ba} - 6\overline{\Xi}_aP + 4\omega\overline{B}_a
\end{aligned}
$$

while for the right-hand side

$$
\begin{aligned}
&\mathbf{D}_c\mathcal{R}_{b4a4}\\
&= e_c(\mathcal{R}(e_b,e_4,e_a,e_4)) - \mathcal{R}(\mathbf{D}_c e_b, e_4, e_a, e_4) - \mathcal{R}(e_b, \mathbf{D}_c e_4, e_a, e_4)\\
&\quad - \mathcal{R}(e_b, e_4, \mathbf{D}_c e_a, e_4) - \mathcal{R}(e_b, e_4, e_a, \mathbf{D}_c e_4)\\
&= e_c(\mathcal{R}(e_b,e_4,e_a,e_4)) - \mathcal{R}(\nabla_c e_b + \frac{1}{2}\chi_{cb}e_3 + \frac{1}{2}\underline{\chi}_{cb}e_4, e_4, e_a, e_4) - \mathcal{R}(e_b, \chi_{cd}e_d - \zeta_c e_4, e_a, e_4)\\
&\quad - \mathcal{R}(e_b, e_4, \nabla_c e_a + \frac{1}{2}\chi_{ca}e_3 + \frac{1}{2}\underline{\chi}_{ca}e_4, e_4) - \mathcal{R}(e_b, e_4, e_a, \chi_{cd}e_d - \zeta_c e_4)\\
&= e_c(\mathcal{R}(e_b,e_4,e_a,e_4)) - \mathcal{R}(\nabla_c e_b, e_4, e_a, e_4) - \frac{1}{2}\chi_{cb}\mathcal{R}(e_3, e_4, e_a, e_4)\\
&\quad - \chi_{cd}\mathcal{R}(e_b, e_d, e_a, e_4) + \zeta_c\mathcal{R}(e_b, e_4, e_a, e_4)\\
&\quad - \mathcal{R}(e_b, e_4, \nabla_c e_a, e_4) - \frac{1}{2}\chi_{ca}\mathcal{R}(e_b, e_4, e_3, e_4) - \chi_{cd}\mathcal{R}(e_b, e_4, e_a, e_d) + \zeta_c\mathcal{R}(e_b, e_4, e_a, e_4)\\
&= \nabla_c\overline{A}_{ba} - \frac{1}{2}\chi_{cb}(2\overline{B}_a) - \chi_{cd}(-i \in_{bd} \overline{B}_a) + \zeta_c\overline{A}_{ba} - \frac{1}{2}\chi_{ca}(2\overline{B}_b) - \chi_{cd}(- \in_{ad} \; {}^*\overline{B}_b) + \zeta_c\overline{A}_{ba}\\
&= \nabla_c\overline{A}_{ba} - \chi_{cb}\overline{B}_a + i\chi_{cd} \in_{bd} \overline{B}_a + \zeta_c\overline{A}_{ba} - \chi_{ca}\overline{B}_b + \chi_{cd} \in_{ad} \; {}^*\overline{B}_b + \zeta_c\overline{A}_{ba}\\
&= \nabla_c\overline{A}_{ba} - \chi_{cb}\overline{B}_a - i(\chi_{cb} \; {}^*)\overline{B}_a + 2\zeta_c\overline{A}_{ba} - \chi_{ca}\overline{B}_b - (\chi_{ca} \; {}^*) \; {}^*\overline{B}_b.
\end{aligned}
$$

Taking the trace over b and c, and using Lemma 2.1.10, we obtain

$$
\begin{aligned}
\mathbf{D}^b\mathbf{R}_{b4a4} &= \mathrm{d}iv\,\overline{A} - \mathrm{tr}\chi\overline{B} + i\,{}^{(a)}\mathrm{tr}\chi\overline{B} + 2\zeta\cdot\overline{A} - \overline{B}\cdot\chi - \; {}^*\overline{B}\cdot\chi\; {}^*\\
&= \frac{1}{2}\mathcal{D}\cdot\overline{A} - \mathrm{tr}\chi\overline{B} + i\,{}^{(a)}\mathrm{tr}\chi\overline{B} + 2\zeta\cdot\overline{A} - \mathrm{tr}\chi\overline{B} + {}^{(a)}\mathrm{tr}\chi\; {}^*\overline{B}\\
&= \frac{1}{2}\mathcal{D}\cdot\overline{A} - \mathrm{tr}\chi\overline{B} + i\,{}^{(a)}\mathrm{tr}\chi\overline{B} + 2\zeta\cdot\overline{A} - \mathrm{tr}\chi\overline{B} + i\,{}^{(a)}\mathrm{tr}\chi\overline{B}\\
&= \frac{1}{2}\mathcal{D}\cdot\overline{A} - 2\mathrm{tr}\chi\overline{B} + 2i\,{}^{(a)}\mathrm{tr}\chi\overline{B} + 2\zeta\cdot\overline{A}\\
&= \frac{1}{2}\mathcal{D}\cdot\overline{A} - 2(\mathrm{tr}\chi - i\,{}^{(a)}\mathrm{tr}\chi)\overline{B} + Z\cdot\overline{A}\\
&= \frac{1}{2}\mathcal{D}\cdot\overline{A} - 2\mathrm{tr}X\overline{B} + Z\cdot\overline{A}
\end{aligned}
$$

Putting it all together gives

$$\frac{1}{2}\left(2\nabla_4\overline{B} - \underline{H}\cdot\overline{A} - 6\Xi P + 4\omega\overline{B}\right) = \frac{1}{2}\mathcal{D}\cdot\overline{A} - 2\mathrm{tr}X\overline{B} + Z\cdot\overline{A}$$

or

$$\nabla_4 B - \frac{1}{2}\overline{\mathcal{D}}\cdot A = -2\overline{\mathrm{tr}X}B - 2\omega B + \frac{1}{2}A\cdot\overline{(2Z+\underline{H})} + 3\overline{P}\Xi$$

Derivation of $\nabla_3 B$ Equation:

We start with $\mathbf{D}^\alpha \mathcal{R}_{\alpha\beta\gamma\delta} = 0$, i.e. $\frac{1}{2}\mathbf{D}_3\mathcal{R}_{43a4} = \mathbf{D}^b\mathcal{R}_{b3a4}$. On the left

$$
\begin{aligned}
\mathbf{D}_3\mathcal{R}_{43a4} =& e_3(\mathcal{R}(e_4, e_3, e_a, e_4)) - \mathcal{R}(\mathbf{D}_3 e_4, e_3, e_a, e_4) - \mathcal{R}(e_4, \mathbf{D}_3 e_3, e_a, e_4)\\
& - \mathcal{R}(e_4, e_3, \mathbf{D}_3 e_a, e_4) - \mathcal{R}(e_4, e_3, e_a, \mathbf{D}_3 e_4)\\
=& e_3(\mathcal{R}(e_4, e_3, e_a, e_4)) - \mathcal{R}(2\underline{\omega} e_4 + 2\eta_b e_b, e_3, e_a, e_4) - \mathcal{R}(e_4, -2\underline{\omega} e_3 + 2\underline{\xi}_b e_b, e_a, e_4)\\
& - \mathcal{R}(e_4, e_3, \nabla_3 e_a + \eta_a e_3 + \underline{\xi}_a e_4, e_4) - \mathcal{R}(e_4, e_3, e_a, 2\underline{\omega} e_4 + 2\eta_b e_b)\\
=& \nabla_3(-2\overline{B}_a) - 2\underline{\omega}\mathcal{R}(e_4, e_3, e_a, e_4) - 2\eta_b\mathcal{R}(e_b, e_3, e_a, e_4)\\
& + 2\underline{\omega}\mathcal{R}(e_4, e_3, e_a, e_4) - 2\underline{\xi}_b\mathcal{R}(e_4, e_b, e_a, e_4)\\
& - \eta_a\mathcal{R}(e_4, e_3, e_3, e_4) - 2\underline{\omega}\mathcal{R}(e_4, e_3, e_a, e_4) - 2\eta_b\mathcal{R}(e_4, e_3, e_a, e_b)\\
=& -2\nabla_3\overline{B}_a - 2\eta_b(-P\delta_{ba} + {}^*P \in_{ba}) - 2\underline{\xi}_b(-\overline{A}_{ba})\\
& - \eta_a(-4P) - 2\underline{\omega}(-2\overline{B}_a) - 2\eta_b(-2 \in_{ab} {}^*P)\\
=& -2\nabla_3\overline{B}_a + 2\eta_a P - 2(\eta_a {}^*) {}^*P + 2\underline{\xi}_b\overline{A}_{ba} + 4\eta_a P + 4\underline{\omega}\overline{B}_a + 4({}^*\eta_a) {}^*P\\
=& -2\nabla_3\overline{B}_a + 6\eta_a P - 6({}^*\eta_a) iP + 2\underline{\xi}_b\overline{A}_{ba} + 4\underline{\omega}\overline{B}_a\\
=& -2\nabla_3\overline{B}_a + 6\overline{H}_a P + \underline{\Xi}_b\overline{A}_{ba} + 4\underline{\omega}\overline{B}_a
\end{aligned}
$$

For the right-hand side

$$
\begin{aligned}
&\mathbf{D}_c\mathcal{R}_{b3a4}\\
&= e_c(\mathcal{R}(e_b, e_3, e_a, e_4)) - \mathcal{R}(\mathbf{D}_c e_b, e_3, e_a, e_4) - \mathcal{R}(e_b, \mathbf{D}_c e_3, e_a, e_4)\\
&\quad - \mathcal{R}(e_b, e_3, \mathbf{D}_c e_a, e_4) - \mathcal{R}(e_b, e_3, e_a, \mathbf{D}_c e_4)\\
&= e_c(\mathcal{R}(e_b, e_3, e_a, e_4)) - \mathcal{R}(\nabla_c e_b + \frac{1}{2}\chi_{cb} e_3 + \frac{1}{2}\underline{\chi}_{cb} e_4, e_3, e_a, e_4) - \mathcal{R}(e_b, \underline{\chi}_{cd} e_d + \zeta_c e_3, e_a, e_4)\\
&\quad - \mathcal{R}(e_b, e_3, \nabla_c e_a + \frac{1}{2}\chi_{ca} e_3 + \frac{1}{2}\underline{\chi}_{ca} e_4, e_4) - \mathcal{R}(e_b, e_3, e_a, \chi_{cd} e_d - \zeta_c e_4)\\
&= e_c(\mathcal{R}(e_b, e_3, e_a, e_4)) - \mathcal{R}(\nabla_c e_b, e_3, e_a, e_4) - \frac{1}{2}\underline{\chi}_{cb}\mathcal{R}(e_4, e_3, e_a, e_4)\\
&\quad - \underline{\chi}_{cd}\mathcal{R}(e_b, e_d, e_a, e_4) - \zeta_c\mathcal{R}(e_b, e_3, e_a, e_4)\\
&\quad - \mathcal{R}(e_b, e_3, \nabla_c e_a, e_4) - \frac{1}{2}\chi_{ca}\mathcal{R}(e_b, e_3, e_3, e_4) - \chi_{cd}\mathcal{R}(e_b, e_3, e_a, e_d) + \zeta_c\mathcal{R}(e_b, e_3, e_a, e_4)\\
&= \nabla_c(-P\delta_{ba} + {}^*P \in_{ba}) - \frac{1}{2}\underline{\chi}_{cb}(-2\overline{B}_a) - \underline{\chi}_{cd}(-i \in_{bd} \overline{B}_a) - \frac{1}{2}\chi_{ca}(2\underline{B}_b) - \chi_{cd}(\in_{ad} {}^*\underline{B}_b)\\
&= \nabla_c(-P\delta_{ba} + {}^*P \in_{ba}) + \underline{\chi}_{cb}\overline{B}_a + i\underline{\chi}_{cd} \in_{bd} \overline{B}_a - \chi_{ca}\underline{B}_b - \chi_{cd} \in_{ad} {}^*\underline{B}_b\\
&= \nabla_c(-P\delta_{ba} + {}^*P \in_{ba}) + \underline{\chi}_{cb}\overline{B}_a - i(\underline{\chi}_{cb} {}^*)\overline{B}_a - \chi_{ca}\underline{B}_b + (\chi_{ca} {}^*) {}^*\underline{B}_b.
\end{aligned}
$$

Therefore, using Lemma 2.1.10 on the 2nd Kerr paper,

$$\begin{aligned}
\mathbf{D}^b \mathcal{R}_{b3a4} &= \nabla_b(-P\delta_{ba} + {}^*P \in_{ba}) + \mathrm{tr}\underline{\chi}\overline{B}_a + i\,{}^{(a)}\mathrm{tr}\underline{\chi}\overline{B}_a - \underline{B}_b\chi_{ba} + {}^*\underline{B}_b\chi_{ba}{}^* \\
&= \nabla_b(-P\delta_{ba} - {}^*P \in_{ab}) + \mathrm{tr}\underline{\chi}\overline{B}_a + i\,{}^{(a)}\mathrm{tr}\underline{\chi}\overline{B}_a - \underline{B}_b\chi_{ba} + {}^*\underline{B}_b\chi_{ba}{}^* \\
&= -(\nabla_a P + {}^*\nabla_a {}^*P) + (\mathrm{tr}\underline{\chi} + i\,{}^{(a)}\mathrm{tr}\underline{\chi})\overline{B}_a - 2\underline{B}_b\widehat{\chi}_{ba} \\
&= -(\nabla_a P - i\,{}^*\nabla_a P) + \overline{\mathrm{tr}\underline{X}}\,\overline{B}_a - \underline{B}_b\overline{\widehat{X}}_{ba} \\
&= -(\nabla_a - i\,{}^*\nabla_a)P + \overline{\mathrm{tr}\underline{X}}\,\overline{B}_a - \underline{B}_b\overline{\widehat{X}}_{ba} \\
&= -\overline{\mathcal{D}}_a P + \overline{\mathrm{tr}\underline{X}}\,\overline{B}_a - \underline{B}_b\overline{\widehat{X}}_{ba}
\end{aligned}$$

Putting it both sides together, and taking the conjugate, yields

$$\nabla_3 B - \mathcal{D}\overline{P} = -\mathrm{tr}\underline{X}\,B + 2\underline{\omega}B + \overline{\underline{B}}\cdot\widehat{X} + 3\overline{P}H + \frac{1}{2}A\cdot\underline{\Xi}$$

Derivation of $\nabla_3 A$ Equation:

According to the Bianchi identities, $\mathbf{D}_{[3}\mathcal{R}_{a4]b4} = 0$ and $\mathbf{D}_{[3}\mathcal{R}_{b4]a4} = 0$. Thus,

$$\begin{aligned}
0 &= 6\left(\mathbf{D}_{[3}\mathcal{R}_{a4]b4} + \mathbf{D}_{[3}\mathcal{R}_{b4]a4}\right) \\
&= \mathbf{D}_3\mathcal{R}_{a4b4} + \mathbf{D}_a\mathcal{R}_{43b4} + \mathbf{D}_4\mathcal{R}_{3ab4} + \mathbf{D}_3\mathcal{R}_{b4a4} + \mathbf{D}_b\mathcal{R}_{43a4} + \mathbf{D}_4\mathcal{R}_{3ba4} \\
&= \mathbf{D}_3\mathcal{R}_{a4b4} - \mathbf{D}_a\mathcal{R}_{b434} - \mathbf{D}_4\mathcal{R}_{a3b4} + \mathbf{D}_3\mathcal{R}_{a4b4} - \mathbf{D}_b\mathcal{R}_{a434} - \mathbf{D}_4\mathcal{R}_{b3a4} \\
&= 2\mathbf{D}_3\mathcal{R}_{a4b4} - (\mathbf{D}_b\mathcal{R}_{a434} + \mathbf{D}_a\mathcal{R}_{b434}) - (\mathbf{D}_4\mathcal{R}_{a3b4} + \mathbf{D}_4\mathcal{R}_{b3a4})
\end{aligned}$$

thus

$$\mathbf{D}_3\mathcal{R}_{a4b4} = \frac{1}{2}(\mathbf{D}_b\mathcal{R}_{a434} + \mathbf{D}_a\mathcal{R}_{b434}) + \frac{1}{2}\mathbf{D}_4(\mathcal{R}_{a3b4} + \mathcal{R}_{b3a4}).$$

For the left-hand side, we compute, as above,

$$
\begin{aligned}
\mathbf{D}_3\mathcal{R}_{a4b4} =& e_3(\mathcal{R}(e_a,e_4,e_b,e_4)) - \mathcal{R}(\mathbf{D}_3 e_a,e_4,e_b,e_4) - \mathcal{R}(e_a,\mathbf{D}_3 e_4,e_b,e_4)\\
& - \mathcal{R}(e_a,e_4,\mathbf{D}_3 e_b,e_4) - \mathcal{R}(e_a,e_4,e_b,\mathbf{D}_3 e_4)\\
=& e_3(\mathcal{R}(e_a,e_4,e_b,e_4)) - \mathcal{R}(\nabla_3 e_a,e_4,e_b,e_4)\\
& - \eta_a\mathcal{R}(e_3,e_4,e_b,e_4) - 4\underline{\omega}\mathcal{R}(e_a,e_4,e_b,e_4) - 2\eta_c\mathcal{R}(e_a,e_c,e_b,e_4)\\
& - \mathcal{R}(e_a,e_4,\nabla_3 e_b,e_4) - \eta_b\mathcal{R}(e_a,e_4,e_3,e_4) - 2\eta_c\mathcal{R}(e_a,e_4,e_b,e_c)\\
=& \nabla_3\overline{A}_{ab} - \eta_a(2\overline{B}_b) - 4\underline{\omega}\overline{A}_{ab} - 2\eta_c(-\in_{ac}\ {}^*\overline{B}_b) - 2\eta_b\overline{B}_a - 2\eta_c(-\in_{bc}\ {}^*\overline{B}_a) = \qquad \nabla_3\overline{A}_{ab} -\\
=& \nabla_3\overline{A}_{ab} - 2\eta_a\overline{B}_b - 2\eta_b\overline{B}_a - 4\underline{\omega}\overline{A}_{ab} - 2(\eta_a\ {}^*)\ {}^*\overline{B}_b - 2(\eta_b\ {}^*)\ {}^*\overline{B}_a\\
=& \nabla_3\overline{A}_{ab} - 2\left(\eta_a\overline{B}_b + \eta_b\overline{B}_a\right) - 4\underline{\omega}\overline{A}_{ab} - 2\left((\eta_a\ {}^*)\ {}^*\overline{B}_b + (\eta_b\ {}^*)\ {}^*\overline{B}_a\right)\\
=& \nabla_3\overline{A}_{ab} - 2\left((\eta\widehat{\otimes}\overline{B})_{ab} + \delta_{ab}(\eta\cdot\overline{B})\right) - 4\underline{\omega}\overline{A}_{ab} - 2\left(((\eta_a\ {}^*)\widehat{\otimes}\ {}^*\overline{B})_{ab} + \delta_{ab}((\eta\ {}^*)\cdot(\ {}^*\overline{B}))\right)\\
=& \nabla_3\overline{A}_{ab} - 2\left((\eta\widehat{\otimes}\overline{B})_{ab} + \delta_{ab}(\eta\cdot\overline{B})\right) - 4\underline{\omega}\overline{A}_{ab} - 2\left(((\eta_a\ {}^*)\widehat{\otimes}\ {}^*\overline{B})_{ab} - \delta_{ab}(\eta\cdot\overline{B})\right)\\
=& \nabla_3\overline{A}_{ab} - 2(\eta\widehat{\otimes}\overline{B})_{ab} - 4\underline{\omega}\overline{A}_{ab} - 2((\eta_a\ {}^*)\widehat{\otimes}\ {}^*\overline{B})_{ab}\\
=& \nabla_3\overline{A}_{ab} - 2(\eta\widehat{\otimes}\overline{B})_{ab} - 4\underline{\omega}\overline{A}_{ab} - 2(\eta\widehat{\otimes}\overline{B})_{ab}\\
=& \nabla_3\overline{A}_{ab} - 4(\eta\widehat{\otimes}\overline{B})_{ab} - 4\underline{\omega}\overline{A}_{ab}\\
=& \nabla_3\overline{A}_{ab} - 2(\overline{H}\widehat{\otimes}\overline{B})_{ab} - 4\underline{\omega}\overline{A}_{ab}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\mathbf{D}_a\mathcal{R}_{b434}\\
&= e_a(\mathcal{R}(e_b,e_4,e_3,e_4)) - \mathcal{R}(\mathbf{D}_a e_b,e_4,e_3,e_4) - \mathcal{R}(e_b,\mathbf{D}_a e_4,e_3,e_4)\\
&\quad - \mathcal{R}(e_b,e_4,\mathbf{D}_a e_3,e_4) - \mathcal{R}(e_b,e_4,e_3,\mathbf{D}_a e_4)\\
&= e_a(\mathcal{R}(e_b,e_4,e_3,e_4)) - \mathcal{R}(\nabla_a e_b + \frac{1}{2}\chi_{ab}e_3 + \frac{1}{2}\underline{\chi}_{ab}e_4,e_4,e_3,e_4) - \mathcal{R}(e_b,\chi_{ac}e_c - \zeta_a e_4,e_3,e_4)\\
&\quad - \mathcal{R}(e_b,e_4,\underline{\chi}_{ac}e_c + \zeta_a e_3,e_4) - \mathcal{R}(e_b,e_4,e_3,\chi_{ac}e_c - \zeta_a e_4)\\
&= \nabla_a(2\overline{B_b}) - \frac{1}{2}\chi_{ab}\mathcal{R}_{3434} - \chi_{ac}\mathcal{R}_{bc34} + \zeta_a\mathcal{R}_{b434} - \underline{\chi}_{ac}\mathcal{R}_{b4c4} - \zeta_a\mathcal{R}_{b434} - \chi_{ac}\mathcal{R}_{b43c} + \zeta_a\mathcal{R}_{b434}\\
&= 2\nabla_a\overline{B_b} - \frac{1}{2}\chi_{ab}\mathcal{R}_{3434} - \chi_{ac}\mathcal{R}_{bc34} + \zeta_a\mathcal{R}_{b434} - \underline{\chi}_{ac}\mathcal{R}_{b4c4} + \chi_{ac}\mathcal{R}_{c3b4}\\
&= 2\nabla_a\overline{B_b} - \frac{1}{2}\chi_{ab}(4P) - \chi_{ac}(2\in_{bc}\ {}^*P) + \zeta_a(2\overline{B_b}) - \underline{\chi}_{ac}\overline{A}_{bc} + \chi_{ac}(-P\delta_{cb} + \ {}^*P\in_{cb})\\
&= 2\nabla_a\overline{B_b} - 2\chi_{ab}P - 2\chi_{ac}\in_{bc}\ {}^*P + 2\zeta_a\overline{B_b} - \underline{\chi}_{ac}\overline{A}_{bc} - \chi_{ab}P + \ {}^*P\in_{cb}\chi_{ac}\\
&= 2\nabla_a\overline{B_b} - 3\chi_{ab}P + 3\ {}^*P\in_{cb}\chi_{ac} + 2\zeta_a\overline{B_b} - \underline{\chi}_{ac}\overline{A}_{bc}\\
&= 2\nabla_a\overline{B_b} - 3\chi_{ab}P - \underline{\chi}_{ac}\overline{A}_{bc} + 3\ {}^*P\chi_{ab}\ {}^* + 2\zeta_a\overline{B_b}.
\end{aligned}
$$

Therefore, using the equations beneath Lemma 2.1.9 in the 2nd Kerr paper.

$$
\begin{aligned}
\frac{1}{2}(\mathbf{D}_b\mathcal{R}_{a434}+\mathbf{D}_a\mathcal{R}_{b434}) =& \frac{1}{2}\Big(2\nabla_b\overline{B_a}-3\chi_{ba}P-\underline{\chi}_{bc}\overline{A}_{ac}+3\ {}^*P\chi_{ba}\ {}^*+2\zeta_b\overline{B_a}\\
&+2\nabla_a\overline{B_b}-3\chi_{ab}P-\underline{\chi}_{ac}\overline{A}_{bc}+3\ {}^*P\chi_{ab}\ {}^*+2\zeta_a\overline{B_b}\Big)\\
=&(\nabla\widehat{\otimes}\overline{B})_{ab}+\delta_{ab}\mathrm{div}\,\overline{B}-3\left(\widehat{\chi}_{ab}+\frac{1}{2}\delta_{ab}\mathrm{tr}\chi\right)P-\frac{1}{2}(\underline{\chi}_{ac}\overline{A}_{bc}+\underline{\chi}_{bc}\overline{A}_{ac})\\
&+\frac{3}{2}\ {}^*P\left(2\widehat{\chi}_{ab}\ {}^*-\delta_{ab}\ {}^{(a)}\mathrm{tr}\chi\right)+(\zeta\widehat{\otimes}\overline{B})_{ab}+\delta_{ab}(\zeta\cdot\overline{B})\\
=&\frac{1}{2}(\mathcal{D}\widehat{\otimes}\overline{B})_{ab}+\frac{1}{2}\delta_{ab}\mathcal{D}\cdot\overline{B}-3\widehat{\chi}_{ab}P-\frac{1}{2}(\underline{\chi}_{ac}\overline{A}_{bc}+\underline{\chi}_{bc}\overline{A}_{ac})\\
&+3\ {}^*P\widehat{\chi}_{ab}\ {}^*+\frac{1}{2}(\overline{Z}\widehat{\otimes}\overline{B})_{ab}+\frac{1}{2}\delta_{ab}(Z\cdot\overline{B})-\frac{3}{2}\delta_{ab}\mathrm{tr}XP\\
=&\frac{1}{2}(\mathcal{D}\widehat{\otimes}\overline{B})_{ab}+\frac{1}{2}\delta_{ab}\mathcal{D}\cdot\overline{B}-\frac{3}{2}(\widehat{X}+\overline{\widehat{X}})P-\frac{1}{2}\left(\frac{\underline{X}_{ac}+\underline{\overline{X}}_{ac}}{2}\overline{A}_{bc}+\frac{\underline{X}_{bc}+\underline{\overline{X}}_{bc}}{2}\overline{A}_{ac}\right)\\
&+\frac{3}{2}P(\widehat{X}-\overline{\widehat{X}})+\frac{1}{2}(\overline{Z}\widehat{\otimes}\overline{B})_{ab}+\frac{1}{2}\delta_{ab}(Z\cdot\overline{B})-\frac{3}{2}\delta_{ab}\mathrm{tr}XP\\
=&\frac{1}{2}(\mathcal{D}\widehat{\otimes}\overline{B})_{ab}+\frac{1}{2}\delta_{ab}\mathcal{D}\cdot\overline{B}-3\overline{\widehat{X}}P-\frac{1}{2}\left(\frac{\underline{X}_{ac}+\underline{\overline{X}}_{ac}}{2}\overline{A}_{bc}+\frac{\underline{X}_{bc}+\underline{\overline{X}}_{bc}}{2}\overline{A}_{ac}\right)\\
&+\frac{1}{2}(\overline{Z}\widehat{\otimes}\overline{B})_{ab}+\frac{1}{2}\delta_{ab}(Z\cdot\overline{B})-\frac{3}{2}\delta_{ab}\mathrm{tr}XP
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathcal{R}_{a3b4}+\mathcal{R}_{b3a4} &= (-P\delta_{ab}+\ {}^*P\in_{ab})+(-P\delta_{ba}+\ {}^*P\in_{ba})\\
&= -P\delta_{ab}+\ {}^*P\in_{ab}-P\delta_{ab}-\ {}^*P\in_{ab}\\
&= -2P\delta_{ab}
\end{aligned}
$$

and so

$$
\frac{1}{2}\mathbf{D}_4(\mathcal{R}_{a3b4}+\mathcal{R}_{b3a4})=\frac{1}{2}\mathbf{D}_4(-2P\delta_{ab})=-\mathbf{D}_4(P\delta_{ab})=-(\mathbf{D}_4P)\delta_{ab}.
$$

Recall that the equation for ∇_4P is

$$
\nabla_4P-\frac{1}{2}\mathcal{D}\cdot\overline{B}=-\frac{3}{2}\mathrm{tr}XP+\frac{1}{2}(2\,\underline{H}+Z)\cdot\overline{B}-\Xi\cdot\underline{B}-\frac{1}{4}\underline{\widehat{X}}\cdot\overline{A}.
$$

So,

$$
\begin{aligned}
\frac{1}{2}\mathbf{D}_4(\mathcal{R}_{a3b4}+\mathcal{R}_{b3a4}) &= -(\mathbf{D}_4 P)\delta_{ab}\\
&= -\left(\frac{1}{2}\mathcal{D}\cdot\overline{B}-\frac{3}{2}\mathrm{tr}XP+\frac{1}{2}(2\,\underline{H}+Z)\cdot\overline{B}-\overline{\Xi}\cdot\underline{B}-\frac{1}{4}\underline{\widehat{X}}\cdot\overline{A}\right)\delta_{ab}\\
&= \left(-\frac{1}{2}\mathcal{D}\cdot\overline{B}+\frac{3}{2}\mathrm{tr}XP-\frac{1}{2}(2\,\underline{H}+Z)\cdot\overline{B}+\overline{\Xi}\cdot\underline{B}+\frac{1}{4}\underline{\widehat{X}}\cdot\overline{A}\right)\delta_{ab}
\end{aligned}
$$

Substituting

$$
\mathbf{D}_3\mathcal{R}_{a4b4}=\frac{1}{2}(\mathbf{D}_b\mathcal{R}_{a434}+\mathbf{D}_a\mathcal{R}_{b434})+\frac{1}{2}\mathbf{D}_4(\mathcal{R}_{a3b4}+\mathcal{R}_{b3a4}).
$$

To finish

$\square$

Appendix A

Wave propagation in Minkowski space

A.1 General Facts about scalar wave equations

A.1.1 Energy-Momentum Tensor

Consider the wave equation,

$$\Box_{\mathbf{g}}\phi \quad = \quad f. \tag{A.1.1}$$

in a time oriented[1] Lorentzian manifold $(\mathbf{M}, \mathbf{g})$. with $\mathbf{D}$ denoting the covariant derivative Let

$$\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\alpha\beta}[\phi] = \mathbf{D}_\alpha\phi\mathbf{D}_\beta\phi - \frac{1}{2}\mathbf{g}_{\alpha\beta}\big(\mathbf{g}^{\mu\nu}\mathbf{D}_\mu\phi\mathbf{D}_\nu\phi\big),$$

be the energy momentum tensor associated to ϕ.

Lemma A.1.1. *The energy momentum tensor $\mathbf{Q}_{\mu\nu}$ is symmetric, verifies the local conservation laws,*

$$\mathbf{D}^\beta\mathbf{Q}_{\alpha\beta} = f\mathbf{D}_\alpha\phi$$

and the positive energy condition, i.e. for all causal, future directed, vector fields X, Y,

$$\mathbf{Q}(X, Y) \geq 0,$$

[1]This means that there exists a globally defined timelike vectorfield T.

A.1.2 Killing and conformal Killing vectorfields

Definition. A diffeomorphism $\Phi : \mathcal{U} \subset \mathbf{M} \to \mathbf{M}$ is said to be a conformal isometry if, at every point p, $\Phi_* \mathbf{g} = \Lambda^2 \mathbf{g}$, that is,

$$(\Phi^* \mathbf{g})(X, Y)|_p = \mathbf{g}(\Phi_* X, \Phi_* Y)|_{\Phi(p)} = \Lambda^2 \mathbf{g}(X, Y)|_p$$

with $\Lambda \neq 0$. If $\Lambda = 1$, Φ is called an isometry of $\mathbf{M}$.

Definition. A vector field K which generates a one parameter group of isometries (respectively, conformal isometries) is called a Killing (respectively, conformal Killing) vector field.

Let K be such a vector field and Φ_t the corresponding one parameter group. Since the $(\Phi_t)_*$ are conformal isometries, we infer that $\mathcal{L}_K \mathbf{g}$ must be proportional to the metric $\mathbf{g}$. Moreover $\mathcal{L}_K \mathbf{g} = 0$ if K is a Killing vector field.

Definition A.1.2. *Given an arbitrary vector field X we denote ${}^{(X)}\pi$ the deformation tensor of X defined by the formula*

$${}^{(X)}\pi_{\alpha\beta} = (\mathcal{L}_X g)_{\alpha\beta} = \mathbf{D}_\alpha X_\beta + \mathbf{D}_\beta X_\alpha \ .$$

The tensor ${}^{(X)}\pi$ measures, in a precise sense, how much the diffeomorphism generated by X differs from an isometry or a conformal isometry. The following simple Proposition holds true

Proposition A.1.3. *The vector field X is Killing if and only if ${}^{(X)}\pi = 0$. It is conformal Killing if and only if ${}^{(X)}\pi$ is proportional to $\mathbf{g}$.*

Lemma A.1.4. *Given an arbitrary vectorfield X with deformation tensor ${}^{(X)}\pi$ we have the identity*

$$\mathbf{D}_\beta \mathbf{D}_\alpha X_\lambda = \mathbf{R}_{\lambda\alpha\beta\sigma} X^\sigma + {}^{(X)}\Gamma_{\alpha\beta\lambda}.$$

where

$${}^{(X)}\Gamma_{\alpha\beta\lambda} = \frac{1}{2}\left(\mathbf{D}_\beta\, {}^{(X)}\pi_{\alpha\lambda} + \mathbf{D}_\alpha\, {}^{(X)}\pi_{\beta\lambda} - \mathbf{D}_\lambda\, {}^{(X)}\pi_{\alpha\beta}\right).$$

Proof.

$$\begin{aligned}
2\,{}^{(X)}\Gamma_{\alpha\beta\lambda} &= \mathbf{D}_\beta\,{}^{(X)}\pi_{\alpha\lambda} + \mathbf{D}_\alpha\,{}^{(X)}\pi_{\beta\lambda} - \mathbf{D}_\lambda\,{}^{(X)}\pi_{\alpha\beta} \\
&= \mathbf{D}_\beta(\mathbf{D}_\alpha X_\lambda + \mathbf{D}_\lambda X_\alpha) + \mathbf{D}_\alpha(\mathbf{D}_\beta X_\lambda + \mathbf{D}_\lambda X_\beta) - \mathbf{D}_\lambda(\mathbf{D}_\alpha X_\beta + \mathbf{D}_\beta X_\alpha) \\
&= \mathbf{D}_\alpha\mathbf{D}_\beta X_\lambda + \mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda + (\mathbf{D}_\alpha\mathbf{D}_\lambda - \mathbf{D}_\lambda\mathbf{D}_\alpha)X_\beta + (\mathbf{D}_\beta\mathbf{D}_\lambda - \mathbf{D}_\lambda\mathbf{D}_\beta)X_\alpha \\
&= 2\mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda + (\mathbf{D}_\alpha\mathbf{D}_\beta - \mathbf{D}_\beta\mathbf{D}_\alpha)X_\lambda + (\mathbf{D}_\alpha\mathbf{D}_\lambda - \mathbf{D}_\lambda\mathbf{D}_\alpha)X_\beta + (\mathbf{D}_\beta\mathbf{D}_\lambda - \mathbf{D}_\lambda\mathbf{D}_\beta)X_\alpha \\
&= 2\mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda + \mathbf{R}_{\lambda\sigma\alpha\beta}X^\sigma + \mathbf{R}_{\beta\sigma\alpha\lambda}X^\sigma + \mathbf{R}_{\alpha\sigma\beta\lambda}X^\sigma \\
&= 2\mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda - \big(\mathbf{R}_{\sigma\lambda\alpha\beta} + \mathbf{R}_{\sigma\beta\alpha\lambda} + \mathbf{R}_{\sigma\alpha\beta\lambda}\big)X^\sigma \\
&= 2\mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda - \big(\mathbf{R}_{\sigma\lambda\alpha\beta} - \mathbf{R}_{\sigma\beta\alpha\lambda} + \mathbf{R}_{\sigma\alpha\beta\lambda}\big)X^\sigma - 2\mathbf{R}_{\sigma\beta\alpha\lambda}X^\lambda \\
&= 2\mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda - \big(\mathbf{R}_{\sigma\lambda\alpha\beta} + \mathbf{R}_{\sigma\beta\lambda\alpha} + \mathbf{R}_{\sigma\alpha\beta\lambda}\big)X^\sigma - 2\mathbf{R}_{\sigma\beta\alpha\lambda}X^\sigma \\
&= 2\mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda - 2\mathbf{R}_{\sigma\beta\alpha\lambda}X^\lambda
\end{aligned}$$

Therefore,

$$\mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda = \mathbf{R}_{\sigma\beta\alpha\lambda}X^\sigma + {}^{(X)}\Gamma_{\alpha\beta\lambda} = \mathbf{R}_{\alpha\lambda\sigma\beta}X^\sigma + {}^{(X)}\Gamma_{\alpha\beta\lambda} = \mathbf{R}_{\lambda\alpha\beta\sigma}X^\sigma + {}^{(X)}\Gamma_{\alpha\beta\lambda}$$

as stated. □

Proposition A.1.5. *On any pseudo-riemannian spacetime $\mathbf{M}$, of dimension $n = p+q$, there can be no more than $\frac{1}{2}(p+q)(p+q+1)$ linearly independent Killing vector fields.*

Proof. If X is a Killing vector field equation

$$\mathbf{D}_\beta(\mathbf{D}_\alpha X_\lambda) = \mathbf{R}_{\lambda\alpha\beta\delta}X^\delta.$$

and this implies, in view of the theorem of existence and uniqueness for ordinary differential equations, that any Killing vector field is completely determined by the $\frac{1}{2}(p+q)(p+q+1)$ values of X and $\mathbf{D}X$ at a given point. □

The n-dimensional Riemannian manifold which possesses the maximum number of Killing vector fields is the Euclidean space $\mathbb{R}^n$. Simmilarily the Minkowski spacetime $\mathbb{R}^{n+1}$ is the Lorentzian manifold with the maximum numbers of Killing vectorfields.

Corollary A.1.6. *If X is a conformal Killing vectorfield on a Ricci flat manifold of dimension $n+1$ and ${}^{(X)}\pi = \Lambda\mathbf{g}$ then, for all $n \geq 1$ $\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta\Lambda = 0$ and, for all $n > 1$, $\mathbf{D}_\alpha\mathbf{D}_\beta\Lambda = 0$.*

Proof. Indeed $\mathbf{D}_\beta\mathbf{D}_\alpha X_\lambda = \mathbf{R}_{\lambda\alpha\beta\delta}X^\delta + {}^{(X)}\Gamma_{\alpha\beta\lambda}$ from which,

$$\Box_{\mathbf{g}} X_\lambda = {}^{(X)}\Gamma_\mu = -\frac{n-1}{2}\mathbf{D}_\lambda\Lambda$$

Note that for Ricci flat spacetimes $\mathbf{D}^\lambda(\square X_\lambda) = \square(\mathbf{D}^\lambda X_\lambda)$. Hence,

$$\square(\mathrm{Div}X) = \frac{1-n}{2}\square\Lambda$$

On the other hand,

$$\mathbf{D}^\lambda X_\lambda = \frac{1}{2}\mathrm{tr}\,{}^{(X)}\pi = \frac{1}{2}(n+1)\Lambda$$

Hence,

$$\frac{1}{2}(n+1)\square\Lambda = \frac{1-n}{2}\square\Lambda$$

from which we deduce,

$$\square\Lambda = 0. \tag{A.1.2}$$

To prove the second part it suffices to commute the equation $\square X_\lambda = -\frac{n-1}{2}\mathbf{D}_\lambda\Lambda$ with covariant derivatives as follows,

$$\begin{aligned}\square\mathbf{D}_\mu X_\lambda &= -\frac{n-1}{2}\mathbf{D}_\mu\mathbf{D}_\lambda\Lambda\\ \square\mathbf{D}_\lambda X_\mu &= -\frac{n-1}{2}\mathbf{D}_\lambda\mathbf{D}_\mu\Lambda\end{aligned}$$

Therefore,

$$-(n-1)\mathbf{D}_\mu\mathbf{D}_\lambda\Lambda = \square\,{}^{(X)}\pi_{\mu\lambda} = \square(\Lambda\mathbf{g}_{\mu\nu}) = 0.$$

$\square$

Corollary A.1.7. *The total number of independent conformal Killing vectorfields on a Ricci flat manifold $\mathbf{M}^{1+n}$, $n \geq 2$, cannot exceed $\frac{(n+1)(n+2)}{2}$.*

A.1.3 Commutation of $\square_g$ with a vectorfield

Lemma A.1.8. *Consider a vectorfield X, with deformation tensor ${}^{(X)}\pi$ and $Q_{\alpha\beta} = \mathbf{D}_\alpha\phi\mathbf{D}_\beta\phi - \frac{1}{2}\mathbf{g}\mathbf{D}^\alpha\phi\mathbf{D}_\lambda\phi$ the energy momentum tensor of the scalar wave operator $\square_\mathbf{g}$. We have:*

$$X(\square_\mathbf{g}\phi) = \square_\mathbf{g}(X\phi) - {}^{(X)}\pi^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta\phi - \Big(2\mathbf{D}^{\beta}\,{}^{(X)}\pi_{\alpha\beta} - \mathbf{D}_\alpha(tr\,{}^{(X)}\pi)\Big)\mathbf{D}^\alpha\phi$$

Proof. Direct computation. This is also an immediate consequence of the general commutation formula of Lemma(3.2.3). □

Corollary A.1.9. *If X is a conformal Killing vectorfield on a Ricci flat manifold $(\mathcal{M}, \mathbf{g})$ we have*

$$[X, \square_{\mathbf{g}}]\phi = -\Lambda\square_{\mathbf{g}}\phi - (n-1)\mathbf{D}^\alpha\Lambda\mathbf{D}_\alpha\phi.$$

Moreover,

$$[X - \frac{n-1}{2}\Lambda, \square_{\mathbf{g}}]\phi = -\frac{n-3}{2}\Lambda\,\square\phi.$$

A.1.4 Generalized Integral currents

The integral current method is based on the following calculation (see the more general formula of Proposition 3.2.9):

Lemma A.1.10. *Given a vectorfield X, a scalar w and a 1-form M, the generalized current*

$$P_\mu := P_\mu[X, w, M] = \mathbf{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\phi\partial_\mu\phi - \frac{1}{4}\partial_\mu w\phi^2 + \frac{1}{4}M_\mu\phi^2$$

verifies

$$\begin{aligned}\mathbf{D}^\mu P_\mu = (X(\phi) + \frac{1}{2}w\phi)\square\phi + \frac{1}{2}\mathbf{Q}_{\mu\nu}{}^{(X)}\pi^{\mu\nu} - \frac{1}{4}\square w\phi^2 + \frac{1}{2}w\mathbf{D}^\mu\phi\mathbf{D}_\mu\phi \\ + \frac{1}{2}M^\mu\phi\partial_\mu\phi + \frac{1}{4}\mathbf{D}_\mu M^\mu\phi^2\end{aligned} \tag{A.1.3}$$

Proof. Direct computation. See also the more general Proposition 3.2.9. □

Corollary A.1.11. *Assume that X is conformal Killing, i.e. ${}^{(X)}\pi = \Omega\mathbf{g}$ for some scalar Ω, and*

$$P_\mu \;=\; \mathbf{Q}_{\mu\nu}X^\nu + \frac{n-1}{4}\Omega\phi\partial_\mu\phi - \frac{n-1}{8}\partial_\mu\Omega\phi^2.$$

Then

$$\mathbf{D}^\mu P_\mu = (X(\phi) + \frac{1}{2}w\phi)\square\phi$$

Lemma A.1.12 (Divergence lemma)**.** *Consider a vectorfield X in domain $\mathcal{D} \subset \mathcal{M}$ with future space-like boundaries $\partial^+\mathcal{D}$ and past boundary $\partial^-\mathcal{D}$. We have*

$$\int_{\partial^+\mathcal{D}} \mathbf{g}(X,N) - \int_{\partial^-\mathcal{D}} \mathbf{g}(X,N) = -\int_{\mathcal{D}} Div(X).$$

where N denote the future normal to the boundary.

Proof. Application of Stokes Theorem. □

A.2 Classical Vectorfield Method in Minkowski space

A.2.1 Symmetries of Minkowski space

Let x^μ be an inertial coordinate system of Minkowski space $\mathbb{R}^{n+1}$. The following are all the isometries and conformal isometries of $\mathbb{R}^{n+1}$.

1. Translations: For any given vector $a = (a^0, a^1, ..., a^n) \in \mathbb{R}^{n+1}$: $x^\mu \to x^\mu + a^\mu$.

2. Lorentz rotations: For any $\Lambda = \Lambda^\rho_\sigma \in \mathbf{O}(1,n)$: $x^\mu \to \Lambda^\mu_\nu x^\nu$.

3. Scalings: For any real number $\lambda \neq 0$: $x^\mu \to \lambda x^\mu$.

4. Inversion: Consider the transformation $x^\mu \to I(x^\mu)$, where $I(x^\mu) = \frac{x^\mu}{(x,x)}$ is defined for all points $x \in \mathbb{R}^{n+1}$ such that $(x,x) \neq 0$.

The first two sets of transformations are isometries of $\mathbb{R}^{n+1}$, the group generated by them is called the Poincarè group. The last two type of transformations are conformal isometries. the group generated by all the above transformations is called the Conformal group. In fact the Liouville theorem, whose infinitesimal version will be proved later on, states that it is the group of all the conformal isometries of $\mathbb{R}^{n+1}$.

We next list the Killing and conformal Killing vector fields which generate the above transformations.

i. The generators of translations in the x^μ directions, $\mu = 0, 1, ..., n$: $\mathbf{T}_\mu = \frac{\partial}{\partial x^\mu}$

ii. The generators of the Lorentz rotations in the (μ, ν) plane:, $\mathbf{L}_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$.

iii. The generators of the scaling transformations: $\mathbf{S} = x^\mu \partial_\mu$.

iv. The generators of the inverted translations: $\mathbf{K}_\mu = 2x_\mu x^\rho \frac{\partial}{\partial x^\rho} - (x^\rho x_\rho)\frac{\partial}{\partial x^\mu}$.

Denoting $\mathcal{P}(1,n)$ the Lie algebra generated by the vector fields $\mathbf{T}_\alpha, \mathbf{L}_{\beta\gamma}$ and $\underline{\mathcal{K}}(1,n)$ the Lie algebra generated by all the vector fields $\mathbf{T}_\alpha, \mathbf{L}_{\beta\gamma}, \mathbf{S}, \mathbf{K}_\delta$ we state the following version of the Liouville theorem,

Theorem A.2.1. *The following statements hold true.*

1) $\mathcal{P}(1,n)$ is the Lie algebra of all Killing vector fields in $\mathbb{R}^{n+1}$.

2) If $n > 1$, $\underline{\mathcal{K}}(1,n)$ is the Lie algebra of all conformal Killing vector fields in $\mathbb{R}^{n+1}$.

3) If $n = 1$, the set of all conformal Killing vector fields in $\mathbb{R}^{1+1}$ is given by the following expression

$$f(x^0 + x^1)(\partial_0 + \partial_1) + g(x^0 - x^1)(\partial_0 - \partial_1)$$

where f, g are arbitrary smooth functions of one variable.

Proof: If X is Kiling Therefore, there exist constants $a_{\mu\nu}, b_\mu$ such that $X^\mu = a_{\mu\nu}x^\nu + b_\mu$. Since X is Killing $\mathbf{D}_\mu X_\nu = -\mathbf{D}_\nu X_\mu$ which implies $a_{\mu\nu} = -a_{\nu\mu}$. Consequently X can be written as a linear combination, with real coefficients, of the vector fields $T_\alpha, L_{\beta\gamma}$.

Let now X be a conformal Killing vector field, i.e.

$$^{(X)}\pi_{\rho\sigma} = \Lambda \mathbf{m}_{\rho\sigma}$$

In view of Corollary A.1.6 $\Box\Lambda = 0$ and moreover, for $n \neq 1$, $D_\mu D_\lambda \Lambda = 0$. This implies that Λ must be a linear function of x^μ. We can therefore find a linear combination, with constant coefficients, $cS + d^\alpha K_\alpha$ such that the deformation tensor of $X - (cS + d^\alpha K_\alpha)$ must be zero. This is the case because $^{(S)}\pi = 2\mathbf{m}$ and $^{(K_\mu)}\pi = 4x_\mu \mathbf{m}$. Therefore $X - (cS + d^\alpha K_\alpha)$ is Killing which, in view of the first part of the theorem, proves the result.

To establish Part 3 we set $X = a\partial_0 + b\partial_1$ and obtain $2D_0X_0 = -\Lambda$, $2D_1X_1 = \Lambda$ and $D_0X_1 + D_1X_0 = 0$. Hence a, b verify the system

$$\frac{\partial a}{\partial x^0} = \frac{\partial b}{\partial x^1}\ ,\ \frac{\partial b}{\partial x^0} = \frac{\partial a}{\partial x^1}\ .$$

Hence the one form $adx^0 + bdx^1$ is exact, $adx^0 + bdx^1 = d\phi$, and $\frac{\partial^2 a}{(\partial x^0)^2} = \frac{\partial^2 b}{(\partial x^1)^2}$, that is $\Box\phi = 0$. In conclusion

$$X = \frac{1}{2}\left(\frac{\partial\phi}{\partial x^0} + \frac{\partial\phi}{\partial x^1}\right)(\partial_0 + \partial_1) + \frac{1}{2}\left(\frac{\partial\phi}{\partial x^0} - \frac{\partial\phi}{\partial x^1}\right)(\partial_0 - \partial_1)$$

which proves the result.

Remark. Expresse relative to the canonical null pair $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$,

$$\mathbf{T}_0 = 2^{-1}(L + \underline{L}), \quad \mathbf{S} = 2^{-1}(\underline{u}\, L + u\, \underline{L}), \quad \mathbf{K}_0 = 2^{-1}(\underline{u}^2\, L + u^2\, \underline{L}). \tag{A.2.1}$$

Both $\mathbf{T}_0 = \partial_t$ and $\mathbf{K}_0 = (t^2 + |x|^2)\partial_t + 2tx^i\partial_i$ are causal[2]. Observe that $\mathbf{S}$ is causal only in $\mathcal{J}^+(0) \cup \mathcal{J}^-(0)$. We note also that ${}^{(S)}\pi = 2\mathbf{m}$, ${}^{(K_0)}\pi = 4t\mathbf{m}$ and therefore, in view of Corollary A.1.9,

$$\begin{aligned} [\mathbf{S}, \Box] &= -2\Box, \\ [\mathbf{K}_0, \Box] &= -4t\Box + 4(n-1)\partial_t \\ [\mathbf{K}_0 + 2(n-1)t\Box] &= -2t\Box \end{aligned}$$

The general vectorfield method applied to the flat wave operator is based on commutation and integral currents.

A.2.2 Wave equation in Minkowski space $\mathbb{R}^{n+1}$

The canonical, inertial, coordinates in $\mathbb{R}^{n+1}$ are denoted by x^μ, $\mu = 0, 1, \ldots, n$ relative to which the Minkowski metric takes the diagonal form $\mathbf{m}_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. We have $x^0 = t$ and $x = (x^1, \ldots, x^n)$ denote the spatial coordinates. We make use of the standard summation convention over repeted indices and those concerning raising and lowering the indices of vectors and tensors. In particular, if $x_\mu = m_{\mu\nu}x^\nu$, we have $x_0 = -t$ and $x_i = x^i$, $i = 1, \ldots, n$. We denote by Σ_{t_0} the spacelike hyperplanes $t = t_0$. The wave operator is defined by $\Box = \mathbf{m}^{\alpha\beta}\partial_{\alpha\beta} = -\partial_t^2 + \sum_i \partial_i^2$. In polar coordinates $t, r, \theta^1, \ldots \theta^n$ the metric takes the form

$$-dt^2 + dr^2 + r^2 d\sigma_{n-1}^2.$$

The functions $u = \frac{1}{2}(t - r)$, $v = \frac{1}{2}(t + r)$ are optical, i.e. they verify the eikonal equation $\mathbf{m}^{\alpha\beta}\partial_\alpha u \partial_\beta u = \mathbf{m}^{\alpha\beta}\partial_\alpha v \partial_\beta v = 0$. We sometimes use $\underline{u}$ to denote v.

In the $u, v, \theta^1, \ldots, \theta^n$ coordinates the Minkowski metric takes the form $-4dudv + r^2 d\sigma_{n-1}^2$. Thus, $\mathbf{g}_{uv} = -2$, $\mathbf{g}^{uv} = -\frac{1}{2}$. The wave operator takes the form,

$$\Box\phi \;=\; \mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta\phi = -\partial_u\partial_v\phi + \frac{n-1}{r}\partial_r\phi + \not{\Delta}_{n-1}\phi.$$

[2] This makes them important in deriving energy estimates.

The standard null pair is given by

$$L = \partial_t + \partial_r = \partial_v, \qquad \underline{L} = \partial_t - \partial_r = \partial_u.$$

The corresponding horizontal structure is, of course, integrable with surfaces of integrability given by the spheres $S_{t,r}$.

Recall that the Minkowski space-time $\mathbb{R}^{n+1}$ is equipped with a family of Killing and conformal Killing vector fields, the translations $\mathbf{T}_\mu = \partial_\mu$, Lorentz rotations $\mathbf{L}_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$, scaling $\mathbf{S} = t\partial_t + x^i \partial_i$ and the inverted translations $\mathbf{K}_\mu = -2x_\mu \mathbf{S} + < x, x > \partial_\mu$.

The Killing vector fields $\mathbf{T}_\mu$ and $\mathbf{L}_{\mu\nu}$ commute with $\Box$ while $\mathbf{S}$ preserves the space of solutions in the sense that $\Box\phi = 0$ implies $\Box\mathbf{S}\phi = 0$ as $[\Box, S] = 2\Box$. One can split the operators $\mathbf{L}_{\mu\nu}$ into the angular rotation operators ${}^{(ij)}\mathbf{O} = x_i\partial_j - x_j\partial_i$ and the boosts ${}^{(i)}\mathbf{L} = x_i\partial_t + t\partial_i$, for $i, j, k = 1, \dots, n$.

A.2.3 Basic Conservation Laws in Minkowski space

The starting point is the pointwise conservation law

$$\mathbf{D}^\mu(\mathbf{Q}_{\mu\nu}X^\nu) = fX(\phi). \tag{A.2.2}$$

To derive an energy type inequality we integrate (A.2.2) on a domain of dependence $\mathcal{D}$, as defined below.

Definition A.2.2. *Given a domain $\Sigma_0 \subset \{t = t_0\}$, $\mathcal{D} = \mathcal{D}(\Sigma_0) \subset \mathbb{R}^n$ is a domain of dependence for Σ_0 if for every $p \in \mathcal{D}$, denoting by $C^-(p)$, the past line cone through p, we have $C^-(p) \cap \{t > t_0\} \subset \mathcal{D}$.*

We consider below the following examples of bounded domains of dependence $\mathcal{D}$ with boundary $\mathcal{D} = \partial^+\mathcal{D} \cup \Sigma_0$.

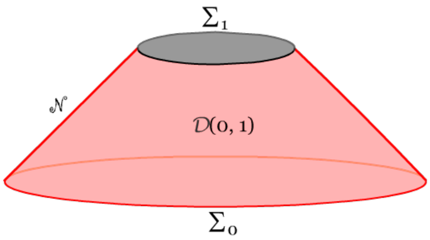
S. The future boundary $\partial\mathcal{D}^+$ is strictly space-like, i.e. the future unit normal N to it is timelike.

C. The domain $\mathcal{D}$ (see Figure A.1) given, for $0 < t_1 < R$,

$$\mathcal{D} = \{|x - x_0| < R - t\} \cap \{t \geq 0\} \cap \{t \leq t_1\}$$

whose future boundary is given by $\mathcal{D}^+ = \mathcal{N} \cap \{t = t_1\}$, with null boundary $\mathcal{N} = \{|x - x_0| = R - t\} \cap \{t \geq 0\} \cap \{t \leq t_1\}$

Figure A.1: Causal domain (domain of dependence) $\mathcal{D}$ with incoming null boundary $\mathcal{N}$ and space-like boundaries Σ_0, Σ_1.



Lemma A.2.3 (Divergence lemma in $\mathbb{R}^{1+n}$)**.** *Integrating the divergence equation* (A.2.2) *on a domain $\mathcal{D}$ we derive*

1. *For a spacelike domain of type (S) we have*

$$\int_{\Sigma_1} P \cdot N = \int_{\Sigma_1} P \cdot T - \int_{\mathcal{D}} F$$

2. *For a causal domain of type (T) we have, with*[3] $L = -\partial^\beta u \partial_\beta$, *i.e.* $\underline{L}^\beta = -\partial^\beta \underline{u}$,

$$\int_{\Sigma_1} P \cdot T + \int_{\mathcal{N}} P \cdot \underline{L} = \int_{\Sigma_0} P \cdot T - \int_{\mathcal{D}} F$$

where,

$$\int_{\mathcal{N}} f = \int_0^{t_1} \int_{|x-x_0| \leq R-t} f(t,x) d\sigma. \tag{A.2.3}$$

Proof. In the spacelike case it follows directly from Lemma A.1.12. Otherwise it requires a simple adaptation. □

Corollary A.2.4. *Given any solution of* $\Box\phi = 0$ *and X Killing we have the conservation law.*

$$\int_{\Sigma_1} Q(X,T) + \int_{\mathcal{N}} Q(X, \underline{L}) = \int_{\Sigma_0} Q(X,T)$$

In the particular case[4] *when $X = T$ we deduce the classical conservation of energy formula*

$$\int_{\Sigma_1} Q(T,T) + \int_{\mathcal{N}} Q(T, \underline{L}) = \int_{\Sigma_0} Q(T,T)$$

[3] Note that L is future null, i.e. $\mathbf{g}(L,L) = 0$, $\mathbf{g}(L,T) = -1$.

[4] Note that the only globally time-like Killing vectorfield in Minkowski space $\mathbb{R}^{1+n}$ is $X = \partial_t$.

where $Q(T,T) = \frac{1}{2}\big(|\partial_t\phi|^2 + |\nabla\phi|^2\big)$.

Thus, any continuous group of isometries of $(\mathcal{M},\mathbf{g})$, generated by a Killing vectorfield X, leads to a conservation law.

Remark A.2.5. *The vectorfield* $X = \mathbf{T} = \partial_t$ *leads to the standard law of consrvation of energy in Minkowski space: In the particular case when* $X = \mathbf{T}_0 = \partial_t$ *we have* ${}^{(X)}\pi = 0$ *and, integrating* (**??**) *on the space-time slab* $[0,T]\times\mathbb{R}^n$ *we derive the usual conservation laws,*

$$\int_{\Sigma_t} |\partial\phi|^2 \;=\; \int_{\Sigma_0} |\partial\phi|^2, \tag{A.2.4}$$

$$\int_{\partial\mathcal{N}^+[0,t]} |\overline{D}\phi| + \int_{\Sigma_t\cap\mathcal{N}^+} |\partial\phi| \;=\; \int_{\Sigma_0\cap\mathcal{N}^+} |\partial\phi|^2 \tag{A.2.5}$$

with $|\partial\phi|^2 := |\partial_t\phi|^2 + |\nabla\phi|^2$ *and* $|\overline{D}\phi|^2 = |L\phi|^2 + |\nabla\phi|^2 = |L\phi|^2 + \sum_{i=1}^{n-1}|e_a\phi|^2$. *Here* $(e_a)_{a=1,\dots,n-1}$ *denote unit vectors at* $p\in H$ *tangent to* H *and the corresponding time slice passing through* p.

Each coordinate vectorfield $X = \partial_i$ *leads to conservation of linear momentum and* $X = O_{ij} = x_i\partial_j - x_j\partial_i$ *leads to conservation of angular momentum.*

A.2.4 Vectorfield method and pointwise decay in Minkowski space

We denote by $E[\phi](t)$ the standard energy norm $E[\phi](t) = \int_{\Sigma_t}|\partial\phi|^2$. We introduce the generalized energy norms:

$$E_k[\phi] = \sum_{X_{i_1},..,X_{i_j}} E[X_{i_1}X_{i_2}...X_{i_j}\phi] \tag{A.2.6}$$

with the sum taken over $0\le j\le k$ and over all Killing vector fields $\mathbf{T},\mathbf{L}_{\mu\nu}$ as well as the scaling vector field $\mathbf{S}$. The crucial point of the commuting vectorfield method is that the quantities E_k, $k\ge 1$ are conserved by solutions to $\Box\phi = 0$. Therefore, if,

$$\sum_{0\le k\le s}\int (1+|x|)^{2k}\Big(|\nabla^{k+1}f(x)|^2 + |\nabla^k g(x)|^2\Big)dx \le C_s < \infty. \tag{A.2.7}$$

then for all t, $E_s[\phi](t)\le C_s$. The desired decay estimates can now be derived from the following global version of the Sobolev inequalities (see [**?**], [Kl:vect2]):

Proposition A.2.6 (Global Sobolev). *Let ϕ be an arbitrary function in R^{n+1} such that $E_s[\phi]$ is finite for some integer $s > \frac{n}{2}$. Then, for $t \geq 0$,*

$$|\partial\phi(t,x)| \lesssim (1+t+|x|)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{1}{2}} \sup_{0\leq t'\leq t} E_s[\phi](t') \tag{A.2.8}$$

for all $t > 0$. Therefore if the data f, g satisfy A.2.7, with $s > \frac{n}{2}$, then for all $t \geq 0$,

$$|\partial\phi(t,x)| \lesssim \frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}} \tag{A.2.9}$$

Moreover, relative to the null frame $L_+ = \partial_t + \partial_r$, $L_- = \partial_t - \partial_r$, $(e_a)_{a=1,\dots n-1}$

$$\begin{aligned} |(L_+, e_a)(\partial\phi)(t,x)| &\lesssim \frac{1}{(1+t+|x|)^{\frac{n+1}{2}}(1+|t-|x||)^{\frac{1}{2}}} \\ |L_-(\partial\phi)(t,x)| &\lesssim \frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{3}{2}}} \end{aligned} \tag{A.2.10}$$

and similarly for higher derivatives.

A.2.5 Global conformal energy identity

We now apply Corollary A.1.11 to the case of Minkowski space and $X = \mathbf{K}_0 = (t^2 + |x|^2)\partial_t + 2tx^i\partial_i$ with $\Omega = 4t$. Thus,

$$P_0 = \mathcal{Q}(\mathbf{K}_0, \mathbf{T}_0) + (n-1)t\phi\partial_t\phi - \frac{n-1}{2}\phi^2.$$

Proposition A.2.7. *The following identity holds in any dimension $n \geq 1$.*

$$\int_{\Sigma_t} P_0 = \frac{1}{4}\int_{\Sigma_t} \left(\underline{u}^2|L'\phi|^2 + 2(t^2+|x|^2)|\nabla\!\!\!\!/\,\phi|^2 + u^2|\underline{L}'\phi|^2\right) \tag{A.2.11}$$

where $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$, $L' = L + (n-1)\underline{u}^{-1}$, $\underline{L}' = \underline{L} + (n-1)u^{-1}$, $u = t - r$ and $\underline{u} = t + r$. Moreover if $n \geq 3$ can prove the following lower bound,

$$c^{-1}\int_{\Sigma_t} P_0 \geq c\int_{\Sigma_t} \left(\underline{u}^2|L\phi|^2 + 2(t^2+|x|^2)|\nabla\!\!\!\!/\,\phi|^2 + u^2|\underline{L}\phi|^2 + \phi^2\right) \tag{A.2.12}$$

for some $c > 0$.

Proof. We review the proof below for the sake of completeness. First obeserve that, $\mathcal{Q}_{LL} = \mathcal{Q}(L, L) = L(\phi)^2$, $\mathcal{Q}_{L\underline{L}} = \mathcal{Q}(L, \underline{L}) = |\nabla\!\!\!/\phi|^2$, $\mathcal{Q}_{\underline{L}\,\underline{L}} = \mathcal{Q}(\underline{L}, \underline{L}) = L(\phi)^2$ and that $\mathbf{K}_0 = \frac{1}{2}(\underline{u}^2 L + u^2 \underline{L})$, $\mathbf{T}_0 = \partial_t = \frac{1}{2}(L + \underline{L})$ and $\mathcal{S} = \frac{1}{2}(\underline{u}L + u\,\underline{L})$. For convenience we also introduce the vectorfield $\underline{\mathcal{S}} = \frac{1}{2}(\underline{u}L - u\,\underline{L}) = r\partial_t + t\partial_r$ Thus,

$$\mathcal{Q}(\mathbf{K}_0, \mathbf{T}_0) = \frac{1}{4}\Big(\underline{u}^2 L(\phi)^2 + (u^2 + \underline{u}^2)|\nabla\!\!\!/\phi|^2 + u^2\,\underline{L}(\phi)^2\Big)$$

and,

$$\begin{aligned} P_0 &= \frac{1}{4}\Big(\underline{u}^2(L\phi)^2 + (u^2 + \underline{u}^2)|\nabla\!\!\!/\phi|^2 + u^2(\underline{L}\phi)^2\Big) + (n-1)t\partial_t\phi\phi - \frac{n-1}{2}\phi^2 \\ &= \frac{1}{2}\Big((\mathcal{S}\phi)^2 + (\underline{\mathcal{S}}\phi)^2 + 2^{-1}(u^2 + \underline{u}^2)|\nabla\!\!\!/\phi|^2\Big) + (n-1)t\partial_t\phi\phi - \frac{n-1}{2}\phi^2 \end{aligned}$$

One then proceeds by a simple integration by parts procedure. Writing $t\partial_t = \mathbf{S} - r\partial_r$ we derive:

$$\int_{\Sigma_t} t\phi\partial_t\phi = \int_{\Sigma_t} (\mathcal{S}\phi - r\partial_r\phi)\cdot\phi = \int_{\Sigma_t} \mathcal{S}\phi\cdot\phi + \frac{n}{2}\int_{\Sigma_t}\phi^2 \qquad \text{(A.2.13)}$$

Therefore,

$$\begin{aligned} \int_{\Sigma_t} P_0 &= \int_{\Sigma_t} \frac{1}{2}\Big((\mathcal{S}\phi)^2 + (\underline{\mathcal{S}}\phi)^2 + 2^{-1}(u^2 + \underline{u}^2)|\nabla\!\!\!/\phi|^2 + (n-1)\mathcal{S}\phi\cdot\phi + \frac{(n-1)^2}{2}\phi^2\Big) \\ &= \frac{1}{2}\int_{\Sigma_t}\Big(\big(\mathcal{S}\phi + (n-1)\phi\big)^2 + (\underline{\mathcal{S}}\phi)^2 + \frac{1}{2}(u^2 + \underline{u}^2)|\nabla\!\!\!/\phi|^2\Big) \\ &= \frac{1}{4}\int_{\Sigma_t}\big(\underline{u}^2|L'\phi|^2 + (u^2 + \underline{u}^2)|\nabla\!\!\!/\phi|^2 + u^2|\underline{L}'\phi|^2\big) \end{aligned}$$

which establishes (A.2.11).

To prove (A.2.12) we use, in addition to (A.2.13), the following modification,

$$\int_{\Sigma_t} t\partial_t\phi = \int_{\Sigma_t} \frac{t}{r}\underline{S}\phi - \int_{\Sigma_t}\frac{t^2}{2r}\partial_r(\phi^2) = \int_{\Sigma_t}\frac{t}{r}\underline{S}\phi + \frac{n-2}{2}\int_{\Sigma_t}\frac{t^2}{r^2}\phi^2 \qquad \text{(A.2.14)}$$

Using positive constants A, B, $A + B = n - 1$, we write,

$$\begin{aligned} \int_{\Sigma_t}(n-1)t\phi\partial_t\phi - \frac{n-1}{2}\phi^2 &= A\int_{\Sigma_t} t\phi\partial_t\phi + B\int_{\Sigma_t} t\phi\partial_t\phi - \int_{\Sigma_t}\frac{n-1}{2}\phi^2 \\ &= \int_{\Sigma_t}\Big(AS\phi\cdot\phi + B\frac{t}{r}\underline{S}\phi\cdot\phi + \big(A\frac{n}{2} - \frac{n-1}{2}\big)\phi^2 + B\frac{n-2}{2}\frac{t^2}{r^2}\phi^2\Big) \end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\Sigma_t} P_0 &= \frac{1}{2}\int_{\Sigma_t}\Big((S\phi)^2 + 2A\phi\cdot S\phi + \big(An-(n-1)\big)\phi^2\Big) + \frac{1}{2}\int_{\Sigma_t}(u^2+\underline{u}^2)|\nabla\!\!\!/\phi|^2 \\ &+ \frac{1}{2}\int_{\Sigma_t}\Big((\underline{S}\phi)^2 + 2A\phi\cdot S\phi + B(n-2)\frac{t^2}{r^2}\phi^2\Big) + \frac{1}{2}\int_{\Sigma_t}(u^2+\underline{u}^2)|\nabla\!\!\!/\phi|^2\end{aligned}$$

Now observe that, if $0 < A < (n-1)$ and $0 < B < n-2$ we can find $c_1, c_2 > 0$ such that

$$\begin{aligned}&(S\phi)^2 + 2A\phi\cdot S\phi + \big(An-(n-1)\big)\phi^2 \geq c_1\big((S\phi)^2+\phi^2\big)\\ &(\underline{S}\phi)^2 + 2A\phi\cdot S\phi + B(n-2)\frac{t^2}{r^2}\phi^2 \geq c_2((\underline{S}\phi)^2 + \frac{t^2}{r^2}\phi^2\big)\end{aligned}$$

If $n \geq 3$ one can find A, B verifying $0 < A < (n-1)$, $0 < B < n-2$ such that $A+B = n-1$. Therefore taking c the minimum of c_1, c_2 we derive,

$$\begin{aligned}c^{-1}\big(\int_{\Sigma_t} P_0 - \frac{1}{2}\int_{\Sigma_t}(u^2+\underline{u}^2)|\nabla\!\!\!/\phi|^2\big) &\geq \int_{\Sigma_t}\big(|S(\phi)|^2 + |\underline{S}(\phi)|^2 + \frac{1}{2}\phi^2\big)\\ &= \frac{1}{2}\int_{\Sigma_t}\big(\underline{u}^2|L(\phi)|^2 + u^2|\,\underline{L}(\phi)|^2 + \phi^2\big)\end{aligned}$$

Hence, for some other $c > 0$,

$$c^{-1}\int_{\Sigma_t} P_0 \geq \int_{\Sigma_t}\big(\underline{u}^2|L(\phi)|^2 + u^2|\,\underline{L}(\phi)|^2 + 2(t^2+|x|^2)|\nabla\!\!\!/\phi|^2 + \phi^2\big)$$

as desired. □

Remark A.2.8. *The second part of the Proposition is typical to the use of Hardy type inequalities to estimate the lower order term in ϕ.*

As a corollary we have the following

Corollary A.2.9. *If $\Box\phi = 0$, $\phi(0) = f$, $\partial_t\phi(0) = g$*

$$\int_{\Sigma_t}\underline{u}^2|L'(\phi)|^2 + u^2|\,\underline{L}'(\phi)|^2 + (t^2+r^2)|\nabla\phi|^2 + \phi^2 \lesssim \int_{\Sigma_0}|f|^2 + |x|^2|\nabla f|^2 + |x|^2|g|^2$$

and $n \geq 3$ we have,

$$\int_{\Sigma_t}\underline{u}^2|L(\phi)|^2 + u^2|\,\underline{L}(\phi)|^2 + (t^2+r^2)|\nabla\phi|^2 + \phi^2 \lesssim \int_{\Sigma_0}|f|^2 + |x|^2|\nabla f|^2 + |x|^2|g|^2$$

A.2.6 Null Conformal energy

Proposition A.2.10. *Consider the domain $\mathcal{D}$ to be the complement of the causal future of $D_R = \{|x| \le R\}$ in $\mathbb{R}^{1+n}_+$, for some $R > 0$. Denote by $\mathcal{D}(\tau)$ the intersection of $\mathcal{D}$ with the time slab $0 \le t \le \tau$. Denote by $\mathcal{H}^+[0,\tau]$ the future boundary of $\mathcal{D}$ intersected with the same time slab. Also denote by $\Sigma(\tau)$ the spacelike hypersurface $t = \tau$.*

The following estimate holds true.

$$C_+[\phi](\tau) + \int_{\partial\mathcal{H}^+[0,\tau]} \frac{1}{2}\big(\underline{u}L\phi - (n-1)\phi\big)^2 + \frac{1}{2}u^2|\nabla\!\!\!/\phi|^2 \lesssim \int_{\Sigma_0\cap\mathcal{H}^+} |f|^2 + |x|^2|\nabla f|^2 + |x|^2|g|^2$$

where $C_+[\phi](\tau)$ is the conformal energy restricted to $\Sigma^+(\tau) = \Sigma(\tau)\cap\mathcal{D}$

$$C_+[\phi](\tau) = \frac{1}{4}\int_{\Sigma(\tau)\cap\mathcal{H}^+} \big(\underline{u}^2|L'\phi|^2 + (u^2+\underline{u}^2)|\nabla\!\!\!/\phi|^2 + u^2|\,\underline{L}'\phi|^2\big).$$

In particular

$$\int_{\partial\mathcal{H}^+[0,\tau]} \frac{1}{2}\big(\underline{u}L\phi - (n-1)\phi\big)^2 + \frac{1}{2}u^2|\nabla\!\!\!/\phi|^2 \;\lesssim\; \int_{\Sigma_0\cap\mathcal{H}^+} |f|^2 + |x|^2|\nabla f|^2 + |x|^2|g|^2$$

Proof. We first consider the case when $\mathcal{H}^+$ is the complement . We apply formula (A.2.15) to $X = \mathbf{K}_0$.

$$\int_{\partial\mathcal{H}^+[0,\tau]} \mathbf{m}(P,L) + \int_{\Sigma(\tau)\cap\mathcal{H}^+} P_0 = \int_{\Sigma_0\cap\mathcal{H}^+} P_0 \tag{A.2.15}$$

where $L = \partial_t + \partial_r$ and

$$\begin{aligned} \mathbf{m}(P,L) &= \mathcal{Q}(\mathbf{K}_0, L) + (n-1)t\phi L\phi - \frac{n-1}{2}\phi^2 \\ P_0 = \mathbf{m}(P,\mathbf{T}_0) &= \mathcal{Q}(\mathbf{K}_0,\mathbf{T}_0) + (n-1)t\phi\partial_t\phi - \frac{n-1}{2}\phi^2. \end{aligned}$$

The integral on $\mathcal{H}$ is defined in the same way as in A.2.3.

We consider first the integral $\int_{\Sigma_\tau\cap\mathcal{H}^+} P_0$ which can be treated exactly as $\int_{\Sigma_t} P_0$ in the previous subsection. The only modification we need to make are in the integration by parts formulas (A.2.13) and (A.2.14) where now need to take into account the boundary terms. Thus (A.2.13) becomes,

$$\int_{\Sigma(\tau)\cap\mathcal{H}^+} t\phi\partial_t\phi \;=\; \int_{\Sigma(\tau)\cap\mathcal{H}^+} \mathcal{S}\phi\cdot\phi + \frac{n}{2}\int_{\Sigma(\tau)\cap\mathcal{H}^+}\phi^2 + \frac{1}{2}\int_{S_{\tau,R}} r\phi^2 d\sigma$$

where $S_{\tau,R}$ is the ball of radius R on $\Sigma(\tau)$ and $d\sigma$ its volume form. Thus, proceeding as in the derivation of (A.2.11) we deduce,

$$\begin{aligned}\int_{\Sigma(\tau)\cap\mathcal{H}^+} P_0 &= \frac{1}{4}\int_{\Sigma(\tau)\cap\mathcal{H}^+} \big(\underline{u}^2|L'\phi|^2 + (u^2+\underline{u}^2)|\nabla\!\!\!\!/\phi|^2 + u^2|\,\underline{L}'\phi|^2\big) \\ &+ \frac{n-1}{2}\int_{S_{\tau,R}} r\phi^2 d\sigma\end{aligned} \tag{A.2.16}$$

We now consider the null boundary integral,

$$\begin{aligned}\int_{\partial\mathcal{H}^+[0,t\tau}\mathbf{m}(P,L) &= \int_{\partial\mathcal{H}^+[0,\tau]}\Big(\mathcal{Q}(\mathbf{K}_0,L) + (n-1)t\phi L\phi - \frac{n-1}{2}\phi^2\Big)d\sigma \\ &= \int_{\partial\mathcal{H}^+[0,\tau]}\frac{1}{2}\big(\underline{u}^2L(\phi)^2 + u^2|\nabla\!\!\!\!/\phi|^2\big) + (n-1)t\phi L\phi - \frac{n-1}{2}\phi^2 \\ &= J + \int_{\partial\mathcal{H}^+[0,\tau]}\frac{1}{2}u^2|\nabla\!\!\!\!/\phi|^2\end{aligned}$$

Now, by a simple integration by parts[5] we deduce,

$$\int_{\partial\mathcal{H}^+[0,\tau]}|x|\phi L\phi = -\frac{n}{2}\int_{\partial\mathcal{H}^+[0,\tau]}\phi^2 + \frac{1}{2}\int_{S_{\tau,R}}|x|\phi^2$$

On the other hand by a simple calculation, recalling that $\underline{u} = t + r$,

$$\begin{aligned}J &= \int_{\partial\mathcal{H}^+[0,\tau]}\frac{1}{2}\underline{u}^2(L\phi)^2 + (n-1)t\phi L\phi - \frac{n-1}{2}\phi^2 \\ &= \int_{\partial\mathcal{H}^+[0,\tau]}\frac{1}{2}\big(\underline{u}L\phi - (n-1)\phi\big)^2 - \frac{n-1}{2}\int_{S_{\tau,R}}|x|\phi^2\end{aligned}$$

Therefore,

$$\int_{\partial\mathcal{H}^+[0,\tau]}\mathbf{m}(P,L) = \int_{\partial\mathcal{H}^+[0,\tau]}\frac{1}{2}\big(\underline{u}L\phi - (n-1)\phi\big)^2 + \frac{1}{2}u^2|\nabla\!\!\!\!/\phi|^2 - \frac{n-1}{2}\int_{S_{\tau,R}}|x|\phi^2$$

Recalling (A.2.16) and (A.2.15) we deduce,

$$\begin{aligned}\int_{\partial\mathcal{H}^+[0,\tau]}\mathbf{m}(P,L) + \int_{\Sigma_\tau\cap\mathcal{H}^+}P_0 &= \int_{\partial\mathcal{H}^+[0,\tau]}\frac{1}{2}\big(\underline{u}L\phi - (n-1)\phi\big)^2 + \frac{1}{2}u^2|\nabla\!\!\!\!/\phi|^2 \\ &+ \frac{1}{4}\int_{\Sigma_\tau\cap\mathcal{H}^+}\big(\underline{u}^2|L'\phi|^2 + (u^2+\underline{u}^2)|\nabla\!\!\!\!/\phi|^2 + u^2|\,\underline{L}'\phi|^2\big) \\ &= \int_{\Sigma_0\cap\mathcal{H}^+}P_0.\end{aligned}$$

[5] $\int_{\partial\mathcal{H}^+(0,\tau)}\phi^2 = \int_0^\tau ds\int_{|y|=1}\phi^2(s,(R+s)y)(R+s)^{n-1}d\sigma_y = \frac{1}{n}\int_{|y|=1}d\sigma_y\int_0\tau\phi^2\frac{d}{ds}(R+s)^n ds$ $= -\frac{1}{n}\int_{|y|=1}d\sigma_y\int_0^\tau\frac{d}{ds}\phi^2\ (R{+}s)^n ds + \frac{1}{n}\int_{|y|=1}(R{+}\tau)^n\phi^2(t,(R{+}\tau)y)d\sigma_y = -\frac{2}{n}\int_0^\tau ds\int_{|x|=s+R}|x|\phi L\phi d\sigma_x +$ $\frac{1}{n}\int_{|x|=\tau+R}|x|\phi^2(t,x)d\sigma_x = -\frac{2}{n}\int_{\mathcal{H}^+(0,\tau)}|x|\phi L\phi + \frac{1}{n}\int_{S_{\tau,R}}|x|\phi^2.$

Therefore

$$\int_{\partial\mathcal{H}^+[0,\tau]} \frac{1}{2}\Big(\big(\underline{u}L\phi-(n-1)\phi\big)^2+u^2|\nabla\!\!\!\!/\,\phi|^2-(n-1)\phi^2\Big)$$
$$+\frac{1}{4}\int_{\Sigma_\tau\cap\mathcal{H}^+}\big(\underline{u}^2|L'\phi|^2+(u^2+\underline{u}^2)|\nabla\!\!\!\!/\,\phi|^2+u^2|\,\underline{L}'\phi|^2\big)=\int_{\Sigma_0\cap\mathcal{H}^+}P_0$$

from which the desire estimate easily follows. □

A.3 Other integral estimates

A.3.1 Morawetz Estimates

Besides the standard Killing and conformal Killing vectorfields of Minkowski space we encounter other useful vectorfields which lead to bulk estimates. The primary example is the so called Morawetz estimate.

Lemma A.3.1. *Consider the vectorfields $Y=\partial_r$ and $X=f(r)\partial_r$ in Minkowski space $\mathcal{R}^{1+n}$.*

1. *The deformation tensor of the vectorfield $Y=\partial_r$ is given by:*

$$^{(Y)}\pi_{00}={}^{(Y)}\pi_{0i}=0,\ {}^{(Y)}\pi_{ij}=\frac{2}{r}(\delta_{ij}-\frac{x_i}{|x|}\frac{x_j}{|x|}),\quad i,j=1,\dots n,\qquad tr\,^{(Y)}\pi=\frac{2(n-1)}{r}.$$

or, relative to a null frame $e_1,e_2,e_3=\underline{L}=\partial_u, e_4=L=\partial_v$ the only nonvanishing components are,

$$^{(Y)}\pi_{ab}=\frac{2}{r}\delta_{ab},\qquad a,b=1,2,\dots n-1.$$

2. *The only nonvanishing components of deformation tensor of the vectorfield $X=f(r)Y=f(r)\partial_r$ are given by*

$$^{(X)}\pi_{rr}=2f'(r),\qquad {}^{(X)}\pi_{ab}=\frac{2f(r)}{r}\delta_{ab},\qquad tr\,^{(X)}\pi=2(f'(r)+\frac{n-1}{r}f)$$

Proof. $^{(Y)}\pi$ can be easily calculated either in cartesian coordinates, since $\partial_r=\frac{x^i}{|x|}\partial_i$ or in polar coordinates. To calculate $^{(X)}\pi$ note that given $X=fY$ we have,

$$^{(X)}\pi_{\alpha\beta}={}^{(fY)}\pi_{\alpha\beta}=f\,^{(Y)}\pi_{\alpha\beta}+\mathbf{D}_\alpha fY_\beta+\mathbf{D}_\beta fY_\alpha,\qquad \mathrm{tr}\,^{(X)}\pi=\mathrm{tr}(^{(fY)}\pi)=f\mathrm{tr}\,^{(Y)}\pi+2Y(f)$$

Note also that the deformation tensor of L is the same as that of $Y = \partial_r$. Hence,

$$^{(V)}\pi_{\alpha\beta} = {}^{(fL)}\pi_{\alpha\beta} = f\,{}^{(L)}\pi_{\alpha\beta} + \mathbf{D}_\alpha f L_\beta + \mathbf{D}_\beta f L_\alpha = f\,{}^{(Y)}\pi_{\alpha\beta} + \mathbf{D}_\alpha f L_\beta + \mathbf{D}_\beta f L_\alpha$$

Hence,

$$^{(V)}\pi_{33} \;=\; 4f'(r),\ \ {}^{(V)}\pi_{34} = -2f'(r),\ \ {}^{(V)}\pi_{44} = {}^{(V)}\pi_{3a} = {}^{(V)}\pi_{4a} = 0,\ \ {}^{(V)}\pi_{ab} = 2r^{-1}f(r)\delta_{ab}$$

as desired. □

We now specialize to $n = 3$ and calculate the term,

$$\square(r^{-1}f(r)) = r^{-1}f''(r) - 4\pi f(r)\delta_0 - 2r^{-2}f'(r)$$

We deduce the following,

Proposition A.3.2. *Given $X = f(r)\partial_r$ and $w = \frac{2}{r}f(r)$, the 1-form in $\mathbb{R}^{1+3}$,*

$$P_\mu[X, w, 0] \;=\; \mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\phi\partial_\mu\phi - \frac{1}{4}\partial_\mu w\phi^2$$

verifies the divergence identity:

$$\begin{aligned}\mathbf{D}^\mu P_\mu = &\frac{1}{2}f'(r)(\partial_t\phi)^2 + \frac{1}{2}f'(r)(\partial_r\phi)^2 + \big(r^{-1}f - \frac{1}{2}f'(r)\big)|\nabla\!\!\!\!/\,\phi|^2\\ &- \frac{1}{2r}f''(r)\phi^2 + f(r)(\partial_r\phi + r^{-1}\phi)\square\phi\end{aligned} \tag{A.3.1}$$

Proof. According to Lemma A.1.10 we have,

$$\mathbf{D}^\mu P_\mu \;=\; (X(\phi) + \frac{1}{2}w\phi)\square\phi + \frac{1}{2}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} - \frac{1}{4}\square w\phi^2 + \frac{1}{2}w\mathbf{g}(d\phi, d\phi).$$

Using Lemma A.3.1 and $\square w = 2\square(r^{-1}f(r)) = 2r^{-1}f''(r) - 8\pi f(r)\delta_0 - 4r^{-2}f'$ we then derive

$$\begin{aligned}\mathbf{D}^\mu P_\mu = &\frac{1}{2}f'(r)(\partial_t\phi)^2 + \frac{1}{2}f'(r)(\partial_r\phi)^2 + \big(r^{-1}f - \frac{1}{2}f'(r)\big)|\nabla\!\!\!\!/\,\phi|^2 - \big(\frac{1}{2r}f''(r) - r^{-2}f'\big)\phi^2\\ &+ \frac{8}{\pi}\phi^2\delta_0 + f(r)(\partial_r\phi + r^{-1}\phi)\square\phi\end{aligned} \tag{A.3.2}$$

□

Proposition A.3.3. *Let $\mathcal{D} = \mathcal{D}(\tau) = \big\{(t,x) \in \mathbb{R}^{1+n} : x \in \mathbb{R}^3,\ 0 \le t \le \tau\big\}$ For every $0 < \delta < 1$, (with a constant dependent of δ), we have*

$$\int_{\mathcal{D}} (1+r)^{-1-\delta}\big(|\partial\phi|^2 + r^{-2}|\phi|^2\big) \lesssim \mathcal{E}[\phi](0) + \int_{\mathcal{D}_\tau} (1+r)^{1+\delta}|\square\phi|^2$$

Proof. Choose $f(r) = 1 - \frac{1}{(1+r)^\delta}$ to the identity (A.3.2). Observe that

$$f'(r) = \frac{\delta}{(1+r)^{1+\delta}}, \quad f''(r) = -\frac{\delta(1+\delta)}{(1+r)^{2+\delta}}, \quad \frac{f(r)}{r} \geq \frac{\delta}{(1+r)^{1+\delta}} = f'(r)$$

and,

$$r^{-1}f, f'(r) \leq (1+r)^{-1-\delta}, \qquad |f''(r)| \lesssim (1+r)^{-2-\delta}, \qquad |f'''(r)| \lesssim (1+r)^{-3-\delta}$$

Finally

$$-\frac{1}{2r}f''(r) + r^{-2}f' = \frac{1}{2r}\frac{\delta(1+\delta)}{(1+r)^{2+\delta}} + r^{-2}\frac{\delta}{(1+r)^{r+\delta}} \geq \frac{1}{2r}\frac{\delta(1+\delta)}{(1+r)^{2+\delta}}$$

Also, since $f(0) = 0$ and $\Box(r^{-1}f(r)) = 2r^{-1}f''(r)$,

$$\begin{aligned} \mathbf{D}^\mu P_\mu &= \frac{1}{2}f'(r)(\partial_t\phi)^2 + \frac{1}{2}f'(r)(\partial_r\phi)^2 + \big(r^{-1}f - \frac{1}{2}f'(r)\big)|\nabla\!\!\!/\phi|^2 - \frac{1}{2r}f''(r)\phi^2 \\ &+ f(r)(\partial_r\phi + r^{-1}\phi)\Box\phi. \end{aligned}$$

We deduce

$$D^\mu P_\mu \geq f(\partial_r\phi + r^{-1}\phi)\Box\phi + \frac{1}{2r}f|\nabla\!\!\!/\phi|^2 + \frac{1}{2}f'(|\partial_r\phi|^2 + |\partial_t\phi|^2) + \frac{1}{2r}\frac{\delta(1+\delta)}{(1+r)^{2+\delta}}|\phi|^2$$

Using the divergence theorem and the positivity of f, f' we deduce

$$\int_{\mathcal{D}}(1+r)^{-1-\delta}\big(|\partial\phi|^2 + r^{-2}|\phi|^2\big) \lesssim \Big|E[\phi](\tau) - E[\phi](0)\Big| + \int_{\mathcal{D}}(1+r)^{1+\delta}|\Box\phi|^2$$

The result then follows by making use of conservation of energy estimate, i.e.

$$E[\phi](\tau) \leq E[\phi](0) + \int_{\mathcal{D}}|\Box\phi|^2.$$

$\Box$

Remark A.3.4. *Choosing $f = 1$ in (A.3.2) we derive*

$$\mathbf{D}^\mu P_\mu = r^{-1}|\nabla\!\!\!/\phi|^2 + \frac{8}{\pi}\phi^2\delta_0 + (\partial_r\phi + r^{-1}\phi)\Box\phi.$$

Thus, after integration,

$$\begin{aligned} \int_{\mathcal{D}(\tau)} r^{-1}|\nabla\!\!\!/\phi|^2 + 8\pi\int_0^\tau|\phi|^2 &= \int_{\Sigma_0}P_0 - \int_{\Sigma_t}P_0 - \int_{\mathcal{D}(\tau)}(\partial_r\phi + r^{-1}\phi)\Box\phi \\ &\lesssim 2E(0) + \int_{\mathcal{D}(\tau)}\big|\partial_r\phi + r^{-1}\phi\big|\big|\Box\phi\big|. \end{aligned}$$

A.3.2 r^p Weighted Flux Estimates

In this section we consider domains $\mathcal{D}(\tau_1, \tau_2)$ as in the picture below.

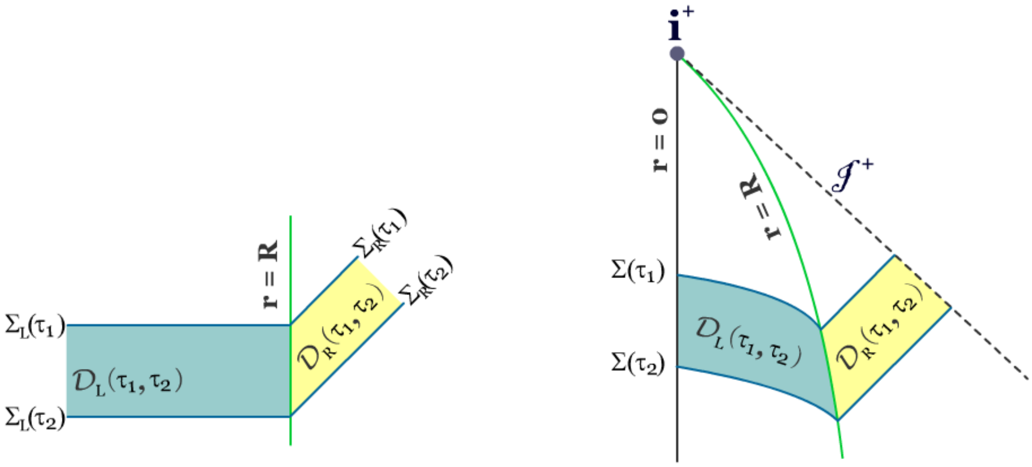


Figure A.2: small Causal Domains $\mathcal{D}(\tau_1, \tau_2)$ with past and future boundaries $\Sigma(\tau_1), \Sigma(\tau_2)$ consisting of the two sides Σ_L, spacelike and Σ_R null.

Theorem A.3.5. *The following weighted flux inequalities hold true, for $0 \leq p \leq 2$:*

$$\begin{aligned}&\int_{\Sigma_R(\tau_2)} r^p(\hat{L}\phi)^2 + \int\int_{\mathcal{D}_R(\tau_1,\tau_2)} r^{p-1}\big(p(\hat{L}\phi)^2 + (2-p)|\nabla\!\!\!\!/\,\phi|^2\big) + \int_{\mathcal{I}^+(\tau_1,\tau_2)} r^p|\nabla\!\!\!\!/\,\phi|^2 \\ &\lesssim \int_{\Sigma_R(\tau_1)} r^p(\hat{L}\phi)^2 + R^p\mathcal{E}[\phi](\tau_1) + I_{p+1}[\Box\phi](\tau_1,\tau_2)\end{aligned} \tag{A.3.3}$$

where,

$$\hat{L}\phi := L\phi + \frac{1}{r}\phi = (\partial_t + \partial_r)\phi + \frac{1}{r}\phi.$$

and,

$$I_{p+1}[\Box\phi](\tau_1,\tau_2) := \int_{\mathcal{D}_R(\tau_1,\tau_2)} r^{p+1}|\Box\phi|^2 + R^p\int_{\mathcal{D}(\tau_1,\tau_2)} (1+r)^{1+\delta}|\Box\phi|^2$$

Remark A.3.6. *In reality the proof gives the estimate,*

$$\begin{aligned}&\int_{\Sigma_R(\tau_2)} r^p(\hat{L}\phi)^2 + \int\int_{\mathcal{D}(\tau_1,\tau_2)} r^{p-1}\big(p(\hat{L}\phi)^2 + (2-p)|\nabla\!\!\!\!/\,\phi|^2\big) + \int_{\mathcal{I}^+(\tau_1,\tau_2)} r^p|\nabla\!\!\!\!/\,\phi|^2 \\ &\lesssim \int_{\Sigma_R(\tau_1)} r^p(\hat{L}\phi)^2 + R^p\mathcal{E}[\phi](\tau_1) + I_{p+1}[\Box\phi](\tau_1,\tau_2)\end{aligned}$$

without making use of the Morawetz integrated decay estimate. We do not expect this to be true in a black hole situation where the integrated decay estimate will have to be used. This would lead to a loss of derivative.

Proof. We make use of the pointwise identities of proposition **??**, which we recall below.

$$\mathrm{Div} P = f(r)\widehat{L}\phi\Box\phi + (r^{-1}f - \frac{1}{2}f')|\nabla\!\!\!\!/\,\phi|^2 + \frac{1}{2}f'(r)(\widehat{L}\phi)^2 \tag{A.3.4}$$

where,

$$\begin{aligned} P_\mu[X, w, M] &= Q_{\mu\nu}X^\nu + \frac{1}{2}w\phi\partial_\mu\phi - \frac{1}{4}\partial_\mu w\phi^2 + \frac{1}{4}M_\mu\phi^2 \\ X &= f(r)(\partial_t + \partial_r) = f(r)L, \qquad w = \frac{2}{r}f, \qquad M = 2r^{-1}f'(r)L \end{aligned}$$

Also,

$$\begin{aligned} P\cdot L &= f(r)(\widehat{L}\phi)^2 - \frac{1}{2}r^{-2}\partial_v\big(rf(r)\phi^2\big) \\ P\cdot \underline{L} &= f(r)|\nabla\!\!\!\!/\,\phi|^2 + \frac{1}{2}r^{-2}\partial_u\big(rf\phi^2\big) \\ P\cdot\partial_t &= \frac{1}{2}f(r)\left[(\widehat{L}\phi)^2 + |\nabla\!\!\!\!/\,\phi|^2\right] - \frac{1}{2}r^{-2}\partial_r\big(rf(r)\phi^2\big) \end{aligned}$$

Start with the formula,

$$\int_{\Sigma(\tau_2)} P\cdot\nu + \int_{\mathcal{I}^+(\tau_1,\tau_2)} P\cdot\nu = \int_{\Sigma(\tau_1)} P\cdot\nu - \int_{\mathcal{D}(\tau_1,\tau_2)} \mathrm{Div}P$$

with $\nu = \partial_t$ along Σ_L and $\nu = L$ along Σ_R. We have,

$$\begin{aligned} \int_{\Sigma(\tau)} P\cdot\nu &= \int_{\Sigma_L(\tau)} P\cdot\partial_t + \int_{\Sigma_R(\tau)} P\cdot L \\ &= \int_{\Sigma_L(\tau)} \frac{1}{2}f(r)\left[(\widehat{L}\phi)^2 + |\nabla\!\!\!\!/\,\phi|^2\right] - \int_{\Sigma_R(\tau)} f(r)(\widehat{L}\phi)^2 \\ &- \frac{1}{2}\int_{\Sigma_L(\tau)} r^{-2}\partial_r\big(rf(r)\phi^2\big) - \frac{1}{2}\int_{\Sigma_R(\tau)} r^{-2}\partial_v\big(rf(r)\phi^2\big) \\ &= \int_{\Sigma_L(\tau)} \frac{1}{2}f(r)\left[(\widehat{L}\phi)^2 + |\nabla\!\!\!\!/\,\phi|^2\right] + \int_{\Sigma_R(\tau)} f(r)(\widehat{L}\phi)^2 - \frac{1}{2}\lim_{V\to\infty}\int_{S_{u_\tau},V} r^{-1}f(r)\phi^2 \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{\mathcal{I}^+(\tau_1,\tau_2)} P\cdot\nu &= \int_{\mathcal{I}^+(\tau_1,\tau_2)} P\cdot\underline{L} = \int_{\mathcal{I}^+(\tau_1,\tau_2)} f(r)|\nabla\!\!\!\!/\,\phi|^2 + \frac{1}{2}\int_{\mathcal{I}^+(\tau_1,\tau_2)} r^{-2}\partial_u\big(rf\phi^2\big) \\ &= \int_{\mathcal{I}^+(\tau_1,\tau_2)} f(r)|\nabla\!\!\!\!/\,\phi|^2 + \frac{1}{2}\lim_{V\to\infty}\int_{S_{u_{\tau_2}},V} r^{-1}f(r)\phi^2 - \frac{1}{2}\lim_{V\to\infty}\int_{S_{u_{\tau_1}},V} r^{-1}f(r)\phi^2 \end{aligned}$$

Hence,

$$\begin{aligned}\int_{\Sigma(\tau_2)} P\cdot\nu + \int_{\mathcal{I}^+(\tau_1,\tau_2)} P\cdot\nu - \int_{\Sigma(\tau_1)} P\cdot\nu &= \int_{\Sigma_R(\tau_2)} f(r)(\widehat{L}\phi)^2 + \int_{\Sigma_L(\tau_2)} \frac{1}{2} f(r)\left[(\widehat{L}\phi)^2 + |\nabla\!\!\!/\phi|^2\right] \\ &- \int_{\Sigma_R(\tau_1)} f(r)(\widehat{L}\phi)^2 - \int_{\Sigma_L(\tau_1)} \frac{1}{2} f(r)\left[(\widehat{L}\phi)^2 + |\nabla\!\!\!/\phi|^2\right]\end{aligned}$$

and we derive,

$$\begin{aligned}\int_{\Sigma_R(\tau_2)} f(r)(\widehat{L}\phi)^2 + \int_{\mathcal{I}^+(\tau_1,\tau_2)} f(r)|\nabla\!\!\!/\phi|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)} \mathrm{Div}P &= \int_{\Sigma_R(\tau_1)} f(r)(\widehat{L}\phi)^2 \\ &+ \int_{\Sigma_L(\tau_1)} \frac{1}{2} f(r)\left[(\widehat{L}\phi)^2 + |\nabla\!\!\!/\phi|^2\right] \\ &- \int_{\Sigma(\tau_2)} \frac{1}{2} f(r)\left[(\widehat{L}\phi)^2 + |\nabla\!\!\!/\phi|^2\right]\end{aligned}$$

On the other hand,

$$\int_{\mathcal{D}(\tau_1,\tau_2)} \mathrm{Div}P = \int_{\mathcal{D}(\tau_1,\tau_2)} (r^{-1}f - \frac{1}{2}f')|\nabla\!\!\!/\phi|^2 + \frac{1}{2}f'(r)(\widehat{L}\phi)^2 + \int_{\mathcal{D}(\tau_1,\tau_2)} f(r)\widehat{L}\phi\Box\phi$$

for $f = r^p$,

$$\int_{\mathcal{D}(\tau_1,\tau_2)} \mathrm{Div}P = \int_{\mathcal{D}(\tau_1,\tau_2)} r^{p-1}\left[\frac{1}{2}(2-p)|\nabla\!\!\!/\phi|^2 + \frac{p}{2}(\widehat{L}\phi)^2\right] + \int_{\mathcal{D}(\tau_1,\tau_2)} r^p\widehat{L}\phi\Box\phi$$

Hence,

$$\begin{aligned}&\int_{\Sigma_R(\tau_2)} r^p(\widehat{L}\phi)^2 + \int_{\mathcal{I}^+(\tau_1,\tau_2)} r^p|\nabla\!\!\!/\phi|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)} r^{p-1}\left[\frac{1}{2}(2-p)|\nabla\!\!\!/\phi|^2 + \frac{p}{2}(\widehat{L}\phi)^2\right] \\ &= \int_{\Sigma_R(\tau_1)} r^p(\widehat{L}\phi)^2 + |\nabla\!\!\!/\phi|^2 + \mathrm{Err}\end{aligned}$$

where

$$\mathrm{Err} = \int_{\Sigma_L(\tau_1)} \frac{1}{2}r^p\left[(\widehat{L}\phi)^2 + |\nabla\!\!\!/\phi|^2\right] - \int_{\Sigma(\tau_2)} \frac{1}{2}r^p\left[(\widehat{L}\phi)^2 + |\nabla\!\!\!/\phi|^2\right] + \int_{\mathcal{D}(\tau_1,\tau_2)} r^p\widehat{L}\phi\Box\phi$$

Clearly,

$$\begin{aligned}|\mathrm{Err}| &\lesssim R^p\big(\mathcal{E}[\phi](\tau_1) + \mathcal{E}[\phi](\tau_2)\big) + \epsilon\int_{\mathcal{D}(\tau_1,\tau_2)} r^{p-1}|\widehat{L}\phi|^2 + \epsilon^{-1}\int_{\mathcal{D}(\tau_1,\tau_2)} r^{p+1}|\Box\phi|^2 \\ &\lesssim 2R^p\mathcal{E}[\phi](\tau_1) + \epsilon\int_{\mathcal{D}(\tau_1,\tau_2)} r^{p-1}|\widehat{L}\phi|^2 + \epsilon^{-1}\int_{\mathcal{D}(\tau_1,\tau_2)} r^{p+1}|\Box\phi|^2\end{aligned}$$

We deduce, for $\epsilon = \frac{p}{4}$,

$$\int_{\Sigma_R(\tau_2)} r^p(\widehat{L}\phi)^2 + \int_{\mathcal{I}^+(\tau_1,\tau_2)} r^p|\nabla\!\!\!/\phi|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)} r^{p-1}\left[\frac{1}{2}(2-p)|\nabla\!\!\!/\phi|^2 + (\frac{p}{4})(\widehat{L}\phi)^2\right]$$
$$\leq 2R^p\mathcal{E}[\phi](\tau_1) + \frac{4}{p}\int_{\mathcal{D}(\tau_1,\tau_2)} r^{p+1}|\Box\phi|^2$$

□

TO BE REVIEWED

A.3.3 Decay of the Energy Flux

Calculus Lemmas

We start with a few simple remarks.

Lemma A.3.7. *Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a C^1 non-negative function verifying, for all $0 \leq t_1 \leq t_2$,*

$$f(t_2) + A\int_{t_1}^{t_2} f(s)ds \leq f(t_1)$$

Then, for all $0 \leq t$,

$$f(t) \quad \leq \quad f(0)e^{-At}$$

Proof. Consider $H(t) := f(t) + A\int_0^t f(s)ds$. Clearly $H(t_2) \leq H(t_1)$ for all $0 \leq t_1 \leq t_2$. Hence $H'(t) \leq 0$ and therefore,

$$e^{-At}\frac{d}{dt}(e^{At}f(t)) = f'(t) + Af(t) \leq 0.$$

□

Lemma A.3.8. *Consider a sequence of continuous functions $f_k : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that,*

$$0 \leq f_0(t) \leq \ldots \leq f_k(t)$$

and, for all $\tau_1 \leq \tau_2 \in [0, T]$ and all $1 \leq i \leq k$,

$$f_i(\tau_2) + \int_{\tau_1}^{\tau_2} f_{i-1}(s)ds \leq f_i(\tau_1) \tag{A.3.5}$$

Then,

$$f_0(T) \leq (T/k)^{-k} f_k(0)$$

Proof. We divide $[0, T]$ in k subintevals of length T/k, i.e. $t_0 = 0 < t_1 \ldots < t_k = T$. In each inteval $I_j = [t_{j-1}, t_j]$ we make use of (A.3.5) i.e.,

$$f_i(t) + \int_{t_{j-1}}^{t} f_{i-1}(s)ds \leq f_i(t_{j-1}), \qquad \forall t \in I_j = [t_{j-1}, t_j]$$

In particular,

$$\int_{I_j} f_{i-1}(s)ds \leq f_i(t_{j-1})$$

and therefore, by the mean value theorem. there exists $\tau \in I_j$ such that,

$$f_{i-1}(\tau) = \frac{1}{|I_j|} \int_{I_j} f_{i-1}(s)ds \leq (T/k)^{-1} f_i(t_{j-1})$$

On the other hand, according to (A.3.5) applied to f_{i-1} we have, since $\tau \leq t_j$,

$$f_{i-1}(t_j) \leq f_{i-1}(\tau)$$

We deduce,

$$f_{i-1}(t_j) \leq (T/k)^{-1} f_i(t_{j-1}) \tag{A.3.6}$$

Consequently,

$$f_0(T) = f_0(t_k) \leq (T/k)^{-1} f_1(t_{k-1}) \leq (T/k)^{-2} f_2(t_{k-2}) \leq \cdots \leq (T/k)^{-k} f_k(t_0) = (T/k)^{-k} f_k(0)$$

as desired.

□

We now generalize the lemma a bit to allow for inhomogeneities.

Proposition A.3.9. *Consider a sequence of continuous functions* $f_k : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *such that,*

$$0 \leq f_0(t) \leq \ldots \leq f_k(t)$$

and, for all $\tau_1 \le \tau_2 \in [0, T]$, $1 \le i \le k$,

$$f_i(\tau_2) + \int_{\tau_1}^{\tau_2} f_{i-1}(s)ds \lesssim f_i(\tau_1) + \int_{\tau_1}^{\tau_2} F_i(s)ds \tag{A.3.7}$$

where F_i are given non-negative continuous functions in $[0, T]$. Then,

$$f_0(T) \quad \lesssim \quad T^{-k} \left[f_k(0) + \sum_{0 \le i \le k} \sup_{\tau \in [0,T]} \int_{\tau}^{T} s^{k-i} F_i(s)ds \right] \tag{A.3.8}$$

In fact

$$f_i(\tau_2) + \int_{\tau_1}^{\tau_2} f_{i-1}(s)ds \lesssim f_i(\tau_1) + f_0(\tau_1) + \int_{\tau_1}^{\tau_2} F_i(s)ds \tag{A.3.9}$$

Proof. We divide $[0, T]$ in $k+1$ subintevals of length $T/(k+1)$, i.e. $t_0 = 0 < t_1 < t_2 \ldots < t_{k+1} = T$. In each inteval $I_j = [t_j, t_{j+1}]$ we make use of (A.3.7) i.e.,

$$f_i(t) + \int_{t_j}^{t} f_{i-1}(s)ds \lesssim f_i(t_j) + \int_{t_j}^{t} F_i(s)ds, \qquad \forall t \in I_j = [t_j, t_{j+1}]$$

In particular,

$$\int_{I_j} f_{i-1}(s)ds \lesssim f_i(t_j) + \int_{I_j} F_i(s)ds$$

and therefore, by the mean value theorem there exists $\tau \in I_j$ such that,

$$f_{i-1}(\tau) \quad = \quad \frac{1}{|I_j|} \int_{I_j} f_{i-1}(s)ds \lesssim (T/(k+1))^{-1} \left[f_i(t_j) + \int_{I_j} F_i(s)ds \right]$$

On the other hand,

$$f_{i-1}(t_{j+1}) + \int_{\tau}^{t_{j+1}} f_{i-2}(s)ds \lesssim f_{i-1}(\tau) + \int_{\tau}^{t_{j+1}} F_{i-1}(s)ds$$

i.e., since all f_i are non-negative,

$$f_{i-1}(t_{j+1}) \lesssim f_{i-1}(\tau) + \int_{\tau}^{t_{j+1}} F_{i-1}(s)ds$$

Hence,

$$\begin{aligned} f_{i-1}(t_{j+1}) &\lesssim \frac{1}{|I_j|}\int_{I_j} f_{i-1}(s)ds \le (T/(k+1))^{-1}\left[f_i(t_j)+\int_{I_j}F_i(s)ds\right]+\int_\tau^{t_{j+1}}F_{i-1}(s)ds \\ &\lesssim (T/(k+1))^{-1}\left[f_i(t_j)+\int_{I_j}F_i(s)ds\right]+\int_{I_j}F_{i-1}(s)ds \end{aligned}$$

Note also that, since $t_j \ge j\frac{k+1}{T}$,

$$\begin{aligned} \int_{I_j}F_i(s)ds &= \int_{t_j}^{t_{j+1}}F_i(s)ds \le t_j^{i-k}\int_{t_j}^{t_{j+1}}s^{k-i}F_i(s)ds \\ &\le \left[\frac{k+1}{jT}\right]^{k-i}F_i^{(k-i)} \end{aligned}$$

where,

$$F_i^{(k-i)} := \sup_{\tau\in[0,T]}\int_\tau^T s^{k-i}F_i(s)ds \tag{A.3.10}$$

Hence,

$$f_{i-1}(t_{j+1}) \lesssim (T/(k+1))^{-1}f_i(t_j)+(T/(k+1))^{-1}\left[\frac{k+1}{jT}\right]^{k-i}F_i^{(k-i)}+\left[\frac{k+1}{jT}\right]^{k-i+1}F_{i-1}^{(k-i+1)}$$

Hence, for all $j\ge 1$, $i \ge 1$

$$f_{i-1}(t_{j+1}) \lesssim T^{-1}f_i(t_j)+T^{-k}T^{i-1}\left[F_i^{(k-i)}+F_{i-1}^{(k-i+1)}\right] \tag{A.3.11}$$

In particular,

$$f_0(T) = f_0(t_{k+1}) \lesssim (T)^{-1}f_1(t_k)+T^{-k}\left[F_1^{(k-1)}+F_0^{(k)}\right]$$

In the same manner,

$$f_1(t_k) \lesssim T^{-1}f_2(t_{k-1})+T^{-k+1}\left[F_2^{(k-2)}+F_1^{(k-1)}\right]$$

Therefore,

$$f_0(T) \lesssim T^{-2}f_2(t_{k-1})+T^{-k}\left[F_2^{(k-2)}+F_1^{(k-1)}+F_0^{(k)}\right]$$

Continuing in the same manner we derive,

$$f_0(T) \lesssim T^{-k} f_k(t_1) + T^{-k} \sum_{0\le i\le k-1} F_i^{(k-i)}$$

or, since,

$$f_k(t_1) \lesssim f_k(0) + \int_{t_0}^{t_1} F_k(s)ds \le f_k(0) + F_k^{(0)}$$

we derive the desired estimate,

$$f_0(T) \lesssim T^{-k} f_k(0) + T^{-k} \sum_{0\le i\le k} F_i^{(k-i)}$$

$\square$

First Decay Theorem

According to the main estimate of theorem A.3.5, for all $0 \le p \le 2$:

$$\begin{aligned}\int_{\Sigma_R(\tau_2)} r^p(\hat{L}\phi)^2 + \int\int_{\mathcal{D}(\tau_1,\tau_2)} r^{p-1}\big(p(\hat{L}\phi)^2 + (2-p)|\nabla\!\!\!/\phi|^2\big) &\lesssim \int_{\Sigma_R(\tau_1)} r^p(\hat{L}\phi)^2 + R^p\mathcal{E}[\phi](\tau_1)\\ &+ I_{p+1}[\square\phi](\tau_1,\tau_2) \qquad \text{(A.3.12)}\end{aligned}$$

where, $\hat{L}\phi := L\phi + \frac{1}{r}\phi = (\partial_t + \partial_r)\phi + \frac{1}{r}\phi$ and,

$$I_{p+1}[\square\phi](\tau_1,\tau_2) := \int_{\mathcal{D}_R(\tau_1,\tau_2)} r^{p+1}|\square\phi|^2 + R^p \int_{\mathcal{D}(\tau_1,\tau_2)} (1+r)^{1+\delta}|\square\phi|^2$$

Let,

$$\begin{aligned} f_i(\tau) &:= \int_{\Sigma_L(\tau)} |\partial\phi|^2 + \int_{\Sigma_R(\tau)} r^i|\widehat{L}\phi|^2, \qquad i = 1, 2,\\ f_0(\tau) &:= \int_{\Sigma_L(\tau)} |\partial\phi|^2 + \int_{\Sigma_R(\tau)} \Big[|\widehat{L}\phi|^2 + |\nabla\!\!\!/\phi|^2\Big]\end{aligned}$$

In view of the Hardy inequality,

$$\mathcal{E}[\phi](\tau) \le f_0(\tau) \lesssim \mathcal{E}[\phi](\tau).$$

where, recall,

$$\mathcal{E}[\phi](\tau) = \int_{\Sigma_L(\tau)} |\partial\phi|^2 + \int_{\Sigma_R(\tau)} |L\phi|^2 + |\nabla\!\!\!/\phi|^2$$

We also define,

$$F_p(\tau) \quad := \quad \int_{\Sigma_R(\tau)} r^{p+1}|\Box\phi|^2 + R^p \int_{\Sigma(\tau)} (1+r)^{1+\delta}|\Box\phi|^2$$

In view of (A.3.12), for all $p = 1, 2$,

$$f_p(\tau_2) + \int_{\tau_1}^{\tau_2} f_{p-1}(\tau)d\tau \quad \lesssim \quad f_p(\tau_1) + f_0(\tau_1) + \int_{\tau_1}^{\tau_2} F_p(\tau)d\tau$$

Hence, as a consequence of proposition A.3.9 we deduce, for all $\tau \in [0, T]$,

$$f_0(\tau) \quad \lesssim \quad \tau^{-2}\left[f_2(\tau) + \sum_{0\le i\le 2} \sup_{\tau\in[0,T]} \int_\tau^T s^{2-i}F_i(s)ds\right]$$

Definition A.3.10. *We introduce the following norms for $\Box\phi$:*

$$\begin{aligned}\mathcal{I}_k[\Box\phi](\tau_0, t) \quad &= \quad \sum_{i=0}^k \sup_{\tau\in[\tau_0,t]} \int\int_{\mathcal{D}_R(\tau,t)} (1+\tau)^{k-i}(1+r)^{1+i}|\Box\phi|^2 \\ &+ \quad \sup_{\tau\in[\tau_0,t]} \int\int_{\mathcal{D}_R(\tau,t)} (1+\tau)^{p-1}(1+r)^{1+\delta}|\Box\phi|^2\end{aligned}$$

and, for a fixed $\epsilon > 0$,

$$\begin{aligned}\mathcal{I}_k^-[\Box\phi](\tau_0, t) \quad &= \quad \sum_{i=0}^k \sup_{\tau\in[\tau_0,t]} \int\int_{\mathcal{D}_R(\tau,t)} (1+\tau)^{k-i}(1+r)^{1+i-\epsilon}|\Box\phi|^2 \\ &+ \quad \sup_{\tau\in[\tau_0,t]} \int\int_{\mathcal{D}_R(\tau,t)} (1+\tau)^{p-1}(1+r)^{1+\delta}|\Box\phi|^2\end{aligned}$$

These considerations prove the first part of the following theorem:

Theorem A.3.11. *Assume $R \approx 1$ and initial data supported in $\Sigma_L(0)$.*

1. *The energy-flux $\mathcal{E}[\phi]$ verifies the decay estimate, for all $0 \le t \le T$,*

$$\mathcal{E}[\phi](t) \quad \lesssim \quad (1+t)^{-2}\left[\mathcal{E}[\phi](0) + \mathcal{I}_{2,t}[\Box\phi]\right] \tag{A.3.13}$$

   where,

$$\mathcal{I}_{2,t}[\Box\phi] \quad := \quad \mathcal{I}_2[\Box\phi](0, t)$$

2. *The incoming flux (through the null hypersurface $\mathcal{N}^V(\tau_1,\tau_2) = \{v = V; u_{\tau_1} \le u \le u_{\tau_2}\}$)*

$$\underline{\mathcal{E}}^V[\phi](\tau,t) \;:=\; \int_{\mathcal{N}^V(\tau,t)} |\partial_u\phi|^2 + |\nabla\!\!\!/\phi|^2$$

verifies,

$$\sup_{\tau\in[0,t]} (1+\tau)^2 \underline{\mathcal{E}}^V[\phi](\tau,t) \lesssim \mathcal{E}[\phi](0) + \mathcal{I}_{3,t}[\Box\phi] \tag{A.3.14}$$

3. *By relaxing the decay assumptions on $\Box\phi$ we have the following slightly weaker decay estimates for the flux.*

$$\mathcal{E}[\phi](t) \;\lesssim\; (1+t)^{-2+\epsilon}\left[\mathcal{E}[\phi](0) + \mathcal{I}^-_{2,t}[\Box\phi]\right] \tag{A.3.15}$$

where,

$$\mathcal{I}^-_{2,t}[\Box\phi] := \;\mathcal{I}^-_2[\Box\phi](0,t)$$

Proof. The proof of (A.3.14) follows easily from the standard energy identity (applied to the region $\mathcal{D}^V(\tau,t) = \mathcal{D}(\tau,t)\cap\{v\le V\}$, for any $\tau\in[0,t]$),

$$\underline{\mathcal{E}}^V[\phi](\tau,t) + \mathcal{E}[\phi](t) \;\le\; \mathcal{E}[\phi](\tau) + \int_{\mathcal{D}^V(\tau,t)} (1+r)^{1+\delta}|\Box\phi|$$

combined with (A.3.13).

It remains to prove (A.3.15). Taking $p = 2-\epsilon$ in the definition of f_p and applying proposition A.3.9 for the functions $f_{1-\epsilon}, f_{2-\epsilon}$ in the interval $[0,t]$, we derive,

$$\int_{\Sigma_R(t)} r^{1-\epsilon}(\widehat{L}\phi)^2 \le f_{1-\epsilon}(t) \lesssim (1+t)^{-1}\left[f_{2-\epsilon}(0) + \mathcal{I}^-_{2,t}[\Box\phi]\right]$$

We also have the estimate (see (A.3.12) with $p = 2-\epsilon$),

$$\begin{aligned}\int_{\Sigma_R(t)} r^{2-\epsilon}(\hat{L}\phi)^2 \;&\lesssim\; \int_{\Sigma_R(0)} r^{2-\epsilon}(\hat{L}\phi)^2 + \mathcal{E}[\phi](0) + \int_{\mathcal{D}_R(0,t)} r^{3-\epsilon}|\Box\phi|^2 + \int_{\mathcal{D}(0,,t))} (1+r)^{1+\delta}|\Box\phi|^2\\ &\le\; \mathcal{E}[\phi](0) + \mathcal{I}^-_{2,t}[\Box\phi]\end{aligned}$$

Interpolating we derive,

$$\int_{\Sigma_R(t)} r(\widehat{L}\phi)^2 \le (1+t)^{-1+\epsilon}\left[\mathcal{E}[\phi](0) + \mathcal{I}^-_{2,t}[\Box\phi]\right]$$

We then proceed as in the proof of proposition A.3.9 to deduce that,

$$\mathcal{E}[\phi](t) \le f_0(t) \lesssim (1+t)^{-2+\epsilon}\left[\mathcal{E}[\phi](0) + \mathcal{I}^-_{2,t}[\Box\phi]\right]$$

as desired. $\Box$

Part III

GCM Constructions

Appendix B

GCM spheres in [K-S:GCM1]

B.1 Stability of Schwarzschild in the polarized case

B.1.1 GCM admissible spacetimes in [K-S:Schw]

In [K-S:Schw], Klainerman and Szeftel proved the nonlinear stability of the Schwarzschild space under axially symmetric polarized perturbations. The final spacetime in [K-S:Schw] was constructed as the limit of a continuous family of finite GCM admissible spacetimes as represented in Figure B.1 below, whose future boundaries consist of the union $\mathcal{A} \cup \underline{\mathcal{C}}_* \cup \mathcal{C}_* \cup \Sigma_*$ where $\mathcal{A}$ and Σ_* are spacelike, $\underline{\mathcal{C}}_*$ is incoming null, and $\mathcal{C}_*$ outgoing null. The boundary $\mathcal{A}$ is chosen so that, in the limit when $\mathcal{M}$ converges to the final state, it is included inside the black hole region of the limit spacetime. The spacetime $\mathcal{M}$ also contains a timelike hypersurface $\mathcal{T}$ which divides $\mathcal{M}$ into an exterior region we call ${}^{(ext)}\mathcal{M}$ and an interior one ${}^{(int)}\mathcal{M}$. Both ${}^{(ext)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$ are foliated by 2–surfaces as follows.

(i) The far region ${}^{(ext)}\mathcal{M}$ is foliated by a geodesic foliation $S(u, s)$ induced by an outgoing optical function u initialized on Σ_* with s the affine parameter along the null geodesic generators of ${}^{(ext)}\mathcal{M}$. We denote by $r = r(u, s)$ the area radius of $S(u, s)$. On the boundary Σ_* of ${}^{(ext)}\mathcal{M}$ we also assume that r is sufficiently large.

(ii) The near region ${}^{(int)}\mathcal{M}$ is foliated by a geodesic foliation induced by an incoming optical function $\underline{u}$ initialized at $\mathcal{T}$ such that its level sets on $\mathcal{T}$ coincide with those of u.

To prove convergence to the final state one has to establish precise decay estimates for all

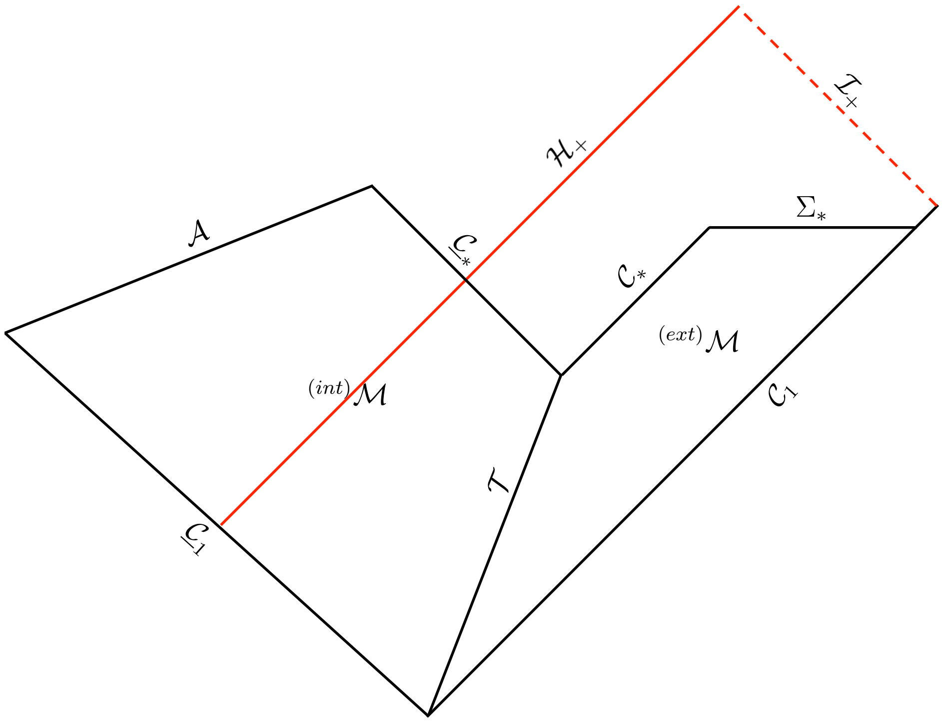


Figure B.1: The GCM admissible space-time $\mathcal{M}$ of [K-S:Schw]

Ricci and curvature coefficients decomposed relative to the null geodesic frames associated to the foliations in ${}^{(ext)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$. The decay properties of both Ricci and curvature coefficients in ${}^{(ext)}\mathcal{M}$ depend heavily on the choice of the boundary Σ_* as well as on the choice of the cuts of the optical function u on it. As such, the central idea of [K-S:Schw] was the introduction and construction of GCM hypersurfaces on which specific geometric quantities take Schwarzschildian values.

Remark B.1.1. *Schwarzschild metric in outgoing Eddington-Finkelstein coordinates has the form*[1]

$$\mathbf{g}_m \quad = \quad -2duds - \Upsilon du^2 + r^2 d\sigma, \qquad \Upsilon = 1 - \frac{2m}{r}, \tag{B.1.1}$$

where $u = t - r_*$, $\frac{dr_*}{dr} = \Upsilon^{-1}$. *For a given sphere* $S(u,s)$, *the expanisons* $\kappa = tr\,\chi$ *and* $\underline{\kappa} = tr\,\underline{\chi}$, *and the mass aspect function* $\mu := -div\,\zeta - \rho + \frac{1}{4}\widehat{\chi}\cdot\widehat{\underline{\chi}}$ *are given by*

$$\kappa = \frac{2}{r}, \qquad \underline{\kappa} = -\frac{2\Upsilon}{r}, \qquad \mu = \frac{2m}{r^3}. \tag{B.1.2}$$

[1] Recall that in standard spherical coordinates, we have $g_m = -\Upsilon dt^2 + \Upsilon^{-1}dr^2 + r^2 d\sigma^2$.

B.1.2 The role played by GCM admissible spacetimes

As mentioned above the final spacetime was constructed as the limit of a continuous family of finite GCM admissible spacetimes. At every stage one assumes that all Ricci and curvature coefficients of a fixed GCM admissible spacetime $\mathcal{M}$ verify precise bootstrap assumptions. One makes use of the GCM admissibility properties of Σ_* and the smallness of the initial conditions to show that all the bounds of the Ricci and curvature coefficients of $\mathcal{M}$ depend only on the size of the initial data and thus, in particular, improve the bootstrap assumptions. This allows to extend the spacetime to a larger one $\mathcal{M}'$ in which the bootstrap assumptions are still valid. To make sure that the extended spacetime is admissible, one has to construct a new GCM hypersurface $\widetilde{\Sigma}_*$ in $\mathcal{M}' \setminus \mathcal{M}$ and use it to define a new extended GCM admissible spacetime $\widetilde{\mathcal{M}}$.

B.2 Review of the main results of [K-S:GCM1]

The main building block of our GCM hypersurface are the GCM spheres constructed in [K-S:GCM1].

B.2.1 Background space

Given an extension $\mathcal{M}'$ of a GCM admissible spacetime $\mathcal{M}$ we restrict our attension to far regions of $\mathcal{R} \subset \mathcal{M}'$ In [K-S:GCM1], one considers such spacetime regions $\mathcal{R}$ foliated by a geodesic foliation $S(u, s)$ induced by an outgoing optical function u with s a properly normalized affine parameter along the null geodesic generators of $L = -\mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha$ where $\mathbf{g}$ is the spacetime metric. We denote by $r = r(u, s)$ the area radius of $S(u, s)$ and let (e_3, e_4, e_1, e_2) be an adapted null frame with e_4 proportional to L and e_1, e_2 tangent to spheres $S = S(u, s)$. The main assumptions made in [K-S:GCM1] are that the Ricci and curvature coefficients, relative to the adapted null frame, have the same asymptotics in powers of r as in Schwarzschild space. The actual size of the perturbation from Kerr is measured with respect to a small parameter $\mathring{\epsilon} > 0$.

B.2.2 Null frame transformation

In general, two null frames (e_3, e_4, e_1, e_2) and (e'_3, e'_4, e'_1, e'_2) are related by a frame transformation of the following form:[2]

$$\begin{aligned} e'_4 &= \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right), \\ e'_a &= \left(\delta_{ab} + \frac{1}{2}\underline{f}_a f_b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2} f_a + \frac{1}{8}|f|^2 \underline{f}_a\right) e_3, \\ e'_3 &= \lambda^{-1}\left(\left(1 + \frac{1}{2} f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b + \frac{1}{4}|\underline{f}|^2 e_4\right), \end{aligned} \tag{B.2.1}$$

where the scalar λ and the 1-forms f and $\underline{f}$ are called the transition coefficients from (e_3, e_4, e_1, e_2) to (e'_3, e'_4, e'_1, e'_2).

The formulas relating Ricci and curvature coefficients of the primed frame in terms of the Ricci and curvature coefficients of the un-primed one are give in 3.1.15. Particulary important for us here are the transformation formulas for $tr\,\chi$, ${}^{(a)}\mathrm{tr}\chi$, $\mathrm{tr}\,\underline{\chi}$, ${}^{(a)}\mathrm{tr}\underline{\chi}, \zeta$ and ρ, ${}^\star\rho$.

$$\begin{aligned} \lambda^{-1}\mathrm{t}r\,\chi' &= \mathrm{t}r\,\chi + \mathrm{d}iv'f + f\cdot\eta + f\cdot\zeta + \mathrm{Err}(\mathrm{t}r\,\chi, \mathrm{t}r\,\chi') \\ \lambda^{-1}\,{}^{(a)}\mathrm{tr}\chi' &= {}^{(a)}\mathrm{tr}\chi + curl'f + f\wedge\eta + f\wedge\zeta + \mathrm{Err}({}^{(a)}\mathrm{tr}\chi, {}^{(a)}\mathrm{tr}\chi'), \end{aligned}$$

$$\begin{aligned} \lambda\mathrm{tr}\,\underline{\chi}' &= \mathrm{tr}\,\underline{\chi} + \mathrm{d}iv'\underline{f} + \underline{f}\cdot\underline{\eta} - \underline{f}\cdot\zeta + \mathrm{Err}(\mathrm{tr}\,\underline{\chi}, \mathrm{tr}\,\underline{\chi}'), \\ \lambda\,{}^{(a)}\mathrm{tr}\underline{\chi}' &= {}^{(a)}\mathrm{tr}\underline{\chi} + curl'\underline{f} + \underline{f}\wedge\underline{\eta} - \zeta\wedge\underline{f} + \mathrm{Err}({}^{(a)}\mathrm{tr}\underline{\chi}, {}^{(a)}\mathrm{tr}\underline{\chi}'), \\ \zeta' &= \zeta - \nabla'(\log\lambda) - \frac{1}{4}\mathrm{tr}\,\underline{\chi} f + \frac{1}{4}\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*f + \omega\underline{f} - \underline{\omega} f + \frac{1}{4}\underline{f}\mathrm{t}r\,\chi + \frac{1}{4}\,{}^*\underline{f}\,{}^{(a)}\mathrm{tr}\chi \\ &\quad +\mathrm{Err}(\zeta, \zeta'), \\ \rho' &= \rho + \mathrm{Err}(\rho, \rho') \\ {}^\star\rho' &= \rho + \mathrm{Err}({}^\star\rho, {}^\star\rho') \end{aligned}$$

B.2.3 Non canonical basis of $\ell = 1$ modes

We introduce the following generalization of the $\ell = 1$ spherical harmonics of the standard sphere[3].

[2]See Lemma 3.1.29 for a precise statement.

[3]Recall that on the standard sphere $\mathbb{S}^2$, in spherical coordinates (θ, φ), these are $J^{(0,\mathbb{S}^2)} = \cos\theta$, $J^{(+,\mathbb{S}^2)} = \sin\theta\cos\varphi$, $J^{(-,\mathbb{S}^2)} = \sin\theta\sin\varphi$.

Definition B.2.1. *On a sphere S, an $\overset{\circ}{\epsilon}$–approximated basis of $\ell = 1$ modes is a triplet of functions $J^{(p)}$ on S verifying*

$$\begin{aligned}
(r^2\Delta + 2)J^{(p)} &= O(\overset{\circ}{\epsilon}), \qquad p = 0, +, -,\\
\frac{1}{|S|}\int_S J^{(p)} J^{(q)} &= \frac{1}{3}\delta_{pq} + O(\overset{\circ}{\epsilon}), \qquad p, q = 0, +, -,\\
\frac{1}{|S|}\int_S J^{(p)} &= O(\overset{\circ}{\epsilon}), \qquad p = 0, +, -,
\end{aligned} \tag{B.2.2}$$

where $\overset{\circ}{\epsilon} > 0$ is a sufficiently small constant.

Remark B.2.2. *$J^{(p)}$ is called a (non-canonical) basis of $\ell = 1$ modes.*

Assuming the existence of such a basis $J^{(p)}$, $p \in \{-, 0, +\}$, we define, for a scalar function h,

$$(h)^S_{\ell=1} \;:=\; \left\{\int_S hJ^{(p)}, \quad p = -, 0, +\right\}. \tag{B.2.3}$$

A scalar function h is said to be supported on $\ell \leq 1$ modes, i.e. $(f)^S_{\ell\geq 2} = 0$, if there exist constants A_0, B_-, B_0, B_+ such that

$$h = A_0 + B_- J^{(-)} + B_0 J^{(0)} + B_+ J^{(+)}. \tag{B.2.4}$$

B.2.4 Definition of GCM spheres

The null expansions $\kappa := \mathrm{tr}\,\chi$ and $\underline{\kappa} := \mathrm{tr}\,\underline{\chi}$ relative to the adapted null frame (e_3, e_4, e_1, e_2) are defined by

$$\mathrm{tr}\,\chi := \mathbf{g}^{ab}\chi_{ab}, \qquad \mathrm{tr}\,\underline{\chi} := \mathbf{g}^{ab}\underline{\chi}_{ab},$$

where

$$\chi_{ab} := \mathbf{g}\left(\mathbf{D}_{e_a} e_4, e_b\right), \qquad \underline{\chi}_{ab} := \mathbf{g}\left(\mathbf{D}_{e_a} e_3, e_b\right).$$

The mass aspect function μ is defined by

$$\mu := -\mathrm{div}\,\zeta - \rho + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}},$$

where the shears $\widehat{\chi}$, $\widehat{\underline{\chi}}$, the torsion ζ and the curvature components ρ are defined by

$$\begin{aligned}
\widehat{\chi}_{ab} &:= \chi_{ab} - \frac{1}{2}\delta_{ab}\kappa, &\qquad \widehat{\underline{\chi}}_{ab} &:= \underline{\chi}_{ab} - \frac{1}{2}\delta_{ab}\underline{\kappa},\\
\zeta_a &:= \frac{1}{2}\mathbf{g}\left(\mathbf{D}_{e_a} e_4, e_3\right), &\qquad \rho &:= \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4).
\end{aligned}$$

In an outgoing geodesic foliation of Schwarzschild spacetime, we have:

$$\kappa = \frac{2}{r}, \qquad \underline{\kappa} = -\frac{2\Upsilon}{r}, \qquad \mu = \frac{2m}{r^3}, \tag{B.2.5}$$

where $\Upsilon = 1 - \frac{2m}{r}$ and r, m denote the area radius and Hawking mass of S, i.e.

$$r := \sqrt{\frac{|S|}{4\pi}}, \qquad \frac{2m}{r} := 1 + \frac{1}{16\pi}\int_{\mathbf{S}} \kappa\underline{\kappa}. \tag{B.2.6}$$

The idea to construct GCM spheres is to mimic the condition (B.2.5) in the perturbed spacetimes. More precisely, the GCM spheres are topological spheres $\mathbf{S}$ embedded in $\mathcal{R}$ endowed with a null frame $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ adapted to $\mathbf{S}$ (i.e. $e_1^{\mathbf{S}}, e_2^{\mathbf{S}}$ tangent to $\mathbf{S}$), relative to which the null expansions $\kappa^{\mathbf{S}} = \mathrm{tr}\,\chi^{\mathbf{S}}$, $\underline{\kappa}^{\mathbf{S}} = \mathrm{tr}\,\underline{\chi}^{\mathbf{S}}$ and mass aspect function $\mu^{\mathbf{S}}$ satisfy:

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \qquad \left(\underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}}\right)_{\ell\geq 2} = 0, \qquad \left(\mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3}\right)_{\ell\geq 2} = 0, \tag{B.2.7}$$

where $r^{\mathbf{S}}$ and $m^{\mathbf{S}}$ denote the area radius and Hawking mass of $\mathbf{S}$.

B.2.5 Deformations of spheres and frame transformations

The construction of GCM spheres in [K-S:GCM1] was obtained by deforming a given sphere $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$ of the background foliation of $\mathcal{R}$. An $O(\overset{\circ}{\delta})$ deformation of $\overset{\circ}{S}$ is defined by a map $\Psi : \overset{\circ}{S} \to \mathbf{S} \subset \mathcal{R}$ of the form

$$\Psi(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2) = \left(\overset{\circ}{u} + U(y^1, y^2), \overset{\circ}{s} + S(y^1, y^2), y^1, y^2\right) \tag{B.2.8}$$

with (U, S) smooth functions on $\overset{\circ}{S}$, vanishing at a fixed point of $\overset{\circ}{S}$, of size proportional to the small constant $\overset{\circ}{\delta}$ and (y^1, y^2) are spherical coordinates on $\overset{\circ}{S}$. Given such a deformation we identify, at any point on $\mathbf{S}$, two important null frames.

1. The null frame (e_3, e_4, e_1, e_2) of the background foliation of $\mathcal{R}$.
2. A null frame $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ obtained from (B.2.1) adapted to $\mathbf{S}$, (i.e. $e_1^{\mathbf{S}}$, $e_2^{\mathbf{S}}$ tangent to $\mathbf{S}$).

Remark B.2.3. *We denote by $(f, \underline{f}, \lambda)$ the transition coefficients from the background frame (e_3, e_4, e_1, e_2) of $\mathcal{R}$ to the null frame $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ adapted to $\mathbf{S}$.*

B.2.6 GCM spheres with $\ell = 1$ modes in [K-S:GCM1]

Here is a short version of the main result in [**?**].

Theorem B.2.4 (Existence of GCM spheres in [**?**]). *Let $\mathcal{R}$ be fixed spacetime region, endowed with an outgoing geodesic foliation $S(u,s)$, verifying specific asymptotic assumptions*[4] *expressed in terms of two parameters $0 < \mathring{\delta} \le \mathring{\epsilon}$. In particular we assume that the GCM quantities of the background spheres in $\mathcal{R}$, i.e.*

$$\kappa - \frac{2}{r}, \qquad \left(\underline{\kappa} + \frac{2\Upsilon}{r}\right)_{\ell\ge 2}, \qquad \left(\mu - \frac{2m}{r^3}\right)_{\ell\ge 2}, \tag{B.2.9}$$

are small with respect to the parameter $\mathring{\delta}$. Let $\mathring{S} = S(\mathring{u}, \mathring{s})$ be a fixed sphere of the foliation with $\mathring{r}$ and $\mathring{m}$ denoting respectively its area radius and Hawking mass, with $\mathring{r}$ sufficiently large. Then, for any fixed triplets $\Lambda, \underline{\Lambda} \in \mathbb{R}^3$ verifying

$$|\Lambda|,\, |\underline{\Lambda}| \ \lesssim\ \mathring{\delta}, \tag{B.2.10}$$

there exists a unique sphere $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$, together with a null frame $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$, which is GCM, i.e. $\mathbf{S}$ is a deformation of $\mathring{S}$, such that[5]

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \qquad \left(\underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}}\right)_{\ell\ge 2} = 0, \qquad \left(\mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3}\right)_{\ell\ge 2} = 0, \tag{B.2.11}$$

and

$$(div^{\mathbf{S}} f)_{\ell=1} = \Lambda, \qquad (div^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}, \tag{B.2.12}$$

where $(f, \underline{f}, \lambda)$ denote the transition coefficients of the transformation (B.2.1) *from the background frame of $\mathcal{R}$ to the frame adapted to $\mathbf{S}$.*

Remark B.2.5. *The conditions* (B.2.9), (B.2.11) *and* (B.2.12) *depend on the definition of $\ell = 1$ modes respectively on $\mathring{S}$ and $\mathbf{S}$. In [K-S:GCM1], once a choice of $\ell = 1$ modes on $\mathring{S}$ is made, it is then extended to $\mathbf{S}$ using the background foliation. As a consequence, the GCM spheres of Theorem B.2.4 depend on the particular choice of $\ell = 1$ modes on $\mathring{S}$.*

[4] Compatible with small perturbations of Kerr.

[5] Note that the GCM conditions (B.2.11) require a choice of $\ell = 1$ modes on $\mathbf{S}$, see Remark B.2.5.

B.3 Sketch of thew proof of Theorem B.2.4

Given a sphere $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$ of this background foliation of $\mathcal{R}$, we look for a $O(\overset{\circ}{\delta})$ deformation of it, i.e a map $\Psi : \overset{\circ}{S} \to \mathbf{S}$ of the form

$$\Psi(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2) = \Big(\overset{\circ}{u} + U(y^1, y^2),\ \overset{\circ}{s} + S(y^1, y^2), y^1, y^2\Big) \tag{B.3.1}$$

with (U, S) smooth functions on $\overset{\circ}{S}$, vanishing at a fixed point of $\overset{\circ}{S}$, of size proportional to the small constant $\overset{\circ}{\delta}$. The goal is then to show that there exist spheres $\mathbf{S}$, described by the functions (U, S), and adapted null pairs $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}})$ such that[6]

$$\kappa^{\mathbf{S}} = \frac{2}{r^{\mathbf{S}}}, \qquad \underline{\kappa}^{\mathbf{S}} = -\frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}}, \qquad \mu^{\mathbf{S}} = \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3}, \tag{B.3.2}$$

where $r^{\mathbf{S}}$ is the area radius of $\mathbf{S}$, $m^{\mathbf{S}}$ is the Hawing mass of $\mathbf{S}$ and $\Upsilon^{\mathbf{S}} = 1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}}$. Note that, given such a deformation, at any point on $\mathbf{S}$ we have two different null frames: the null frame (e_3, e_4, e_1, e_2) of the background foliation of $\mathcal{R}$ and the null frame $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$. In general, two null frames (e_3, e_4, e_1, e_2) and (e_3', e_4', e_1', e_2') are related by a frame transformation of the form (B.2.1).

The condition that the horizontal part of the frame (e_1', e_2') is tangent to $\mathbf{S}$ also leads to a relation between the gradients of U, S, defined in (B.3.1), and $(f, \underline{f})$. We thus expect to derive a coupled system of the form

$$\begin{aligned}
\partial_{y^a} S &= \Big(\big(\mathcal{S}(f, \underline{f}, \Gamma)\big)_a^{\#}\Big), \quad a = 1, 2,\\
\partial_{y^a} U &= \Big(\big(\mathcal{U}(f, \underline{f}, \Gamma)\big)_a^{\#}\Big), \quad a = 1, 2,\\
\mathcal{D}^{\mathbf{S}}(f, \underline{f}, \overset{\circ}{\lambda}) &= \mathcal{G}(\Gamma) + \mathcal{H}(f, \underline{f}, \overset{\circ}{\lambda}, \Gamma),
\end{aligned} \tag{B.3.3}$$

where the terms $\mathcal{S}, \mathcal{U}, \mathcal{H}, \mathcal{G}, \mathcal{D}^{\mathbf{S}}$ have the following meaning.

1. The expressions $\mathcal{S}(f, \underline{f}, \Gamma), \mathcal{U}(f, \underline{f}, \Gamma)$ are 1-forms depending on $f, \underline{f}$ and Γ, with Γ denoting the Ricci coefficients of the background foliation of $\mathcal{R}$ and with # denoting the pull back by the map Ψ defined in (B.3.1).

2. The expression $\mathcal{H}$ refers to a system of scalar functions on $\mathbf{S}$ depending on $(f, \underline{f}, \overset{\circ}{\lambda})$ and Γ, where $\overset{\circ}{\lambda} = \lambda - 1$.

[6] It needs recalling that in reality we only impose these conditions for the $\ell \geq 2$ modes of $\underline{\kappa}$ and μ.

3. The expressions $(\mathcal{U}, \mathcal{S})$ and $\mathcal{H}$ satisfy, schematically, the following.

$$\big|\mathcal{S}, \mathcal{U}\big| \lesssim \big|(f, \underline{f})\big| + \big|(f, \underline{f})\big|^2, \qquad \big|\mathcal{H}\big| \lesssim \big(r^{-1} + \overset{\circ}{\epsilon}\big)\big|(f, \underline{f}, \overset{\circ}{\lambda})\big| + \big|(f, \underline{f}, \overset{\circ}{\lambda})\big|^2.$$

4. The expression $\mathcal{D}^{\mathbf{S}}$ denotes a linear differential operator on $\mathbf{S}$.

5. The term $\mathcal{G}(\Gamma)$ denotes a system of scalars involving the GCM quantities for the $\mathcal{R}$-foliation appearing in (**??**).

The construction of a GCM sphere can thus be reduced to the problem of finding solutions $(U, S, f, \underline{f}, \overset{\circ}{\lambda})$ to the system (B.3.3) of size $\overset{\circ}{\delta}$. There are however various difficulties in solving (B.3.3) which we emphasize below.

B.3.1 Integrability

Note that the transition coefficients have in fact five degrees of freedom while (B.3.2) provides us with only three scalar constraints. The additional degrees of freedom of the triplet $(f, \underline{f}, \lambda)$ have to be constrained by integrability conditions, that is integrability in the sense of Frobenius. Indeed, since the vectorfields (e_1', e_2') have to be tangent to the sphere $\mathbf{S}$, the distribution generated by them has to be integrable[7]. Given an arbitrary frame (e_1', e_2', e_3', e_4'), related to the background frame (e_1, e_2, e_3, e_4) by the formula (B.2.1), the lack of of integrability of the distribution generated by (e_1', e_2') translates into lack of symmetry for the null second fundamental forms,

$$\chi'_{ab} \;=\; \mathbf{g}(\nabla_{e_a'} e_4', e_b'), \qquad \underline{\chi}'_{ab} = \mathbf{g}(\nabla_{e_a'} e_3', e_b'),$$

which can be measured by the scalar functions[8],

$$^{(a)}\mathrm{tr}\chi' = \in^{ab} \chi'_{ab}, \qquad ^{(a)}\mathrm{tr}\underline{\chi}' = \in^{ab} \underline{\chi}'_{ab}.$$

We note that in the axial polarized situation of [K-S:Schw], we can always choose the primed frame (e_3', e_4', e_1', e_2') such that e_2' is collinear to the axially symmetric Killing vectorfield $\mathbf{Z}$ and all other elements of the frame commute with $\mathbf{Z}$. This automatically ensures the integrability of the frame without any additional conditions.

To deal with the issue of integrability, in the general case, we are led to add two more conditions to (B.3.2)

$$^{(a)}\mathrm{tr}\chi^{\mathbf{S}} = \, ^{(a)}\mathrm{tr}\underline{\chi}^{\mathbf{S}} = 0, \tag{B.3.4}$$

[7] Recall that a distribution generated by linearly independent vectorfields X, Y is integrable if the commutator $[X, Y]$ belongs to the distribution.

[8] See precise definitions in section **??**.

translating into two additional differential relations for $f, \underline{f}$ which can be incorporated in the definition of $\mathcal{D}^{\mathbf{S}}$ above. This provides us with the correct number of equations in the last row of (B.3.3), but, as we discuss below, it does not ensure that the kernel of $\mathcal{D}^{\mathbf{S}}$ is trivial which would be a necessary condition for solvability.

subsectionNon-triviality of $\ker \mathcal{D}^{\mathbf{S}}$

Upon inspection, the linear operator $\mathcal{D}^{\mathbf{S}}$, though elliptic, has a non-trivial kernel. To circumvent this difficulty we need to modify the conditions (B.3.2) by requiring instead that only the $\ell \geq 2$ modes[9] of $\mathrm{tr}\,\underline{\chi}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}}$ and $\mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3}$ are set to vanish. As a consequence, we have the freedom to fix the $\ell = 1$ modes of $f, \underline{f}$. These modifications allow us to assume that $\mathcal{D}^{\mathbf{S}}$ is both elliptic and coercive.

B.3.2 Solvability

Note that the first two equations in (B.3.3) require a compatibility condition i.e.

$$\partial_{y^b}\Big(\mathcal{S}(f, \underline{f}, \Gamma)\Big)^{\#}_a = \partial_{y^a}\Big(\mathcal{S}(f, \underline{f}, \Gamma)\Big)^{\#}_b.$$

In the axial polarized case, this can be avoided by a simple symmetry reduction argument, but in the general case, this becomes an issue. We deal with it by modifying the first two equations in (B.3.3), i.e. we consider instead the system[10]

$$\begin{aligned} \Delta^{\mathring{S}} S &= \mathrm{d}iv^{\mathring{S}}\Big(\big(\mathcal{S}(f, \underline{f}, \Gamma)\big)^{\#}\Big), \\ \Delta^{\mathring{S}} U &= \mathrm{d}iv^{\mathring{S}}\Big(\big(\mathcal{U}(f, \underline{f}, \Gamma)\big)^{\#}\Big), \\ \mathcal{D}^{\mathbf{S}}(f, \underline{f}, \mathring{\lambda}) &= \mathcal{G}(\Gamma) + \mathcal{H}(f, \underline{f}, \mathring{\lambda}, \Gamma). \end{aligned} \tag{B.3.5}$$

We also fix the values of U, S to be zero at a given point of $\mathring{S}$ to ensure uniqueness.

subsectionNonlinear implicit nature of (B.3.5)

To disentangle the highly nonlinear and implicit nature of (B.3.5), we proceed by an

[9] We refer here to a generalization of the spherical harmonics of the standard sphere $\mathbb{S}^2$. This is itself an additional difficulty one has to overcome, i.e. to define a suitable generalization of modes for deformed spheres.

[10] Note that the equations for (U, S) in (B.3.5) do not imply the ones in (B.3.3). It is thus a priori not clear that solving (B.3.5) will lead to a GCM sphere. The fact that it does is discussed in section B.3.3.

iterative procedure which starts with the trivial quintet

$$\mathcal{Q}^{(0)} := (U^{(0)}, S^{(0)}, \overset{\circ}{\lambda}{}^{(0)}, f^{(0)}, \underline{f}^{(0)}) = \big(0, 0, 0, 0, 0\big),$$

corresponding to the un-deformed sphere $\overset{\circ}{S}$, and, making us of the n-th iterate $\mathcal{Q}^{(n)}$, produces

$$\mathcal{Q}^{(n+1)} = \Big(U^{(n+1)}, S^{(n+1)}, \overset{\circ}{\lambda}{}^{(n+1)}, f^{(n+1)}, \underline{f}^{(n+1)}\Big)$$

as follows.

- The pair $(U^{(n)}, S^{(n)})$ defines the deformation sphere $\mathbf{S}(n)$ and the corresponding pull back map $\#_n$ given by the map $\Psi^{(n)} : \overset{\circ}{S} \to \mathbf{S}(n)$,

  $$(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2) \longrightarrow (\overset{\circ}{u} + U^{(n)}(y^1, y^2), \overset{\circ}{s} + S^{(n)}(y^1, y^2), y^1, y^2).$$

- We define the triplet $(f^{(n+1)}, \underline{f}^{(n+1)}, \overset{\circ}{\lambda}{}^{(n+1)})$ as the solution of the following linear system

  $$\mathcal{D}^{\mathbf{S}(n)}(f^{(n+1)}, \underline{f}^{(n+1)}, \overset{\circ}{\lambda}{}^{(n+1)}) \quad = \mathcal{G}(\Gamma) + \mathcal{H}(f^{(n)}, \underline{f}^{(n)}, \overset{\circ}{\lambda}{}^{(n)}, \Gamma).$$

  Note that $\mathcal{D}^{\mathbf{S}(n)}$ is defined with respect to the geometric structure of $\mathbf{S}(n)$.

- We use the new pair $(f^{(n+1)}, \underline{f}^{(n+1)})$ to solve the equations on $\overset{\circ}{S}$,

  $$\begin{aligned} \Delta^{\overset{\circ}{S}} U^{(n+1)} &= \mathrm{d}iv^{\overset{\circ}{S}}\Big(\big(\mathcal{U}(f^{(n+1)}, \underline{f}^{(n+1)}, \Gamma)\big)^{\#_n}\Big), \\ \Delta^{\overset{\circ}{S}} S^{(n+1)} &= \mathrm{d}iv^{\overset{\circ}{S}}\Big(\big(\mathcal{S}(f^{(n+1)}, \underline{f}^{(n+1)}, \Gamma)\big)^{\#_n}\Big), \end{aligned} \tag{B.3.6}$$

  with $U^{(n+1)}, S^{(n+1)}$ vanishing at the same given point of $\overset{\circ}{S}$ and where the pull back $\#_n$ is defined with respect to the map $\Psi^{(n)} : \overset{\circ}{S} \to \mathbf{S}(n)$. The new pair $(U^{(n+1)}, S^{(n+1)})$ defines the new sphere $\mathbf{S}(n+1)$ and we can proceed with the next step of the iteration.

B.3.3 Have we produced a GCM sphere?

If $\overset{\circ}{\epsilon}$ is sufficiently small one can show that the iterative procedure mentioned above leads to a solution $\big(U^{(\infty)}, S^{(\infty)}, \overset{\circ}{\lambda}{}^{(\infty)}, f^{(\infty)}, \underline{f}^{(\infty)}\big)$ verifying the system

$$\begin{aligned}\overset{\circ}{\Delta} U^{(\infty)} &= \overset{\circ}{\mathrm{d}iv}\left(\left(\mathcal{U}(f^{(\infty)}, \underline{f}^{(\infty)}, \Gamma)\right)^{\#_\infty}\right),\\ \overset{\circ}{\Delta} S^{(\infty)} &= \overset{\circ}{\mathrm{d}iv}\left(\left(\mathcal{S}(f^{(\infty)}, \underline{f}^{(\infty)}, \Gamma)\right)^{\#_\infty}\right),\\ \mathcal{D}^\infty(f^{(\infty)}, \underline{f}^{(\infty)}, \overset{\circ}{\lambda}^{(\infty)}) &= \mathcal{G}(\Gamma) + \mathcal{H}(f^{(\infty)}, \underline{f}^{(\infty)}, \overset{\circ}{\lambda}^{(\infty)}, \Gamma),\end{aligned} \tag{B.3.7}$$

where the elliptic operator $\mathcal{D}^\infty$ is defined on the sphere $\mathbf{S}(\infty)$, i.e. the deformation of $\overset{\circ}{S}$ induced by $(U^{(\infty)}, S^{(\infty)})$. Is $\mathbf{S}(\infty)$ the desired solution to the problem, i.e. is it a GCM sphere in the sense discussed above? This is a priori not clear as the equations for $(U^{(\infty)}, S^{(\infty)})$ in (B.3.7) do not imply those in (B.3.3). As a result, we have potentially two different frames associated to $\mathbf{S} = \mathbf{S}(\infty)$.

- The frame $\left(e_1^{(\infty)}, e_2^{(\infty)}, e_3^{(\infty)}, e_4^{(\infty)}\right)$ induced by the transition functions $(\overset{\circ}{\lambda}^{(\infty)}, f^{(\infty)}, \underline{f}^{(\infty)})$, with the quintet $\left(U^{(\infty)}, S^{(\infty)}, \overset{\circ}{\lambda}^{(\infty)}, f^{(\infty)}, \underline{f}^{(\infty)}\right)$ verifying the limiting system (B.3.7).
- The geometric frame[11] $(e_1^{\mathbf{S}}, e_2^{\mathbf{S}}, e_3^{\mathbf{S}}, e_4^{\mathbf{S}})$, adapted to $\mathbf{S}$.

The main remaining hurdle is to show that these two null frames coincide.

B.3.4 Applications to the construction of intrinsic GCM spheres

In [K-S:GCM2], we derive a far reaching corollary of Theorem B.2.4 where we replace the $\ell = 1$ conditions (B.2.12) on $\mathrm{d}iv^{\mathbf{S}}(f)$ and $\mathrm{d}iv^{\mathbf{S}}(\underline{f})$ by the vanishing of the canonical $\ell = 1$ modes of $\mathrm{d}iv^{\mathbf{S}}\beta^{\mathbf{S}}$ and $\widetilde{\mathrm{t}r\,\chi}^{\mathbf{S}}$. Tthe definition of canonical $\ell = 1$ modes depends on an effective version of the uniformization theorem which we also develop in [K-S:GCM2]. The horizontal 1-form $\beta^{\mathbf{S}}$ is a curvature component of the Riemann curvature tensor with respect to the null frame adapted to $\mathbf{S}$, constructed in Theorem B.2.4. Here is a short version of that result.

Theorem B.3.1 (Intrinsic GCM spheres with canonical $\ell = 1$ modes [K-S:GCM2])**.** *Under slightly stronger assumptions on the background foliation of $\mathcal{R}$, there exists a unique*[12] *GCM deformation of $\overset{\circ}{S}$ verifying, in addition to* (**??**),

$$(div^{\mathbf{S}}\beta^{\mathbf{S}})_{\ell=1} = 0, \qquad \widetilde{tr\,\chi}^{\mathbf{S}}_{\ell=1} = 0, \tag{B.3.8}$$

relative to the canonical $\ell = 1$ modes of $\mathbf{S}$.

[11]With a proper normalization for the null pair $e_3^{\mathbf{S}}, e_4^{\mathbf{S}}$, in fact the one corresponding to $\lambda = \lambda^{(\infty)}$.

[12]Up to a rotation of $\mathbb{S}^2$.

Remark B.3.2. *In [K-S:GCM2] one also makes use of Theorem B.3.1 to define a quasi-local notion of angular momentum.*

Remark B.3.3. *We note that a related notion of preferred spheres, of constant mean curvature, in an asymptotically euclidean Riemannian 3-manifold has been introduced in [Hu-Yau]. Note that the spheres in [Hu-Yau] have codimension 1, while the GCM spheres have codimension 2 in a 4 dimensional Lorentzian manifold.*

Remark B.3.4. *The assumptions on the spacetime region $\mathcal{R}$ in Theorem B.3.1 are in particular satisfied in Kerr for r sufficiency large. We can thus apply Theorem B.3.1 in that context, and obtain the existence of intrinsic GCM spheres $\mathbf{S}_{Kerr}$ in Kerr for r sufficiency large, see Corollary 7.9 in [K-S:GCM2]. The intrinsic GCM spheres $\mathbf{S}$ of Theorem B.3.1 thus correspond to the analog of $\mathbf{S}_{Kerr}$ in perturbations of Kerr for r sufficiency large.*

B.4 GCM hypersurfaces, following [Shen]

B.4.1 First version of the main theorem

In [Shen] one constructs hypersurfaces which are suitable concatenations of the spheres of Theorem B.2.4.

Here is a simple version of the main theorem in [Shen].

Theorem B.4.1 (Existence of GCM hypersurfaces, first version)**.** *Let $\mathcal{R}$ be fixed spacetime region, endowed with an outgoing geodesic foliation $S(u,s)$, verifying same assumptions as Theorem B.2.4. Assume in addition that $e_3(J^{(p)})$, $(div\,\eta)_{\ell=1}$, $(div\,\underline{\xi})_{\ell=1}$, $r-s$ and $e_3(r)-e_3(s)$ are small with respect to the parameter $\overset{\circ}{\delta}$.*

Let $\mathbf{S}_0$ be a fixed sphere included in the region $\mathcal{R}$, let a pair of triplets $\Lambda_0, \underline{\Lambda}_0 \in \mathbb{R}^3$ such that

$$|\Lambda_0|,\, |\underline{\Lambda}_0| \lesssim \overset{\circ}{\delta},$$

and let $J^{(\mathbf{S}_0,p)}$ a basis of $\ell=1$ modes on $\mathbf{S}_0$, such that we have on $\mathbf{S}_0$

$$\kappa^{\mathbf{S}_0} - \frac{2}{r^{\mathbf{S}_0}} = 0, \qquad \left(\underline{\kappa}^{\mathbf{S}_0} + \frac{2\Upsilon^{\mathbf{S}_0}}{r^{\mathbf{S}_0}}\right)_{\ell\geq 2} = 0, \qquad \left(\mu^{\mathbf{S}_0} - \frac{2m^{\mathbf{S}_0}}{(r^{\mathbf{S}_0})^3}\right)_{\ell\geq 2} = 0,$$

and

$$(div^{\mathbf{S}_0} f)_{\ell=1} = \Lambda_0, \qquad (div^{\mathbf{S}_0} \underline{f})_{\ell=1} = \underline{\Lambda}_0,$$

where $(f, \underline{f})$ *denote the transition coefficients of the transformation* (B.2.1) *from the background frame of* $\mathcal{R}$ *to the frame adapted to* $\mathbf{S}_0$.

Then, there exists a unique, local, smooth, spacelike hypersurface Σ_0 *passing through* $\mathbf{S}_0$, *a scalar function* $u^{\mathbf{S}}$ *defined on* Σ_0, *whose level surfaces are topological spheres denoted by* $\mathbf{S}$, *a smooth collection of triplets of constants* $\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}}$ *and a triplet of functions* $J^{(\mathbf{S},p)}$ *defined on* Σ_0 *verifying,*

$$\Lambda^{\mathbf{S}_0} = \Lambda_0, \qquad \underline{\Lambda}^{\mathbf{S}_0} = \underline{\Lambda}_0, \qquad J^{(\mathbf{S},p)}\big|_{\mathbf{S}_0} = J^{(\mathbf{S}_0,p)},$$

such that the following conditions are verified:

1. *The surfaces* $\mathbf{S}$ *of constant* $u^{\mathbf{S}}$, *together with an adapted null frame* $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$, *verify*

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \qquad \left(\underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}}\right)_{\ell\geq 2} = 0, \qquad \left(\mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3}\right)_{\ell\geq 2} = 0, \tag{B.4.1}$$

and

$$(div^{\mathbf{S}} f)_{\ell=1} = \Lambda^{\mathbf{S}}, \qquad (div^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}^{\mathbf{S}}, \tag{B.4.2}$$

for the triplets of constants $\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}}$ *and* $\ell = 1$ *modes* $J^{(\mathbf{S},p)}$.

2. *The following transversality conditions*

$$\xi^{\mathbf{S}} = 0, \qquad \omega^{\mathbf{S}} = 0, \qquad \underline{\eta}^{\mathbf{S}} = -\zeta^{\mathbf{S}}, \tag{B.4.3}$$

and

$$e_4^{\mathbf{S}}(u^{\mathbf{S}}) = 0, \qquad e_4^{\mathbf{S}}(r^{\mathbf{S}}) = 1 \tag{B.4.4}$$

are assumed on Σ_0.

3. *We have, for some constant* c_0,

$$u^{\mathbf{S}} + r^{\mathbf{S}} = c_0, \quad along \quad \Sigma_0. \tag{B.4.5}$$

4. *Let $\nu^{\mathbf{S}}$ be the unique vectorfield tangent to the hypersurface Σ_0, normal to $\mathbf{S}$, and normalized by $\mathbf{g}(\nu^{\mathbf{S}}, e_4^{\mathbf{S}}) = -2$. Let $\beta^{\mathbf{S}}$ be the unique scalar function on Σ_0 such that $\nu^{\mathbf{S}}$ is given by*

$$\nu^{\mathbf{S}} = e_3^{\mathbf{S}} + b^{\mathbf{S}} e_4^{\mathbf{S}}. \tag{B.4.6}$$

   The following normalization condition holds true

$$\overline{b^{\mathbf{S}}} = -1 - \frac{2m_{(0)}}{r^{\mathbf{S}}}, \tag{B.4.7}$$

   where $\overline{b^{\mathbf{S}}}$ is the average value of $b^{\mathbf{S}}$ over $\mathbf{S}$ and $m_{(0)}$ is a constant.

5. *We have the following identities on Σ_0:*

$$(div^{\mathbf{S}} \eta^{\mathbf{S}})_{\ell=1} = 0, \qquad (div^{\mathbf{S}} \underline{\xi}^{\mathbf{S}})_{\ell=1} = 0. \tag{B.4.8}$$

6. *The transition functions $(f, \underline{f}, \lambda)$, area radius $r^{\mathbf{S}}$ and Hawking mass $m^{\mathbf{S}}$ verify appropriate estimates.*

Remark B.4.2. *Theorem B.4.1 is the generalization of Theorem 9.52 in [K-S:Schw] in the absence of symmetry. It plays a central role in the proof of Theorem M6 and M7 in [K-S:Kerr] , see sections 8.4 and 8.5 in [K-S:Kerr].*

Remark B.4.3. *We provide below more explanations for the statements 1-5 in Theorem B.4.1:*

1. *Since we concatenate a family of GCM spheres $\mathbf{S}(\Lambda^{\mathbf{S}}, \underline{\Lambda}^{\mathbf{S}})$ emanating from $\mathbf{S}_0$ to construct the GCM hypersurfaces Σ_0, by Theorem B.2.4, we have automatically* (B.4.1) *and* (B.4.2) *on every* $\mathbf{S}$.

2. *The transversality conditions* (B.4.3) *and* (B.4.4) *are consistent with a local extension by an outgoing geodesic foliation initialized on Σ_0, see item 1 in Remark B.4.5. The role of these transversality conditions is to make sense of $\eta^{\mathbf{S}}$ and $\underline{\xi}^{\mathbf{S}}$ on Σ_0, see item 5 below.*

3. *$u^{\mathbf{S}}$ should be chosen to be constant on the GCM spheres foliating Σ_0. The choice* (B.4.5) *is simple and fulfills this condition but other choices making Σ_0 spacelike are possible.*

4. *The value $\overline{\beta^{\mathbf{S}}}$ is free and should be prescribed. Note that the choice* (B.4.7) *coincides with the value for the hypersurface $\{u + r = c_0\}$ in Schwarzschild spacetime.*

5. *In* (B.4.8), $\eta^{\mathbf{S}}$ *and* $\underline{\xi}^{\mathbf{S}}$ *are defined intrinsically on* Σ_0 *by*

$$\eta^{\mathbf{S}}_a = \frac{1}{2}\mathbf{g}\left(\mathbf{D}_{\nu^{\mathbf{S}}}e^{\mathbf{S}}_4, e^{\mathbf{S}}_a\right), \qquad \underline{\xi}^{\mathbf{S}}_a = \frac{1}{2}\mathbf{g}\left(\mathbf{D}_{\nu^{\mathbf{S}}}e^{\mathbf{S}}_3, e^{\mathbf{S}}_a\right) + \beta^{\mathbf{S}}\zeta^{\mathbf{S}}_a.$$

These definitions are consistent with the standard ones provided Σ_0 *satisfies the transversality condition* (B.4.3) *which is equivalent to extending* Σ_0 *locally by an outgoing geodesic foliation, see item 2 in Remark B.4.5. The condition* (B.4.8) *will be enforced thanks to a special choice of* $\Lambda^{\mathbf{S}}$ *and* $\underline{\Lambda}^{\mathbf{S}}$.

Remark B.4.4. *As in Section 9.8 of [K-S:Schw],* Σ_0 *is chosen to be spacelike. One may wonder whether* Σ_0 *could be chosen to be null*[13]*. The reason for choosing it to be spacelike is that it allows more flexibility: all spheres foliating* Σ_0 *in Theorem B.4.1 are GCM spheres, while only one could be a GCM sphere on a null hypersurface.*

Remark B.4.5. *We provide below justifications for introducing the transversality conditions* (B.4.3) *and* (B.4.4) *:*

1. (B.4.3) *and* (B.4.4) *are consistent with a local extension by an outgoing geodesic foliation initialized on* Σ*. The use of transversality conditions instead of a local extension is chosen here to have intrinsic definitions on* Σ.

2. *The role of the transversality conditions* (B.4.3) *is to make sense of the Ricci coefficients* $\eta^{\mathbf{S}}$, $\underline{\xi}^{\mathbf{S}}$ *and* $\underline{\omega}^{\mathbf{S}}$ *on* Σ *through the formulae:*[14]

$$\begin{aligned}
\mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}}e^{\mathbf{S}}_4, e^{\mathbf{S}}_a) &= 2\eta^{\mathbf{S}}_a + 2\beta^{\mathbf{S}}\xi^{\mathbf{S}}_a = 2\eta^{\mathbf{S}}_a,\\
\mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}}e^{\mathbf{S}}_3, e^{\mathbf{S}}_a) &= 2\underline{\xi}^{\mathbf{S}}_a + 2\beta^{\mathbf{S}}\underline{\eta}^{\mathbf{S}}_a = 2\underline{\xi}^{\mathbf{S}}_a - 2\beta^{\mathbf{S}}\zeta^{\mathbf{S}}_a,\\
\mathbf{g}(\mathbf{D}_{\nu^{\mathbf{S}}}e^{\mathbf{S}}_3, e^{\mathbf{S}}_4) &= 4\underline{\omega}^{\mathbf{S}} - 4\beta^{\mathbf{S}}\omega^{\mathbf{S}} = 4\underline{\omega}^{\mathbf{S}}.
\end{aligned} \tag{B.4.9}$$

3. *The role of the transversality conditions* (B.4.4) *is to make sense of* $e^{\mathbf{S}}_3(r^{\mathbf{S}})$ *and* $e^{\mathbf{S}}_3(u^{\mathbf{S}})$ *on* Σ *through the formulae:*

$$\begin{aligned}
e^{\mathbf{S}}_3(r^{\mathbf{S}}) &= \nu^{\mathbf{S}}(r^{\mathbf{S}}) - b^{\mathbf{S}}e^{\mathbf{S}}_4(r^{\mathbf{S}}) = \nu^{\mathbf{S}}(r^{\mathbf{S}}) - \beta^{\mathbf{S}},\\
e^{\mathbf{S}}_3(u^{\mathbf{S}}) &= \nu^{\mathbf{S}}(u^{\mathbf{S}}) - b^{\mathbf{S}}e^{\mathbf{S}}_4(u^{\mathbf{S}}) = \nu^{\mathbf{S}}(u^{\mathbf{S}}).
\end{aligned} \tag{B.4.10}$$

[13] In the context of the stability of Minkowski, the last slice in the original proof by Christodoulou and Klainerman in [Ch-Kl] is spacelike, while it is null in the proof by Klainerman and Nicolò in [Kl-Ni1] in the case of the exterior of an outgoing null cone.

[14] Note that the L.H.S. of (B.4.9) are well defined on Σ since $\nu^{\mathbf{S}}$ is tangent to Σ.

B.4.2 Sketch of the proof of the main theorem

The idea of the proof is to construct Σ_0 as a 1–parameter union of GCM spheres.

Step 1. For every given:

- Background sphere $S(u,s)$ in $\mathcal{R}$ endowed with a basis of $\ell = 1$ modes $J^{(p)}$,
- Triplets $(\Lambda, \underline{\Lambda})$
- Triplet of functions $\widetilde{J}^{(p)}$ satisfying[15]

$$\sum_{p=0,+,-} \|J^{(p)} - \widetilde{J}^{(p)}\|_{\mathfrak{h}_{s_{max}}(S(u,s))} \lesssim r\overset{\circ}{\delta},$$

we associate, by Theorem B.2.4, a unique GCM sphere $\mathbf{S}[u, s, \Lambda, \underline{\Lambda}, \widetilde{J}^{(p)}]$, as a deformation of $S(u,s)$ with $\ell = 1$ modes in the definition of (B.2.11) and (B.2.12) computed w.r.t. $\widetilde{J}^{(p)}$. In particular, (B.2.11) and (B.2.12) are verified and we have $\mathbf{S}_0 = \mathbf{S}[\overset{\circ}{u}, \overset{\circ}{s}, \Lambda_0, \underline{\Lambda}_0, \widetilde{J}^{(p)}]$, provided we choose $\widetilde{J}^{(p)}\big|_{\mathbf{S}_0} = J^{(\mathbf{S}_0,p)}$.

Step 2. Given $(\Psi(s), \Lambda(s), \underline{\Lambda}(s))$ such that

$$\Psi(\overset{\circ}{s}) = \overset{\circ}{u}, \qquad \Lambda(\overset{\circ}{s}) = \Lambda_0, \qquad \underline{\Lambda}(\overset{\circ}{s}) = \underline{\Lambda}_0,$$

We construct, relying on Step 1 and a Banach fixed-point argument, see Theorem **??**, a family of basis of $\ell = 1$ modes $\widetilde{J}(s)$ and of GCM spheres $\mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \widetilde{J}(s)]$ verifying

$$\nu^{\mathbf{S}}(\widetilde{J}(s)) = 0 \quad \text{on } \Sigma, \qquad \widetilde{J}^{(p)}(\overset{\circ}{s}) = J^{(\mathbf{S}_0,p)},$$

where the hypersurface Σ is given by

$$\Sigma = \bigcup_s \mathbf{S}(s) = \bigcup_s \mathbf{S}[\Psi(s), s, \Lambda(s), \underline{\Lambda}(s), \widetilde{J}(s)],$$

and where $\nu^{\mathbf{S}}$ is the unique vectorfield tangent to Σ with $\mathbf{g}(\nu^{\mathbf{S}}, e_4^{\mathbf{S}}) = -2$ and normal to $\mathbf{S}(s)$.

[15] $\mathfrak{h}_{s_{max}}(S(u,s))$ denotes the Sobolev space on $S(u,s)$ of order s_{max}

Step 3. We then derive for $(\Psi, \Lambda, \underline{\Lambda})$ an ODE system of the following type:

$$\begin{aligned}\frac{1}{\Psi'(s)}\Lambda'(s) =&(\mathrm{d}iv^{\mathbf{S}}\eta^{\mathbf{S}})_{\ell=1} - \frac{1}{2}r^{-1}\Lambda(s) - \frac{1}{2}r^{-1}\underline{\Lambda}(s) + \text{ l.o.t.},\\ \frac{1}{\Psi'(s)}\underline{\Lambda}'(s) =&(\mathrm{d}iv^{\mathbf{S}}\underline{\xi}^{\mathbf{S}})_{\ell=1} + \text{ l.o.t.},\\ \Psi'(s) =& -1 - \frac{1}{2}\left(\overline{b^{\mathbf{S}}} + 1 + \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}}\right) + \text{ l.o.t.},\end{aligned} \tag{B.4.11}$$

where l.o.t. denote lower order terms, see Section **??** for the precise statement.

Step 4. We look for a special choice $(\breve{\Psi}(s), \breve{\Lambda}(s), \breve{\underline{\Lambda}}(s))$ such that the additional GCM conditions (B.4.7) and (B.4.8) are verified. These conditions lead, in view of Step 3, to an ODE system for $(\breve{\Psi}(s), \breve{\Lambda}(s), \breve{\underline{\Lambda}}(s))$, with prescribed initial conditions at $\mathring{s}$ which allows us to uniquely determine the desired hypersurface Σ_0.

Remark B.4.6. *The proof of Theorem B.4.1 is largely analogous to that of Theorem 9.52 in [K-S:Schw]. Below, we compare the proof in this paper and that in Section 9.8 of [K-S:Schw].*

- *In* **Step 1** *and* **Step 2**, *we show that, in general, one can choose the approximate basis of $\ell = 1$ modes so that they are transported along the normal direction to the GCM spheres $\mathbf{S}(s)$ on Σ. This contrasts with [K-S:Schw] where the basis of $\ell = 1$ modes is fixed by the polarized symmetry.*
- *Once the choice of $\ell = 1$ modes is made, in* **Step 3**, *we derive the system of ODEs* (B.4.11). *Note that the coefficients of linear terms of $\Lambda(s)$ and $\underline{\Lambda}(s)$ on the R.H.S. of* (B.4.11) *are different from that of (9.8.74) in [K-S:Schw], which is due to the different choice of $\ell = 1$ modes.*[16]
- **Steps 1-3** *are significantly more involved than the corresponding in [K-S:Schw] due the absence of symmetry, while* **Step 4** *is similar to that in [K-S:Schw].*

[16]More precisely, the basis of $\ell = 1$ modes in [K-S:Schw], fixed by polarized symmetry, is not transported along the vectorfield $\nu^{\mathbf{S}}$.

Appendix C

GCM spheres and hypersurfaces in [K-S:GCM2] and [Shen]

C.1 GCM spheres in [K-S:Schw]

C.2 Review of uniformization results for the sphere

C.2.1 Uniformization for metrics on $\mathbb{S}^2$

We start with the following well known calculation.

Lemma C.2.1. *Let S a surface and let g^S be a Riemannian metric on S. For a scalar function u on S, the Gauss curvature of $e^{2u}g^S$ is connected to that of g^S by the formula*

$$K(e^{2u}g^S) = e^{-2u}\big(K^S - \Delta_S u\big) \tag{C.2.1}$$

where K^S denotes the Gauss curvature of S and Δ_S the Laplace-Beltrami on S.

According to the classical uniformization theorem, if S is a closed, oriented and connected surface of genus 0 and g^S is a Riemannian metric on S, then, there exists a smooth diffeomorphism $\Phi : \mathbb{S}^2 \to S$ and a smooth conformal factor u on $\mathbb{S}^2$ such that

$$\Phi^{\#}(g^S) = e^{2u}\gamma_0 \tag{C.2.2}$$

where γ_0 is the canonical metric on the standard sphere $\mathbb{S}^2$. In view of Lemma C.2.1 above, if we denote by g the metric on $\mathbb{S}^2$, $g = \Phi^\#(g^S) = e^{2u}\gamma_0$, we derive

$$\Delta_0 u + K(g)e^{2u} = 1 \tag{C.2.3}$$

where we have used the notation

$$\Delta_0 \;\; := \;\; \Delta_{\gamma_0}.$$

We thus have the following corollary of Lemma C.2.1.

Corollary C.2.2. *If $\Phi^\# g^S = e^{2u}\gamma_0$, then u verifies the equation*

$$\Delta_0 u + (K^S \circ \Phi)e^{2u} = 1, \tag{C.2.4}$$

where K^S is the Gauss curvature of g^S on S.

Proof. The proof follows from (C.2.3) in view of the fact that $K\big(\Phi^\# g^S\big) = K(g^S)\circ\Phi$. $\square$

Definition C.2.3. *Let* **M** *denote the group of conformal transformations of $\mathbb{S}^2$, i.e. the set of diffeomorphisms Φ of $\mathbb{S}^2$ such that $\Phi^\#\gamma_0 = e^{2u}\gamma_0$ for some scalar function u on $\mathbb{S}^2$.*

Remark C.2.4. *Let $\Phi \in \mathbf{M}$ so that $\Phi^\#\gamma_0 = e^{2u}\gamma_0$. Then, u satisfies*[1]

$$u = \frac{1}{2}\log|\det d\Phi|.$$

Also, in view of Corollary C.2.2, we have

$$\Delta_0 u + e^{2u} = 1. \tag{C.2.5}$$

C.2.2 Conformal isometries of $\mathbb{S}^2$ and the Möbius group

We represent the standard sphere $\mathbb{S}^2$ as $\big\{x \in \mathbb{R}^3, |x|^2 = 1\big\}$. Let $N = (0,0,1)$ denote the north pole of $\mathbb{S}^2$. Through the stereographic projection from the North pole to the equatorial plane plane (x^1, x^2) we consider the complex coordinate

$$z = \frac{x^1 + ix^2}{1 - x^3} \tag{C.2.6}$$

[1]This follows immediately from writing $\Phi^\#\gamma_0 = e^{2u}\gamma_0$ in matrix form, by evaluating on an orthonormal frame, and then taking the absolute value of the determinant on both sides. Also, recall that $|\det d\Phi|$ is an intrinsic scalar on $\mathbb{S}^2$, i.e. it does not depend on the particular choice of orthonormal frame.

with the inverse transformation

$$x^1 = \frac{2}{1+|z|^2}\Re z, \quad x^2 = \frac{2}{1+|z|^2}\Im z, \quad x^3 = \frac{|z|^2-1}{|z|^2+1} \tag{C.2.7}$$

and pull-back of the standard metric γ_0 on $\mathbb{S}^2$

$$4\big(1+|z|^2\big)^{-2}|dz|^2. \tag{C.2.8}$$

The conformal isometry group $\mathbf{M}$ of $\mathbb{S}^2$ consists in fact of Möbius transforms and conjugation of Möbius transforms, see for example Theorem 18.10.4 and section 18.10.2.4 in [**?**], and can thus be identified with $SL(2,\mathbb{C})$,

$$z \to \frac{az+b}{cz+d}, \qquad z \to \frac{a\overline{z}+b}{c\overline{z}+d}, \qquad ad-bc=1, \quad a,b,c,d \in \mathbb{C}. \tag{C.2.9}$$

The particular case of Möbius transforms where $d = a^{-1} > 0$ and $b = c = 0$ will play an important role. Given $t > 0$ and a point $p \in \mathbb{S}^2$, we can choose coordinates such that p is at the north pole and obtain scale transformations defined by

$$\Phi_{p,t} z = tz. \tag{C.2.10}$$

C.2.3 Effective uniformization

Definition C.2.5. *We define the center of mass of e^{2u} to be*

$$CM[e^{2u}] \;=\; \frac{\int_{\mathbb{S}^2} xe^{2u}}{\int_{\mathbb{S}^2} e^{2u}} \tag{C.2.11}$$

where $x = (x^1, x^2, x^3)$ on the sphere $\mathbb{S}^2$. Also, we define the spaces of functions[2]

$$\begin{aligned} \mathcal{S} :=& \Big\{ u \in H^1(\mathbb{S}^2) \;\; \textit{such that} \;\; CM[e^{2u}] = 0 \Big\}, \\ \mathcal{S}_0 :=& \left\{ u \in \mathcal{S} \;\; \textit{such that} \;\; \int_{\mathbb{S}^2} e^{2u} = 4\pi \right\}. \end{aligned} \tag{C.2.12}$$

Theorem C.2.6 (Effective uniformization)**.** *Let (S, g^S) be a fixed sphere with $|S| = 4\pi$. There exists, up to isometries*[3] *of $\mathbb{S}^2$, a unique diffeomorphism $\Phi : \mathbb{S}^2 \to S$ and a unique centered conformal factor u, i.e. $u \in \mathcal{S}$, such that $\Phi^\#(g^S) = e^{2u}\gamma_0$. Moreover, under the almost round condition*

$$\|K^S - 1\|_{L^\infty} \le \epsilon \tag{C.2.13}$$

where $K^S = K(g^S)$, the following properties are verified for sufficiently small $\epsilon > 0$.

[2] To the best of our knowledge the condition $\int_{\mathbb{S}^2} xe^{2u} = 0$ appears first in Aubin.

[3] i.e. all the solutions are of the form $(\Phi \circ O, u \circ O)$ for $O \in O(3)$.

1. We have

$$\|u \circ \Phi^{-1}\|_{L^\infty(S)} \lesssim \epsilon. \tag{C.2.14}$$

2. If in addition $\|K^S - 1\|_{H^s(S)} \leq \epsilon$ *for some* $s \geq 0$*, then*

$$\|u \circ \Phi^{-1}\|_{H^{2+s}(S)} \lesssim \epsilon. \tag{C.2.15}$$

Proof. See section 3.3 in [K-S:GCM2]. □

C.2.4 Effective uniformization for nearly round spheres of arbitrary area

Let (S, g^S) be a fixed sphere, and let r^S denote its area radius, i.e. r^S satisfies

$$|S| = 4\pi(r^S)^2.$$

Given a positive integer s, we introduce the following norm on S

$$\|f\|_{\mathfrak{h}_s(S)} := \sum_{i=0}^{s} \|(r^S \nabla^S)^i f\|_{L^2(S)}. \tag{C.2.16}$$

The goal of the following corollary is to extend Theorem C.2.6 to the case $r^S \neq 1$.

Corollary C.2.7. *Let* (S, g^S) *be a fixed sphere. There exists, up to isometries*[4] *of* $\mathbb{S}^2$*, a unique diffeomorphism* $\Phi : \mathbb{S}^2 \to S$ *and a unique centered conformal factor* u*, i.e.* $u \in \mathcal{S}$*, such that*

$$\Phi^\#(g^S) = (r^S)^2 e^{2u} \gamma_0.$$

Moreover, under the almost round condition

$$\left\| K^S - \frac{1}{(r^S)^2} \right\|_{L^\infty(S)} \leq \frac{\epsilon}{(r^S)^2}, \tag{C.2.17}$$

the following properties are verified for sufficiently small $\epsilon > 0$.

1. We have

$$\|u \circ \Phi^{-1}\|_{L^\infty(S)} \lesssim \epsilon. \tag{C.2.18}$$

[4] i.e. all the solutions are of the form $(\Phi \circ O, u \circ O)$ for $O \in O(3)$.

2. If in addition

$$\left\| K^S - \frac{1}{(r^S)^2} \right\|_{\mathfrak{h}_s(S)} \leq \frac{\epsilon}{r^S} \tag{C.2.19}$$

for some $s \geq 0$, then

$$\left\| u \circ \Phi^{-1} \right\|_{\mathfrak{h}_{s+2}(S)} \lesssim \epsilon r^S. \tag{C.2.20}$$

C.2.5 Canonical basis of $\ell = 1$ modes on S

Let S be an almost round sphere, i.e. verifying (C.2.17). The goal of this section is to define on S a canonical generalization of the $\ell = 1$ spherical harmonics.

Recall that the $\ell = 1$ spherical harmonics $J^{\mathbb{S}^2} = (J^{(-,\mathbb{S}^2)}, J^{(0,\mathbb{S}^2)}, J^{(+,\mathbb{S}^2)})$ are given by the restriction of x^1, x^2, x^3 to $\mathbb{S}^2$. More precisely, in polar coordinates,

$$J^{(0,\mathbb{S}^2)} = x^3 = \cos\theta, \qquad J^{(+,\mathbb{S}^2)} = x^1 = \sin\theta\cos\varphi, \qquad J^{(-,\mathbb{S}^2)} = x^2 = \sin\theta\sin\varphi. \tag{C.2.21}$$

Lemma C.2.8. *We have, for $p, q \in \{-, 0, +\}$,*

$$\begin{aligned} \Delta_0 J^{(p,\mathbb{S}^2)} &= -2J^{(p,\mathbb{S}^2)}, \\ \int_{\mathbb{S}^2} J^{(p,\mathbb{S}^2)} J^{(q,\mathbb{S}^2)} da_{\gamma_0} &= \frac{4\pi}{3}\delta_{pq}, \\ \int_{\mathbb{S}^2} J^{(p,\mathbb{S}^2)} da_{\gamma_0} &= 0. \end{aligned} \tag{C.2.22}$$

Proof. Straightforward verification. □

Definition C.2.9 (Basis of canonical $\ell = 1$ modes on S)**.** *Let (S, g^S) be an almost round sphere, i.e. verifying* (C.2.17)*. Let (Φ, u) the unique, up to isometries of $\mathbb{S}^2$, uniformization pair given by Corollary C.2.7, i.e.,*

$$\Phi : \mathbb{S}^2 \longrightarrow S, \qquad \Phi^\#(g^S) = (r^S)^2 e^{2u}\gamma_0, \qquad u \in \mathcal{S}.$$

We define the basis of canonical $\ell = 1$ modes on S by

$$J^S := J^{\mathbb{S}^2} \circ \Phi^{-1}, \tag{C.2.23}$$

where $J^{\mathbb{S}^2}$ denotes the $\ell = 1$ spherical harmonics, see (C.2.21)*.*

Remark C.2.10. *Note that the canonical basis is unique up to a rotation on $\mathbb{S}^2$.*

Lemma C.2.11. *Consider (S, g^S) a sphere of area radius r^S verifying the almost round condition* (C.2.17)*. Let (Φ, u) the unique, up to isometries of $\mathbb{S}^2$, uniformization pair given by Corollary C.2.7. Let J^S denote the basis of canonical $\ell = 1$ modes on S of Definition C.2.9. Then, we have*

$$\begin{aligned}
\Delta_S J^{(p,S)} &= -\frac{2}{(r^S)^2} J^{(p,S)} + \frac{2}{(r^S)^2}\big(1 - e^{-2u\circ\Phi^{-1}}\big) J^{(p,S)},\\
\int_S J^{(p,S)} J^{(q,S)} da_g &= \frac{4\pi}{3}(r^S)^2\delta_{pq} + \int_S J^{(p,S)} J^{(q,S)}\big(1 - e^{-2u\circ\Phi^{-1}}\big) da_{g^S},\\
\int_S J^{(p,S)} da_g &= 0,
\end{aligned} \tag{C.2.24}$$

with Δ^S the Laplace-Beltrami of the metric g^S. Moreover we have,

$$\begin{aligned}
\Delta_S J^{(p,S)} &= \left(-\frac{2}{(r^S)^2} + O\left(\frac{\epsilon}{(r^S)^2}\right)\right) J^{(p,S)},\\
\int_S J^{(p,S)} J^{(q,S)} da_g &= \frac{4\pi}{3}(r^S)^2\delta_{pq} + O(\epsilon (r^S)^2),
\end{aligned} \tag{C.2.25}$$

where $\epsilon > 0$ is the smallness constant appearing in the almost round condition (C.2.17)*.*

Proof. See section 3.5 in [K-S:GCM2] $\square$

Corollary C.2.12. *Let (S, g^S) verifying* (C.2.17)*. Let (Φ, u) the unique, up to isometries of $\mathbb{S}^2$, uniformization pair given by Corollary C.2.7. Let J^S denote the basis of canonical $\ell = 1$ modes on S of Definition C.2.9. Then, for sufficiently small $\epsilon > 0$, the following holds*[5]

$$\int_S \left(K^S - \frac{1}{(r^S)^2}\right) J^{(p,S)} \quad = \quad O(\epsilon^2), \qquad p = 0, +, -, \tag{C.2.26}$$

where K^S and r^S denote respectively the Gauss curvature and the area radius of S.

C.3 Stability of uniformization for nearby spheres

Consider two almost round spheres spheres (S_1, g^{S_1}) and (S_2, g^{S_2}), i.e. verifying (C.2.17), and their respective uniformization pairs (Φ_1, u_1), (Φ_2, u_2), i.e.

$$\begin{aligned}
\Phi_1 : \mathbb{S}^2 \longrightarrow S_1, \qquad g_1 := \Phi_1^\#(g^{S_1}) = (r^{S_1})^2 e^{2u_1}\gamma_0,\\
\Phi_2 : \mathbb{S}^2 \longrightarrow S_2, \qquad g_2 := \Phi_2^\#(g^{S_2}) = (r^{S_2})^2 e^{2u_2}\gamma_0,
\end{aligned} \tag{C.3.1}$$

[5]Note that, a priori, one would expect the right-hand side of (C.2.26) to be $O(\epsilon)$. The fact that it is actually $O(\epsilon^2)$ is an application of Corollary C.2.7 and (C.2.25), see the proof below.

and u_1, u_2 defined on $\mathbb{S}^2$ verifying the conclusions of Corollary C.2.7. We assume in addition given a smooth diffeomorphism $\Psi : S_1 \to S_2$ such that the metrics g^{S_1} and $\Psi^\#(g^{S_2})$ are close to each other in S^1 with respect to the coordinate chart provided by Φ_1, i.e. for some $0 < \delta \le \epsilon$,

$$\left\|g^{S_1} - \Psi^\#(g^{S_2})\right\|_{L^\infty(S_1)} + \frac{1}{(r^{S_1})}\left\|g^{S_1} - \Psi^\#(g^{S_2})\right\|_{\mathfrak{h}_4(S^1)} \le (r^{S_1})^2\delta. \tag{C.3.2}$$

The goal of this section is to show the existence of a canonical diffeomorphism $\widehat{\Psi} : \mathbb{S}^2 \to \mathbb{S}^2$ which relates the two uniformization maps. More precisely we prove the following.

Theorem C.3.1. *Under the assumptions above, let $\widehat{\Psi} : \mathbb{S}^2 \to \mathbb{S}^2$ be the unique smooth diffeomorphism such that $\Psi \circ \Phi_1 = \Phi_2 \circ \widehat{\Psi}$. Then, the following holds true.*

1. *The diffeomorphism $\widehat{\Psi}$ is smooth and there exists $O \in O(3)$ such that*

$$\|\widehat{\Psi} - O\|_{L^\infty(\mathbb{S}^2)} + \|\widehat{\Psi} - O\|_{H^1(\mathbb{S}^2)} \lesssim \delta. \tag{C.3.3}$$

2. *The conformal factors u_1, u_2 verify*

$$\left\|u_1 - \widehat{\Psi}^\# u_2\right\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta. \tag{C.3.4}$$

Remark C.3.2. *Let us note the following concerning assumption* (C.3.2).

- *It is clearly not sharp in terms of regularity. Sharper results could be obtained by working in Hölder spaces. On the other hand, in view of our applications, $\mathfrak{h}_s(S)$ are the natural spaces.*
- *It is coordinate dependent. Though it is sufficient for our applications, it would be nice find a coordinate independent condition sufficient to recover the conclusions of Theorem C.3.1.*

Proposition C.3.3. *Assume, in addition to the assumptions of Theorem C.3.1, that*

$$\left\|g^{S_1} - \Psi^\#(g^{S_2})\right\|_{\mathfrak{h}_{4+s}(S^1)} \le (r^{S_1})^3\delta \tag{C.3.5}$$

for some $s \ge 0$. Then, the following higher regularity analogs of (C.3.3), (C.3.4) *hold true.*

1. *The diffeomorphism $\widehat{\Psi}$ is smooth and there exists $O \in O(3)$ such that*

$$\|\widehat{\Psi} - O\|_{H^{5+s}(\mathbb{S}^2)} \lesssim \delta. \tag{C.3.6}$$

2. *The conformal factors u_1, u_2 verify*

$$\left\|u_1 - \widehat{\Psi}^\# u_2\right\|_{H^{4+s}(\mathbb{S}^2)} \lesssim \delta. \tag{C.3.7}$$

C.3.1 Calibration of uniformization maps between spheres

In order to eliminate the arbitrariness with respect to isometries of $\mathbb{S}^2$ in Theorem C.3.1 we calibrate the effective uniformization maps[6] $\Phi_1 : \mathbb{S}^2 \to S_1$, $\Phi_2 : \mathbb{S}_2 \to S_2$, for given diffeomorphism $\Psi : S_1 \to S_2$, as follows.

Definition C.3.4. *On $\mathbb{S}^2$ we fix[7] a point N and a unit vector v in the tangent space $T_N\mathbb{S}^2$. Given $\Psi : S_1 \to S_2$, we say that the effective uniformization maps $\Phi_1 : \mathbb{S}^2 \to S_1$, $\Phi_2 : \mathbb{S}^2 \to S_2$ are calibrated relative to Ψ if the map $\widehat{\Psi} := (\Phi_2)^{-1} \circ \Psi \circ \Phi_1 : \mathbb{S}^2 \to \mathbb{S}^2$ is such that*

1. *The map $\widehat{\Psi}$ fixes the point N, i.e. $\widehat{\Psi}(N) = N$.*
2. *The tangent map $\widehat{\Psi}_\#$ fixes the direction of v, i.e. $\widehat{\Psi}_\#(v) = a_{1,2}v$ where $a_{1,2} > 0$.*
3. *The tangent map $\widehat{\Psi}_\#$ preserves the orientation of $T_N\mathbb{S}^2$.*

Lemma C.3.5. *Given $\Psi : S_1 \to S_2$ and a fixed effective uniformization map $\Phi_1 : \mathbb{S}^2 \to S_1$ for S_1. Then, there exists a unique, effective, uniformization for S_2 calibrated with that of S_1 relative to Ψ.*

Corollary C.3.6. *In addition to the assumptions of Proposition C.3.3, assume that the maps Φ_1, Φ_2 are calibrated relative to Ψ according to Definition C.3.4. Then $\widehat{\Psi}$ verifies*

$$\|\widehat{\Psi} - I\|_{H^{5+s}(\mathbb{S}^2)} \lesssim \delta. \tag{C.3.8}$$

The conformal factors u_1, u_2 verify

$$\left\|u_1 - \widehat{\Psi}^\# u_2\right\|_{H^{4+s}(\mathbb{S}^2)} \lesssim \delta. \tag{C.3.9}$$

Proof. In view of Proposition C.3.3, there exists $O \in O(3)$ such that $\widehat{\Psi}$ satisfies

$$\|\widehat{\Psi} - O\|_{L^\infty(\mathbb{S}^2)} + \|\widehat{\Psi} - O\|_{H^{5+s}(\mathbb{S}^2)} \lesssim \delta$$

and (C.3.9) holds. It remains to check (C.3.8) which follows immediately from the bound $|O - I| \lesssim \delta$ established in the proof of Corollary **??**. □

Lemma C.3.7 (Transitivity of calibrations)**.** *Let $\Phi_1 : \mathbb{S}^2 \to S_1$, $\Phi_2 : \mathbb{S}^2 \to S_2$ and $\Phi_3 : \mathbb{S}^2 \to S_3$ three effective uniformization maps. Let $\Psi_{12} : S_1 \to S_2$ and $\Psi_{13} : S_1 \to S_3$ satisfying* (C.3.2) *and assume that Φ_1, Φ_2 are calibrated relative to Ψ_{12}, while Φ_1, Φ_3 are calibrated relative to Ψ_{13}. Then, Φ_2, Φ_3 are calibrated relative to $\Psi_{23} := \Psi_{13} \circ \Psi_{12}^{-1}$.*

[6] Given by Corollary C.2.7.

[7] In particular, one can choose $N = (0, 0, 1)$ and $v = (1, 0, 0)$.

Proof. Since Φ_1, Φ_2 are calibrated relative to Ψ_{12}, and Φ_1, Φ_3 are calibrated relative to Ψ_{13}, $\widehat{\Psi}_{12} := (\Phi_2)^{-1} \circ \Psi_{12} \circ \Phi_1$ and $\widehat{\Psi}_{13} := (\Phi_3)^{-1} \circ \Psi_{13} \circ \Phi_1$ satisfy the three properties of Definition C.3.4. Then, introducing

$$\Psi_{23} = \Psi_{13} \circ \Psi_{12}^{-1}, \qquad \widehat{\Psi}_{23} := (\Phi_3)^{-1} \circ \Psi_{23} \circ \Phi_2,$$

we have $\widehat{\Psi}_{23} = \widehat{\Psi}_{13} \circ \widehat{\Psi}_{12}^{-1}$ so that $\widehat{\Psi}_{23}$ also satisfy the three properties of Definition C.3.4. Hence, Φ_2, Φ_3 are calibrated relative to Ψ_{23} as desired. □

C.3.2 Comparison of $\ell = 1$ modes between two spheres

Consider, as in Theorem C.3.1, two almost round spheres (S_1, g^{S_1}) and (S_2, g^{S_2}) and a smooth map $\Psi : S_1 \to S_2$ such that, as in (C.3.2), that the metrics g^{S_1} and $\Psi^{\#}(g^{S_2})$ are close to each other in S^1. Assume that (Φ_1, u_1) , (Φ_2, u_2) are effective uniformization maps of S_1 and S_2, calibrated as in definition C.3.4. We define

$$J^i = J^{S_i} = J^{\mathbb{S}^2} \circ \Phi_i^{-1}, \qquad i = 1, 2,$$

to be the $\ell = 1$ canonical modes of S_1, S_2 according to Definition C.2.9. We want to compare J^1 with $J^2 \circ \Psi$. We prove the following result.

Proposition C.3.8. *Under the assumptions of Corollary* **??***, we have*

$$\sup_{S_1} \Big| J^1 - J^2 \circ \Psi \Big| \;\lesssim\; \delta. \tag{C.3.10}$$

Also, under the assumptions of Corollary C.3.6, we have

$$\Big\| J^1 - J^2 \circ \Psi \Big\|_{\mathfrak{h}_{5+s}(S_1)} \;\lesssim\; r^{S_1}\delta. \tag{C.3.11}$$

Proof. Indeed, using that $\Psi \circ \Phi_1 = \Phi_2 \circ \widehat{\Psi}$,

$$J^2 \circ \Psi = J^{\mathbb{S}^2} \circ \Phi_2^{-1} \circ \Psi = J^{\mathbb{S}^2} \circ \widehat{\Psi} \circ \Phi_1^{-1}.$$

Hence,

$$J^1 - J^2 \circ \Psi = J^{\mathbb{S}^2} \circ \Phi_1^{-1} - J^{\mathbb{S}^2} \circ \widehat{\Psi} \circ \Phi_1^{-1}.$$

This implies, together with (**??**),

$$\Big| J^1 - J^2 \circ \Psi \Big| \lesssim \sup_{\mathbb{S}^2} \Big| I - \widehat{\Psi} \Big| \lesssim \delta$$

and, together with (C.3.8),

$$\Big\| J^1 - J^2 \circ \Psi \Big\|_{\mathfrak{h}_{5+s}(S_1)} \lesssim r^{S_1}\delta$$

as stated. □

Bibliography

[Al] S. Alinhac, *Energy multipliers for perturbations of the Schwarzschild metric*, Comm. Math. Phys. 288 (2009), 199–224.

[An-Luk] X. An and J. Luk, *Trapped surfaces in vacuum arising dynamically from mild incoming radiation*, ArXiv:1409.6270v2.

[A-B] L. Andersson and P. Blue, *Hidden symmetries and decay for the wave equation on the Kerr spacetime*. Ann. of Math. (2) **182** (2015), 787–853.

[AWPW] L. Andersson, S. Ma, C. Paganini and B. Whiting, *Mode stability on the real axis*, J. Math. Phys. **58** (2017).

[ABBMa2019] L. Andersson, T. Bäckdahl, P. Blue and S. Ma, *Stability for linearized gravity on the Kerr spacetime*, arXiv:1903.03859.

[AArGa] Y. Angelopoulos, S. Aretakis and D. Gajic, *A vector field approach to almost-sharp decay for the wave equation on spherically symmetric, stationary spacetimes*, Ann. PDE **4** (2018), Art. 15, 120 pp.

[Bi] L. Bieri, *An extension of the stability theorem of the Minkowski space in general relativity*, Thesis, ETH, 2007.

[BiZi] L. Bieri and N. Zipser, *Extensions of the stability theorem of the Minkowski space in general relativity.* AMS/IP Studies in Advanced Mathematics, 2009. xxiv+491 pp.

[BFJT] L. Bigorgne, D. Fajman, J. Joudioux, J. Smulevici and M. Thaller, *Asymptotic stability of Minkowski spacetime with non-compactly massless Vlasov matter*, ARMA **242** (2021), 1–147.

[B-S1] P. Blue and A. Soffer, *Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates*, Adv. Differential Equations **8** (2003), 595–614.

[B-S2] P. Blue and A. Soffer, *Errata for "Global existence and scattering for the nonlinear Schrödinger equation on Schwarzschild manifolds" , "Semilinear wave equations on the Schwarzschild manifold I: Local Decay Estimates", and " The wave equation on the Schwarzschild metric II: Local Decay for the spin 2 Regge Wheeler equation"* , gr-qc/0608073, 6 pages.

[B-St] P. Blue and J. Sterbenz, *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space*, Comm. Math. Phys. **268** (2006), 481–504.

[Carter] B. Carter, *Global structure of the Kerr family of gravitational fields*, Phys. Rev. **174** (1968), 1559–1571.

[Chand2] S. Chandrasekhar, *On the equations governing the perturbations of the Schwarzschild black hole*, P. Roy. Soc. Lond. A Mat. **343** (1975), 289–298.

[Chand3] S. Chandrasekhar, *An introduction to the theory of the Kerr metric and its perturbations*, in General Relativity, an Einstein centenary survey, ed. S.W. Hawking and W. Israel, Cambridge University Press (1979).

[Chand4] S. Chandrasekhar, *Truth and Beauty: Aesthetics and Motivations in Science*, The University of Chicago Press (1987).

[Chen] P.-N. Chen, M.-T. Wang and S.-T. Yau, *Quasilocal angular momentum and center of mass in general relativity*, Adv. Theor. Math. Phys. **20** (2016), 671–682.

[Chen2] P.-N. Chen, M.-T. Wang, Y.-K. Wang and S.-T. Yau, *Supertranslation invariance of angular momentum*, Adv. Theor. Math. Phys. **25** (2021), 777–789.

[Br] Y. Choquet-Bruhat, *Théoréme d'existence pour certains systémes d'equations aux dérivées partielles non-linéaires.*, Acta Math. **88** (1952), 141–225.

[Chr] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), 267–282.

[Chr-BH] D. Christodoulou, *The formation of Black Holes in General Relativity*, EMS Monographs in Mathematics, 2009.

[Ck-Kl0] D. Christodoulou and S. Klainerman, *Asymptotic properties of linear field theories in Minkowski space*, Comm. Pure Appl. Math. **43** (1990), 137–199.

[Ch-Kl] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton University Press (1993).

[DaRo1] M. Dafermos and I. Rodnianski, *The red-shift effect and radiation decay on black hole spacetimes*, Comm. Pure Appl. Math. **62** (2009), 859–919.

[DaRo2] M. Dafermos and I. Rodnianski, *A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds*, Invent. Math. **185** (2011), 467–559.

[Da-Ro3] M. Dafermos and I. Rodnianski, *A new physical-space approach to decay for the wave equation with applications to black hole spacetimes*, XVIth International Congress on Mathematical Physics, World Sci. Publ., Hackensack, NJ, 2010, 421–432.

[D-R-SR] M. Dafermos, I. Rodnianski and Y. Shlapentokh-Rothman, *Decay for solutions of the wave equation on Kerr exterior spacetimes iii: The full subextremal case* $|a| < m$, Ann. of Math. **183** (2016), 787–913.

[DHR] M. Dafermos, G. Holzegel and I. Rodnianski, *Linear stability of the Schwarzschild solution to gravitational perturbations*, Acta Math. **222** (2019), 1–214.

[DHR-Kerr] M. Dafermos, G. Holzegel and I. Rodnianski, *Boundedness and decay for the Teukolsky equation on Kerr spacetimes I: The case* $|a| \ll M$, Ann. PDE (2019), 118 pp.

[DHRT] M. Dafermos, G. Holzegel, I. Rodnianski and M. Taylor, *The non-linear stability of the Schwarzschild family of black holes*, arXiv:2104.0822.

[DKSW] R. Donninger, J. Krieger, J. Szeftel and W. Wong, *Codimension one stability of the catenoid under the vanishing mean curvature flow in Minkowski space*, Duke Math. Journal **165** (2016), 723–791.

[FJS] D. Fajman, J. Joudioux and J. Smulevici, *The stability of the Minkowski space for the Einstein-Vlasov system*, Anal. PDE 14 (2021), 425–531.

[Fang1] A. J. Fang, *Nonlinear stability of the slowly-rotating Kerr-de Sitter family* arXiv:2112.07183.

[Fang2] A. J. Fang, *Linear stability of the slowly-rotating Kerr-de Sitter family* arXiv:2207.07902.

[Fins1] F. Finsler, N. Kamran, J. Smoller and S.-T. Yau. *Decay of solutions of the wave equation in the Kerr geometry*, Communications in Mathematical Physics **264** (2006), 465–503.

[Fins2] F. Finsler and J. Smoller, *Linear stability of the non-extreme Kerr black hole*, Adv. Theor. Math. Phys. **21** (2017), 1991–2085.

[GHP] R. Geroch, A. Held and R. Penrose, *A space-time calculus based on pairs of null directions*, Journal of Mathematical Physics **14** (1973), 874–881.

[GKS-2020] E. Giorgi, S. Klainerman and J. Szeftel, *A general formalism for the stability of Kerr*, arXiv:2002.02740.

[Giorgi] E. Giorgi, *The linear stability of Reissner-Nordström spacetime for small charge*, Ann. PDE **6**, 8 (2020).

[GKS-2022] E. Giorgi, S. Klainerman and J. Szeftel, *Wave equations estimates and the nonlinear stability of slowly rotating Kerr black holes*, arXiv:2205.14808.

[Giorgi:KN] Giorgi E., *Electromagnetic-gravitational perturbations of Kerr-Newman spacetime: the Teukolsky and Regge-Wheeler equations*, arXiv:2002.07228, to appear in Journal of Hyperbolic Differential Equations.

[Graf] O. Graf, *Global nonlinear stability of Minkowski space for spacelike-characteristic initial data*, arXiv:2010.12434.

[HHV] D. Häfner, P. Hintz and A. Vasy, *Linear stability of slowly rotating Kerr black holes*, Invent. Math., **223** (2021), 1227–1406.

[H-V1] P. Hintz and A. Vasy, *The global non-linear stability of the Kerr-de Sitter family of black holes*, Acta Math. **220** (2018), 1–206.

[HV2] P. Hintz and A. Vasy, *Stability of Minkowski space and polyhomogeneity of the metric*, Ann. PDE **6**, 2 (2020).

[Huneau] C. Huneau, *Stability of Minkowski spacetime with a translation space-like Killing field*, Ann. PDE **4** (2018), Art. 12, 147 pp.

[Hu-Yau] G. Huisken, S.-T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature.* Invent. Math. **124** (1996), 281–311.

[HKW] P. K. Hung, J. Keller and M. T. Wang, *Linear stability of Schwarzschild spacetime: decay of metric coefficients*, J. Diff. Geom **116** (2020), 481–541.

[IK-review] A. Ionescu and S. Klainerman *Rigidity results in General Relativity: A review*, Surveys in Differential Geometry 20, 123–156, 2015.

[I-Kl] A. Ionescu and S. Klainerman, *On the global stability of the wave-map equation in Kerr spaces with small angular momentum*, Ann. PDE **1** (2015), Art. 1, 78 pp.

[IP] A. Ionescu and B. Pausader, *The Einstein-Klein-Gordon coupled system: global stability of the Minkowski solution*, Annals of Math Studies, 213. Princeton University Press, Princeton NJ, 2022, xi+297 pp.

[Johnson] T. W. Johnson, *The linear stability of the Schwarzschild solution to gravitational perturbations in the generalised wave gauge*, Ann. PDE **5** (2019), no. 2, Art. 13, 92 pp.

[K-Wald] B. S. Kay and R. M. Wald, *Linear stability of Schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere*, Classical Quantum Gravity 4 (1987), 893–898.

[Kerr] R. P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, Physical Review Letters **11** (1963), 237.

[Kl-vect1] S. Klainerman, *Uniform decay estimates and the Lorentz invariance of the classical wave equations*, Comm. Pure Appl. Math. **38** (1985), 321–332.

[Kl:vect2] S. Klainerman, *Remarks on the global Sobolev inequalities*, Communications on Pure and Applied Mathematics, Vol. 40, 111-117 (1987).

[Kl-ICM] S. Klainerman, *Long time behavior of solutions to nonlinear wave equations*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 1209–1215, PWN, Warsaw, 1984.

[Kl-null] S. Klainerman, *The Null Condition and global existence to nonlinear wave equations*, Lect. in Appl. Math. **23** (1986), 293–326.

[Kl-vect2] S. Klainerman, *Remarks on the global Sobolev inequalities*, Comm. Pure Appl. Math. **40** (1987), 111–117.

[Kl-L-R] S. Klainerman, J. Luk and I. Rodnianski *A Fully Anisotropic Mechanism for Formation of Trapped Surfaces in Vacuum*, Inventiones **198** (2014), 1–26.

[Kl-Ni1] S. Klainerman and F. Nicolo, The evolution problem in general relativity. Progress in Mathematical Physics **25**, Birkhauser, Boston, 2003, +385 pp.

[Kl-Ni2] S. Klainerman and F. Nicolo, Peeling properties of asymptotic solutions to the Einstein vacuum equations, Class. Quantum Grav. **20** (2003), 3215–3257.

[Kl-Rod1] S. Klainerman and I. Rodnianski, *On the break-down criterion in General Relativity*, J. Amer. Math. Soc. **23** (2010), 345–382.

[Kl-Rod2] S. Klainerman and I. Rodnianski, *On the formation of trapped surfaces*, Acta Mathematica, **208** (2012), 211- 333.

[K-S:Schw] S. Klainerman and J. Szeftel, *Global Non-Linear Stability of Schwarzschild Spacetime under Polarized Perturbations*, Annals of Math Studies, 210. Princeton University Press, Princeton NJ, 2020, xviii+856 pp.

[K-S:GCM1] S. Klainerman and J. Szeftel, *Construction of GCM spheres in perturbations of Kerr* arXiv:1911.00697. Ann. PDE, 8, Art. 17, 153 pp., 2022.

[K-S:GCM2] S. Klainerman and J. Szeftel, *Effective results in uniformization and intrinsic GCM spheres in perturbations of Kerr*, arXiv:1912.12195. Ann. PDE, **8** (2), Art. 18, 89 pp., 2022.

[K-S:Kerr] S. Klainerman and J. Szeftel, *Kerr stability for small angular momentum*, Pure and Appl. Math. Quaterly (PAMQ), Volume 19(2023), 3, pp. 791-1678,

[K-S:review] S. Klainerman and J. Szeftel, *Brief introduction to the stability of Kerr*, arXiv:2210.14400, to appear in PAMQ.

[Ko-Is] H. Kodama and A. Ishibashi, *A master equation for gravitational pertur bations of maximally symmetric black holes in higher dimensions.* Progress of Theoretical Physics 110:701–722, 2003.

[Lf-Ma] LeFloch and Y. Ma, *The global nonlinear stability of Minkowski space for self-gravitating massive fields*, Comm. Math. Phys. **346** (2016), 603–665.

[Lind] H. Lindblad, *On the asymptotic behavior of solutions to Einsteins vacuum equations in wave coordinates*, Comm. Math. Phys. **353**, (2017), 135–184.

[Lind-Rodn] H. Lindblad and I. Rodnianski, *Global existence in the Einstein Vacuum equations in wave co-ordinates.* Comm. Math. Phys. **256** (2005), 43–110.

[Lind-Ta] H. Lindblad and M. Taylor, *Global stability of Minkowski space for the EinsteinVlasov system in the harmonic gauge*, ARMA **235** (2020), 517–633.

[Luk] J. Luk, *The null condition and global existence for nonlinear wave equations on slowly rotating Kerr spacetimes*, JEMS **15** (2013), 1629–1700.

[Ma] S. Ma, *Uniform energy bound and Morawetz estimate for extreme components of spin fields in the exterior of a slowly rotating Kerr black hole II: linearized gravity*, Comm. Math. Phys. **377** (2020), 2489–2551.

[Ma-Me] Y. Martel and F. Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal. **157** (2001), 219–254.

[Ma-Me-Ta-To] J. Marzuola, J. Metcalfe, D. Tataru and M. Tohaneanu, *Strichartz estimates on Schwarzschild black hole backgrounds*, Comm. Math. Phys. **293** (2010), 37–83.

[Millet] P. Millet, *Optimal decay for solutions of the Teukolsky equation on the Kerr metric for the full subextremal range* $|a| < M$, arXiv:2302.06946.

[Me-R] F. Merle and P. Raphaël, *On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation*, Invent. Math. **156** (2004), 565–672.

[Moncr] V. Moncrief, *Gravitational perturbations of spherically symmetric systems. I. The exterior problem*, Ann. Phys. **88** (1975), 323–342.

[Mor1] C. Morawetz, *Decay of solutions of the exterior initial boundary value problem for the wave equation.* Comm. Pure and App. Math. **14** (1961), 561–568.

[Mor2] C. Morawetz, J. V. Ralston and W. A. Strauss, *Decay of solutions of the wave equation outside non-trapping obstacles*, Comm. Pure Appl. Math. **30** (1977), 447–508.

[NP] E. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys. **3** (1962), 566–578.

[P-T] W. Press and S. A. Teukolsky, *Perturbations of a rotating black hole. II. Dynamical stability of the Kerr metric*, Astrophys. J. **185** (1973), 649–673.

[Re-W] T. Regge and J. A. Wheeler, *Stability of a Schwarzschild singularity*, Phys. Rev. (2), 108:1063–1069, 1957.

[Rizzi] A. Rizzi, *Angular Momentum in General Relativity: A new Definition*, Phys. Rev. Letters, vol 81, no6, 1150–1153, 1998.

[Vo] V. Schlue, *Decay of the Weyl curvature in expanding black hole cosmologies*, Ann. PDE, 8, Art. 9, 125 pp., 2022.

[Shen] D. Shen, *Construction of GCM hypersurfaces in perturbations of Kerr*, arXiv:2205.12336. Annals of PDE 9, Art. 11, 2023.

[Shen:Mink-ext] D. Shen, *Stability of Minkowski spacetime in exterior regions*, arXiv:2211.15230.

[Shen:Kerr-ext] D. Shen, *Kerr stability in external regions*, arXiv:2303.12758.

[SR] Y. Shlapentokh-Rothman, *Quantitative Mode Stability for the Wave Equation on the Kerr Spacetime*, Ann. Henri Poincaré. **16** (2015), 289–345.

[SR-Te1] Y. Shlapentokh-Rothman and R. Teixeira da Costa, *Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range $|a| < M$: frequency space analysis*, arXiv:2007.07211.

[SR-Te2] Y. Shlapentokh-Rothman and R. Teixeira da Costa, *Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range $|a| < M$: physical space analysis*, arXiv:2302.08916.

[St] J. Stogin, *Princeton PHD thesis*, 2017.

[Sz] L. B. Szabados, *Quasi-Local Energy-Momentum and Angular Momentum in General Relativity*, Living Rev. Relativity, 12, (2009), 4.

[Ta-To] D. Tataru and M. Tohaneanu, *A Local Energy Estimate on Kerr Black Hole Backgrounds*, Int. Math. Res. Not., 2 (2011), 248–292.

[Te] R. Teixeira da Costa, *Mode stability for the Teukolsky equation on extremal and subextremal Kerr spacetimes*, Comm. Math. Phys. **378** (2020), 705–781.

[Teo] E. Teo, *Spherical photon orbits around a Kerr black hole.* General Relativity and Gravitation, **35** (2003), 1909–1926.

[Teuk] S. A. Teukolsky, *Perturbations of a rotating black hole. I. Fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations*, Astrophys. J. **185** (1973), 635–648.

[Vishev] C. V. Vishveshwara, *Stability of the Schwarzschild metric*, Phys. Rev. D, 1 (1970), 2870–2879.

[Wald] R. M. Wald, *Construction of solutions of gravitational, electromagnetic, or other perturbation equations from solutions of decoupled equations*, Phys. Rev. Lett. **41** (1978), 203–206.

[Wa] Q. Wang, *An intrinsic hyperboloid approach for Einstein Klein-Gordon equations*, J. Diff. Geom. **115** (2020), 27–109.

[Whit] B. Whiting, *Mode stability of the Kerr black hole*, J. Math. Phys. **30** (1989), 1301–1305.

[Ze] F. J. Zerilli, *Effective potential for even-parity Regge-Wheeler gravitational perturbation equations*, Phys. Rev. Lett. **24** (1970), 737–738.