# UNBOUNDED APOLLONIAN CIRCLE PACKINGS, SELF-SIMILARITY AND RESIDUAL POINTS

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Abstract. We begin with discussing Apollonian circle packings by introducing the notion of a Descartes configuration of four mutually tangent circles. We study Apollonian circle packings and the configurations of four mutually tangent circles they contain algebraically via the Apollonian group and geometrically via Möbius transformations. We primarily examine unbounded circle packings and non-trivial residual points, innately geometric concepts, through algebraic means. Using the generators of the Apollonian group and the principles behind the reduction algorithm, we construct sequences of Apollonian group generators with infinite length to identify self-similar unbounded packings and connect them with non-trivial residual points.

## 1. Introduction

We define a circle in the extended complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, to be a circle in $\mathbb{C}$ or a line in $\mathbb{C}$ together with $\infty$. Each circle comes with a choice of interior between the two connected components of its complement in $\widehat{\mathbb{C}}$. A collection of circles is said to be *mutually tangent* when any two intersect tangentially and all of their interiors are disjoint. Given a triplet of mutually tangent circles $C_1$, $C_2$, and $C_3$, there are exactly two possible circles, $C_4$ and $C_4'$, that can lie tangent to the triplet. Choosing one of these possible circles, we form an *ordered Descartes configuration* $\mathcal{D} = (C_1, C_2, C_3, C_4)$, which is an ordered set of four mutually tangent circles lying in $\widehat{\mathbb{C}}$. Note that a Descartes configuration may have a circle that contains the other three (the "interior" of this circle then contains $\infty$). In this case, we say that these circles have *internal contact*. We say the circles of a given Descartes configuration $\mathcal{D}$ belongs to generation 0, and we make new generations of circles by replacing $C_i$ with $C_i'$, the other circle mutually tangent to the circles excluding $C_i$. We say that the circles $C_1'$, $C_2'$, $C_3'$, and $C_4'$ make up generation 1. By replacing circles in the first $n$ generations with the other circles $C'$ forming a Descartes configuration, we obtain circles in the $(n+1)$st generation. The collection of circles from all generations 0, 1, ... is an *Apollonian circle packing* $\mathcal{P}$. For any three mutually tangent circles in $\mathcal{P}$, the circles tangent to all three will also lie in $\mathcal{P}$.

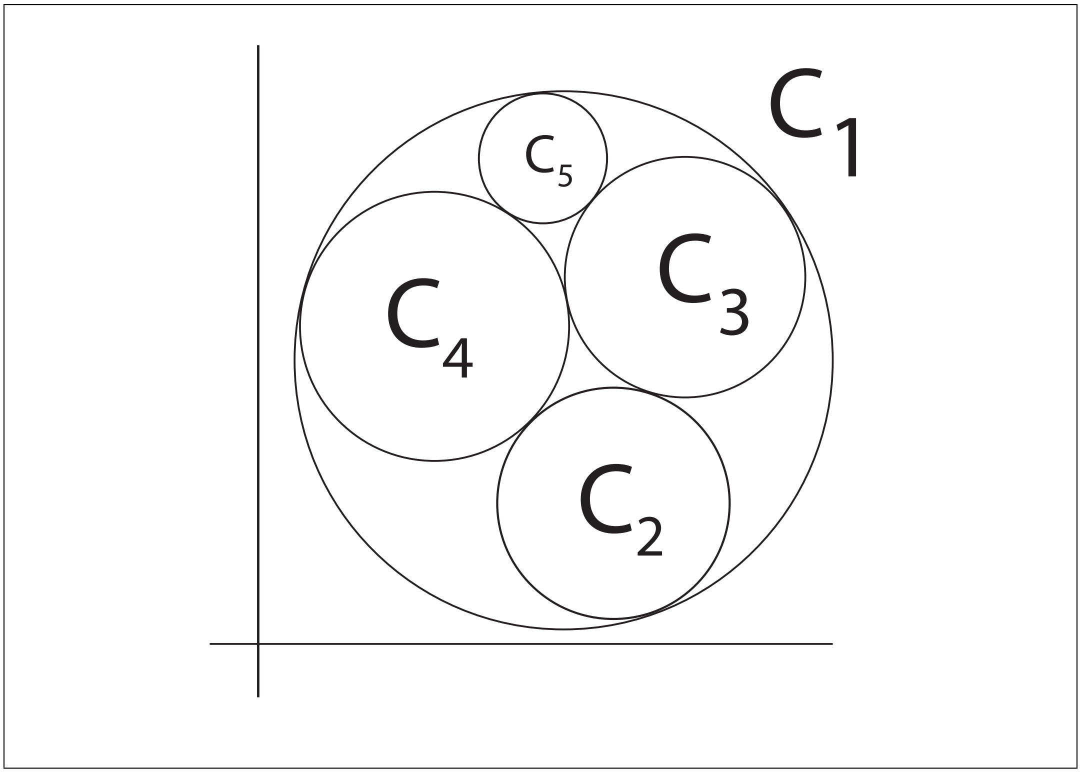



**Figure 1:** The quadruple $(C_1, C_2, C_3, C_4)$ is a Descartes configuration lying in $\widehat{\mathbb{C}}$ where $C_5$ is the other circle mutually tangent to $C_1$, $C_3$ and $C_4$.

Each Descartes configuration $\mathcal{D} = (C_1, C_2, C_3, C_4)$ has a corresponding *Descartes quadruple* $\mathbf{q} = (a, b, c, d)$ describing the curvature of each $C_i$ in $\mathcal{D}$. The curvature of a circle is the inverse of its radius when that circle's interior does not contain $\infty$ and otherwise is the negative of this number. Descartes' theorem states that each such quadruple is contained in the kernel of the following quadratic form

$$Q_D(w, x, y, z) \colon = 2(w^2 + x^2 + y^2 + z^2) - (w + x + y + z)^2$$

**Definition.** The projective Descartes variety, $\mathbb{P}D$, is the set of non-zero solutions to the Descartes equation up to scaling defined thusly:

$$\mathbb{P}D = \{(a, b, c, d) \in \mathbb{R}^4 \colon 2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2\}/\sim \quad ,$$

where

$$(a, b, c, d) \sim (x, y, z, w) \leftrightarrow \exists \lambda \in \mathbb{R}^\times \text{ s.t. } \lambda a = x, \lambda b = y, \lambda c = z, \lambda d = w$$

Lagarias et al have shown that a group of transformations Möb(2) can be identified with an index 2 subgroup of *Aut($Q_D$)*. [5]

**Definition.** We define the *general Möbius group* as Möb(2)=Möb(2)$_+$∪Möb(2)$_-$, where

$$M\ddot{o}b(2)_+ := \{(z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C}, ad-bc \neq 0\},$$

is the set of orientation-preserving Möbius transformations and where

$$M\ddot{o}b(2)_- := \{(z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d} : a,b,c,d \in \mathbb{C}, ad-bc \neq 0\}.$$

is the set of orientation-reversing Möbius transformations.

Since Möbius transformations take circles to circles and preserve angles, they take an ordered Descartes configuration to another Descartes configuration and in fact can take any one configuration to any other.[5] As a result, these Möbius transformations have a corresponding action on Descartes quadruples. We observe that the Descartes configurations in each Apollonian packing form an orbit of a single Descartes configuration under a certain finitely generated discrete subgroup of Aut($Q_D$), which we call the *Apollonian group*.

**Definition.** The *Apollonian group* $\mathcal{A}$ is the subgroup of $GL_4(\mathbb{Z})$ generated by the following four integer $4 \times 4$ matrices

$$S_1 = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix}$$

These generators form a Coxeter group with only the following relations: $S_1^2 = S_2^2 = S_3^2 = S_4^2 = I$.[4] The *length* of an Apollonian group member is the number of generators that appear in its representation as $S_{i_n} \dots S_{i_2} S_{i_1}$ such that $S_{i_j} \neq S_{i_{j+1}}$. For instance, we say $S_2S_3S_4S_2S_1$ has length 5 while length $S_2S_3S_3S_4S_1$ has length 3 because $S_3S_3 = I$.

If we view real Descartes quadruples $\mathbf{v} = (a,b,c,d)^T$ as column vectors, then the Apollonian group $\mathcal{A}$ acts by matrix multiplication, sending $\mathbf{v}$ to $M\mathbf{v}$, for any $M \in \mathcal{A}$. The geometric interpretation of the action of $\mathcal{A}$ on a Descartes configuration $\mathcal{D}$ has each $S_i$ corresponding to inversion of the circle passing through the three tangent points that do not include the $i$th circle of the configuration. Since this inversion fixes the three circles and moves the fourth circle to the unique $C_i'$ that is tangent to the other three.[5] In this sense, an Apollonian packing is just the $\mathcal{A}$-orbit of a Descartes configuration

We say an Apollonian circle packing is *integral* if all the circles have an integer curvature. For each integral Apollonian circle packing and a given Descartes quadruple associated with it, there is a reduction procedure described by Lagarias et al that yields a sequence of

Descartes quadruples manifested in the packing and halting at a unique *root quadruple*.[4] We will cover the definition of the the reduction algorithm and the root quadruple in section (2). Geometrically, the Apollonian packing generated by actions of $\mathcal{A}$ on the root quadruple $(-1, 2, 2, 3)$ can be represented by figure (1), which shows the curvature of a circle in its interior.

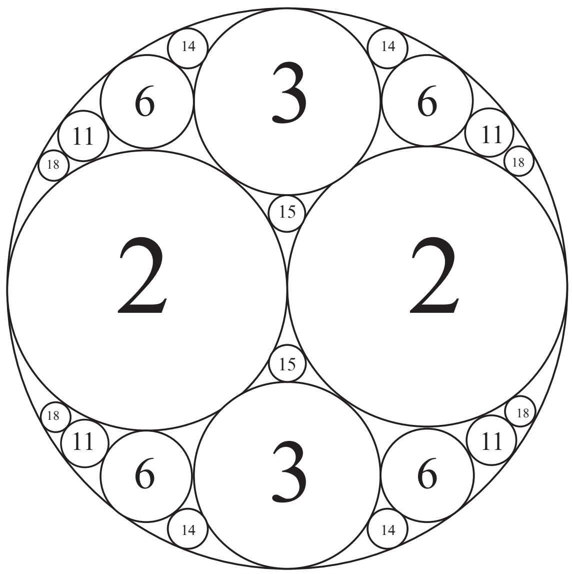


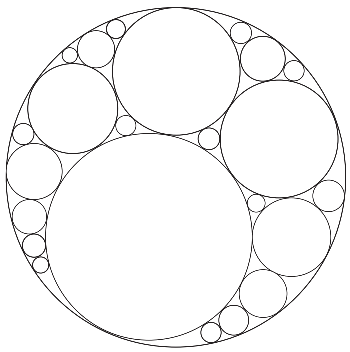

**Figure 2:** This is the geometric representation of an Apollonian packing with a root quadruple of (-1,2,2,3)

Of particular interest to us will be the *residual points* of $\mathcal{P}$.

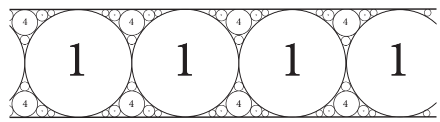
**Definition.** The *residual set* $\Lambda(\mathcal{P};\mathcal{D})$ of an Apollonian packing $\mathcal{P}$ with the Descartes configuration $\mathcal{D}$ is the complement of the interiors of all circles of $\mathcal{P}$. A point $z_0 \in \Lambda(\mathcal{P};\mathcal{D})$ is a *non-trivial* residual point if and only if $z_0$ is not on a circle of $\mathcal{P}$.

If we treat $\mathcal{P}$ as a collection of open disks in the plane, $\Lambda(\mathcal{P};\mathcal{D})$ is the plane with all of these disks removed. The main focus of this paper is to describe how to find the non-trivial residual points that make up $\Lambda(\mathcal{P};\mathcal{D})$. We know that there are non-trivial residual points as previous research due to David Boyd has shown that $\mathcal{P}$ has a Hausdorff dimension greater than 1.[1]

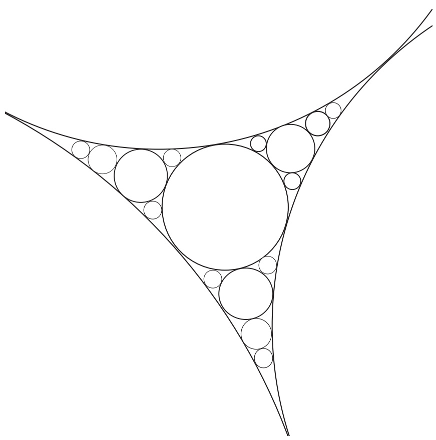
Any Apollonian circle packing $\mathcal{P}$ comes in four basic shapes: *bounded*, *strip*, *half-plane*, and *full-plane*, as illustrated in figures $3(a), 3(b), 3(c), 3(d)$. Packings with a circle having $\infty$ in its interior are *bounded*. Otherwise, a packing is a *strip*, *half-plane* or *full-plane* packing when it contains 2, 1 or 0 lines respectively. In practice, it is not difficult to generate examples of bounded, strip or half-plane packings and their properties were studied extensively by Ching



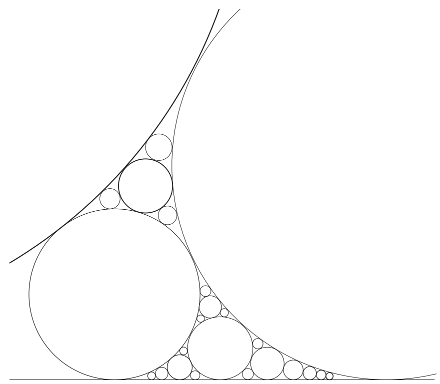
(a) This is a Bounded Packing




(b) This is a Strip Packing



(c) This is an Unbounded Packing



(d) This is a Half Plane Packing

and Doyle in [2]. On the other hand, it is not immediately clear that full-plane packings exist. We are interested in constructing such full-plane unbounded packings, and plan on demonstrating their connection with residual points. While *a priori* the two do not relate, we seek to show that the residual points of a fixed packing are in a bijective correspondence with unbounded packings with a marked configuration up to similarity.

**Definition.** Similarity conditions between packings can be represented through a collection of transformations, specifically translation, rotation, dilation, and reflection. More formally the following sets represent orientation-preserving similarities and orientation-reversing similarities of $\widehat{\mathbb{C}}$:

$$\mathcal{S}_+ = \{az + b : a, b \in \mathbb{C}, a \neq 0\} \quad \mathcal{S}_- = \{a\bar{z} + b : a, b \in \mathbb{C}, a \neq 0\}$$

and we take $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$

**Definition.** Two packings $\mathcal{P}_1$, $\mathcal{P}_2$ are *similar* with respective to marked Descartes configurations $\mathcal{D}_1$ and $\mathcal{D}_2$ if there exists some transformation $T$ such that $T(\mathcal{D}_1) = \mathcal{D}_2$ (implying $T(\mathcal{P}_1) = \mathcal{P}_2$) where $T$ belongs to $\mathcal{S}$. A packing $\mathcal{P}$ is *self-similar* if there exists a transformation $T'$ such that:

$$T'(\mathcal{P}) = \mathcal{P}$$

Where $T'$ is again in $\mathcal{S}$.

We will be studying the collection of all unbounded packings with marked Descartes configurations.

**Definition.** Let $\mathcal{U}$ be set of unbounded packings in $\widehat{\mathbb{C}}$ with a marked Descartes configuration up to similarity transformation. That is

$$\mathcal{U} = \{(\mathcal{P}', Q') \mid \mathcal{P}' \text{ is an unbounded packing, } Q' \text{ is a Descartes configuration of } \mathcal{P}'\}/\sim$$

where $\sim$ denotes similarity.

Another useful tool to studying Apollonian circle packings will be infinite sequences in the generators of $\mathcal{A}$, which we call *Apollonian sequences.*

**Definition.** The set of *Apollonian sequences* $\mathcal{A}_\infty$ is defined as follows:

$$\mathcal{A}_\infty = \{...S_{i_3}S_{i_2}S_{i_1} \mid i_1, i_2, \ldots \in \{1,2,3,4\}, S_{i_{j+1}} \neq S_{i_j}\}.$$

$\overline{\mathcal{A}}$ is the set of *Apollonian sequences* that eventually repeat. That is

$$\overline{\mathcal{A}} = \{...S_{i_3}S_{i_2}S_{i_1} \in \mathcal{A}_\infty \mid \exists\, n \text{ such that } \forall\, j >> 0, i_j = i_{j+n}\}.$$

Given a Descartes configuration $\mathcal{D}$ of a packing $\mathcal{P}$, we associate circles contained in $\mathcal{P} \setminus \mathcal{D}$ to some $a \in \mathcal{A}\backslash\{1\}$.

**Definition.** Let $(\mathcal{P}, \mathcal{D})$ be a circle packing with marked configuration. Let $a = S_{i_n}...S_{i_2}S_{i_1}$ be an element of $\mathcal{A}$. We may associate to $a$ a circle $C$ which is the circle in $S_{i_n}...S_{i_1}(\mathcal{D})$ but not in $S_{i_{n-1}}...S_{i_1}(\mathcal{D})$. So we may write that $C = S_{i_n}...S_{i_1}(\mathcal{D})\backslash S_{i_{n-1}}...S_{i_1}(\mathcal{D})$.

The sequence of circles associated with an Apollonian sequence $a_\infty = ...S_{i_3}S_{i_2}S_{i_1}$ is just the sequence of circles associated to $S_{i_1}$, $S_{i_2}S_{i_1}$, $S_{i_3}S_{i_2}S_{i_1}$,... with respect to the given configuration $\mathcal{D}$. In particular, the $n$th circle is associated to the $n$th generator in $a_\infty$.

Our primary tool in this paper will be a sequence of $\mathcal{A}$-*equivariant* maps of this form:

$$\overline{\mathcal{A}} \hookrightarrow \mathcal{A}_\infty \xrightarrow{F_1} \Lambda(\mathcal{P};\mathcal{D}) \xrightarrow{F_2} \mathcal{U} \xrightarrow{F_3} \mathbb{P}D \xrightarrow{F_4} S^2$$

With these tools we prove that each Apollonian group member of length $\geq 2$ has an eigenvector which is a Descartes quadruple of a self-similar unbounded packing and that all unbounded self-similar packings appear in this fashion. Then using $F_2^{-1}$ in the diagram above, one can obtain non-trivial residual points from full-plane unbounded packings.

In section 2, we establish the map $F_1$ and study some of its properties using reduction algorithm techniques. In section 3, we connect the studies of unbounded Apollonian packings and residual points using $F_2$. In section 4, we complete our diagram with $F_3$ and show how to connect Apollonian sequences to Descartes quadruples and their reduction algorithms. In section 5, we study the action of the Apollonian group on Descartes quadruples, unbounded packings, residual points and Apollonian sequences and demonstrate how to move between these realms $\mathcal{A}$-equivariantly. In section 6, we restrict our attention to Apollonian sequences that are eventually repeating and show how they are related to self-similar, unbounded Apollonian circle packings.

## 2. The Reduction Algorithm and Arriving at Residual Points of an Apollonian Packing

Through a unique sequence of actions of the generators of $\mathcal{A}$ on a Descartes quadruple $\mathbf{q}$, we can successively reduce the curvatures of $\mathbf{q}$ while staying in some packing $\mathcal{P}$. This process can be found for a given quadruple by the reduction algorithm provided by Lagarias et al in [4]:

**Definition.** Given a real Descartes quadruple $(a, b, c, d)$ with $a + b + c + d > 0$,

(1) Test to see in order $1 \leq i \leq 4$ whether some $S_i$ decreases the sum $(a + b + c + d)$. If so, apply that generator to the quadruple $\mathbf{q}$ to produce a new quadruple, and continue the algorithm.

(2) If no $S_i$ decreases the sum, halt.

When the reduction algorithm terminates it must end at a *root quadruple*.

**Definition.** A *root quadruple* $\mathbf{r} = (a, b, c, d)$ is a quadruple belonging to some packing $\mathcal{P}$ that has the following properties:

(1) $a \leq 0 \leq b \leq c \leq d$

(2) $a + b + c \geq d$

A *root configuration* is any Descartes configuration whose curvatures are a root quadruple.

The following facts about the reduction algorithm were shown to be true:

(1) The reduction algorithm is unique; there exists only one configuration that decreases the curvature at each step.

(2) If the algorithm terminates, it must terminate at a unique root quadruple. If a root quadruple exists, the algorithm must reach it.

(3) The algorithm does not halt if and only if the packing is an unbounded packing that isn't the strip packing.

Note that when applying generators that do not follow the reduction algorithm, that we are necessarily following a reduction algorithm in reverse. By considering the actions of an infinite chain of generators of $\mathcal{A}$ on a quadruple in this fashion, we arrive at the following theorem:

**Theorem 1.** *Given a bounded Apollonian circle packing $\mathcal{P}$ and an element $a_\infty = ...S_{i_3}S_{i_2}S_{i_1}$ in $\mathcal{A}_\infty$, the following are true:*

(1) *Given two points $z_n$ and $z'_n$ on the circles associated with the element $S_{i_n}S_{i_{n-1}}\dots S_{i_1}$ in the packing $\mathcal{P}$ with respect to a root configuration $\mathcal{D}$, the following limits exist*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} z'_n$$

*And converge to a residual point in the packing $\mathcal{P}$.*

(2) *The number of $S_i$'s appearing finitely many times in $a_\infty$ is equal to the number of circles of $\mathcal{P}$ through $\lim\limits_{n\to\infty} z_n$.*

*Proof.*
(1) By looking at the generators $\{S_1, S_2, S_3, S_4\}$ of the Apollonian group $\mathcal{A}$, we can see that given an infinite chain of such generators and having them act on a Descartes root quadruple $\mathbf{r}=(a, b, c, d)$, we are performing the reduction algorithm backwards. In other words, as the generators in the chain $a_\infty$ act on $\mathbf{r}$ successively, each generator increases the curvature value of some element. To see that this infinite sequence converges to a specific point, we look to [4], which considers a slowest-case situation in which the authors conclude that the minimal growth in $n$ of length $2n$ Apollonian group elements $W_{2n}$ acting on a quadruple $\mathbf{v}$ is:

$$|W_{2n}\mathbf{v}|_\infty = n(n+1)(a+b) - nc + (n-1)d$$

which clearly grows quadratically with $n$. We see that even the slowest growing case converges. This follows from the fact that

$$|z_n - z_m| < 2\sum_{j=n}^{m}(\text{radii of the } j^{th} \text{ circle})$$

which goes to zero, making $\{z_n\}$ a Cauchy sequence and converging since $\widehat{\mathbb{C}}$ is complete.

(2) By applying the generators that appear only a finite number of times to a quadruple $\mathbf{q}$, we arrive at some new configuration on which we would now like to apply the generators that appear an infinite number of times. By choosing $z_n$ to be points on circles whose curvature will no longer be changed by the sequence of generators, we see that the limit of the sequence of points on the circles associated with the infinite sequence will converge to a point on the circles whose curvatures are not altered by the infinite sequence of generators (since the number of generators that appear infinitely many times must be at least two, it follows that this point will remain fixed). Therefore, the number of generators that appear a finite number of times designates the number of circles that pass through the residual point associated with the infinite sequence of generators. □

As we can see, this then allows us to construct a map

$$F_1 : \mathcal{A}_\infty \to \Lambda(\mathcal{P};\mathcal{D}) \quad a_\infty \in \mathcal{A}_\infty \mapsto a_\infty\mathcal{D} \in \Lambda(\mathcal{P};\mathcal{D})$$

By associating the residual point in $\Lambda(\mathcal{P};\mathcal{D})$ with the residual point that the element $a_\infty$ arrives at in theorem 1.

**Corollary 2.** *Theorem 1 holds for a packing $\mathcal{P}$ that is not necessarily bounded with a Descartes configuration $\mathcal{D}$ not necessarily a root configuration.*

*Proof.* Recall that any two Apollonian packings are equivalent under some Möbius transformation. By taking a bounded packing to an unbounded packing via some transformation, the sequence of generators that approached a residual point starting from some root configuration $\mathcal{D}$ must still approach a residual point in the new unbounded packing with Descartes configuration $\mathcal{D}'$. From this, our corollary follows. □

Thus $F_1$ is defined even when $\mathcal{P}$ is unbounded or $\mathcal{D}$ is not a root configuration.

2.1. **Injectivity and Surjectivity of $F_1$.** It is natural to ask if the map $F_1$ is surjective. Based on a similar argument to Theorem 1, we obtain the following proposition:

**Proposition 3.** *The map $F_1 : \mathcal{A}_\infty \to \Lambda(\mathcal{P};\mathcal{D})$ is surjective.*

*Proof.* Consider some generation $n$ in a packing. Based on the definition of a residual point, it is clear that residual points are in the lunes constructed by the circles of the packing in the first $n$ generations. Then consider the next generation $(n+1)$; it is clear that the three lunes created by inscribing a circle into that lune is contained in the original lune. Once again, based on the definition of a residual point, we know that whatever points are in the packing are contained in one of the three smaller lunes. From this we can construct an infinite sequence of lunes, which gets smaller as $n$ grows. To show that this sequence of lunes must converge to a point, we briefly introduce the concept of a *dual* circle. The dual circle is the new circle constructed by taking the three tangent points of three mutually tangential circles as points on a circle. This is represented in figure **3**. Since the diameter

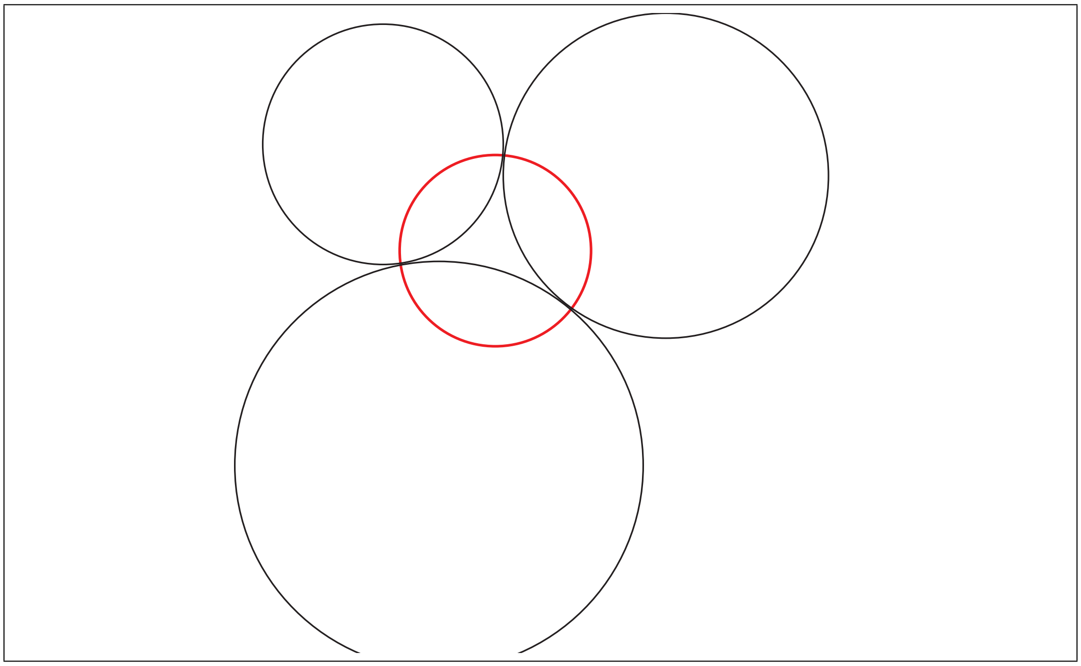


**Figure 3:** Here the Red Circle represents the Dual circle of the three black circles

of any lune is less than the diameter of the dual circle encapsulating it, it suffices to show that the diameter of the dual circle goes to zero. This follows from the diameter of the dual circle being represented by the elements in the Apollonian packing you started with in the following way:

$$d'^2 = ab + bc + ac$$

where $d'$ is the curvature of the dual circle to circles with curvatures $a$, $b$ and $c$.[3] From this relation we can conclude that the diameter of the dual circle goes to zero, since the curvatures of at least one of these circles is increasing to $\infty$ as we look at later generations. This then tells us that the lunes of our original packing must also converge to a point. □

Another natural question that follows from the construction of the map $F_1$ is whether or not this map is injective. What this boils down to is whether or not there exists a unique path of tangent circles to a specific residual point.

As is geometrically obvious from figure 4, it is clear that there exist distinct paths of circles that converge to the same residual point. More formally, this case can be expressed as

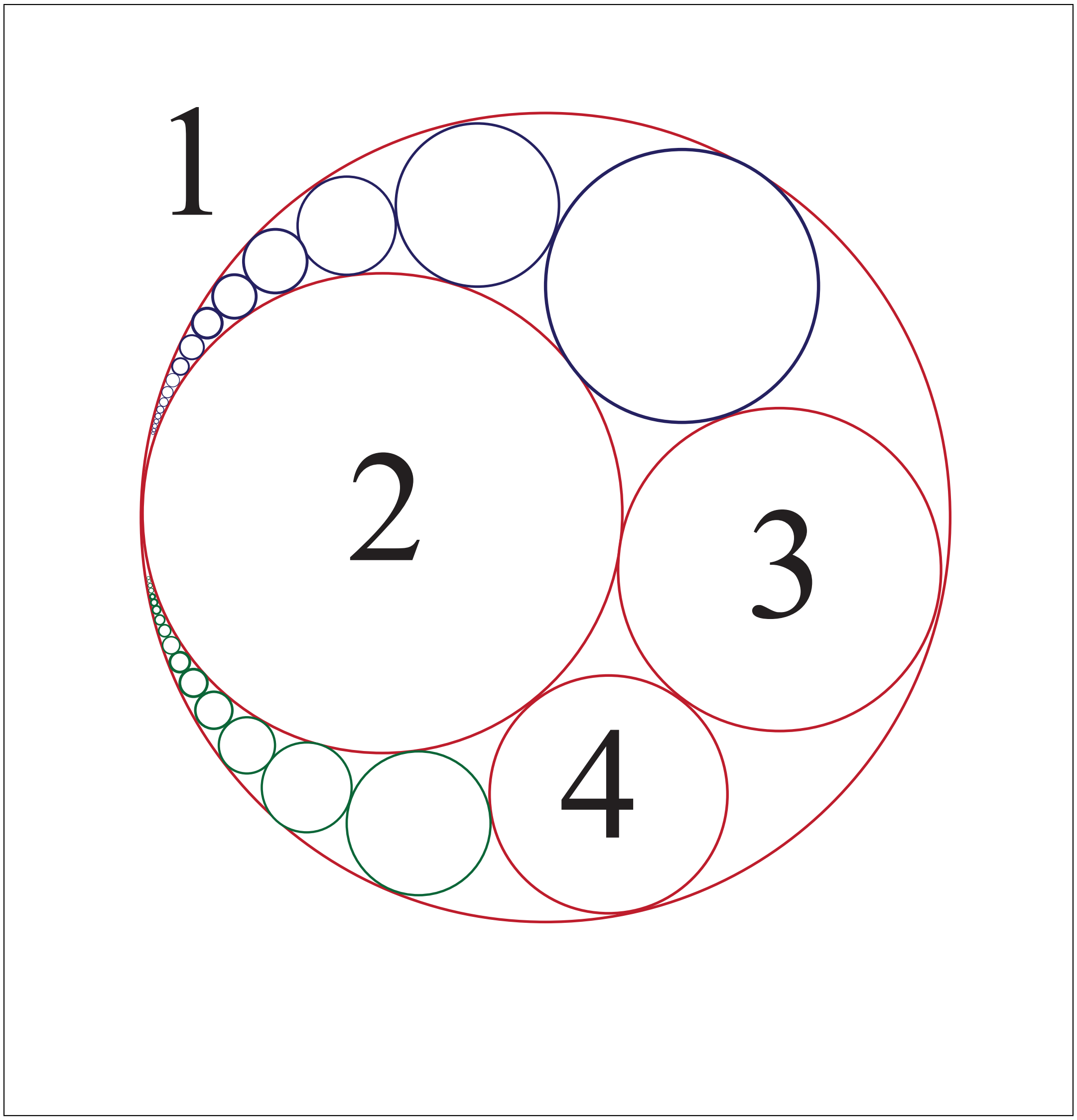



**Figure 4:** This is a packing generated by the root quadruple represented by circles (1,2,3,4)

actions on the quadruple $\mathbf{v} = (1, 2, 3, 4)$ in the following way:

$$\lim_{n\to\infty} (S_3 S_4)^n \mathbf{v} = \lim_{n\to\infty} (S_4 S_3)^n \mathbf{v}$$

This rules out any hope for injectivity on elements of $\mathcal{A}_\infty$ that have only two generators appearing infinitely many times. From this we have the following proposition:

**Proposition 4.** *The map $F_1 : \mathcal{A}_\infty \to \Lambda(\mathcal{P}; \mathcal{D})$ is 2-1 on elements in $\mathcal{A}_\infty$ where only two generators appear infinitely many times.*

*Proof.* Circles are in bijective correspondence with lunes they are inscribed in when introduced to the packing generationally. Tangency points are contained in two lunes after the first generation in which they are introduced, so there are exactly two choices for a circle to approach the residual point. The next circle must be tangent to the first one and so must be contained the lunes created by adding the first circle. There is exactly one of those lunes that still contains our residual point. Inductively, the remaining circles approaching the residual point are uniquely determined. □

A harder question to answer is if the map $F_1$ is injective with respect to the elements in $\mathcal{A}_\infty$ where at least three generators are repeated infinitely many times. We have the following proposition:

**Proposition 5.** *The Map* $F_1 : \mathcal{A}_\infty \to \Lambda(\mathcal{P};\mathcal{D})$ *is injective on elements in* $\mathcal{A}_\infty$ *that have at least three generators appearing infinitely many times.*

*Proof.* Unlike the case in which the residual point is at the intersection of two circles in the packing, the case where the point is on the boundary of exactly one or no circles cannot be contained in two lunes since the intersection of two lunes at most consists of one point, which is at the intersection of two circles. Because the circles used to approach the residual point are determined by the lunes containing that residual point, there is exactly one sequence of circles in the packing that approaches our point. Thus, one cannot construct two distinct paths arriving at the same point from lunes without breaking our assumption that the point we arrive at is not at the intersection of two circles in the packing. □

## 3. Arriving at Unbounded Packings from Residual Points

For this section, fix an Apollonian packing $\mathcal{P}$ and one of its Descartes configurations $\mathcal{D}$. Let $\Lambda(\mathcal{P};\mathcal{D})$ denote the set of residual points of $\mathcal{P}$ (specifying $\mathcal{D}$ will help us define an action of $\mathcal{A}$ on these residual points). To each residual point $z_0$ in $\Lambda(\mathcal{P};\mathcal{D})$, we may associate a Möbius transformation $\phi_{z_0} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined as follows

$$\phi_{z_0}(z) = \frac{1}{z - z_0} \quad \text{with} \quad \phi_{z_0}(z_0) = \infty$$

As in section 1, we define $\mathcal{U}$ as follows:

$$\mathcal{U} = \{(\mathcal{P}', \mathcal{D}') \mid \mathcal{P}' \text{ is an unbounded packing, } \mathcal{D}' \text{ is a Descartes configuration of } \mathcal{P}'\}/\sim$$

where $\sim$ denotes similarity. Given a residual point $z_0$, we may now define the map $F_2 : \Lambda(\mathcal{P},\mathcal{D}) \to \mathcal{U}$ as follows:

$$z_0 \mapsto (\phi_{z_0}(\mathcal{P}), \phi_{z_0}(\mathcal{D}))$$

Thus, $F_2$ maps a residual point $z_0$ to the unbounded packing obtained by moving $\mathcal{P}$ such that $z_0$ goes to $\infty$. This map is bijective tying the study of residual points and the study of unbounded Apollonian packings together.

**Proposition 6.** *The map* $F_2 : \Lambda(\mathcal{P};\mathcal{D}) \to \mathcal{U}$ *is bijective.*

*Proof.* We define a map $G_2 : \mathcal{U} \to \Lambda(P, D)$ having the property that $G_2(F_2(z_0)) = z_0$ and $F_2(G_2(\mathcal{P}', \mathcal{D}')) = (\mathcal{P}', \mathcal{D}')$ for all $z_0$ in $\Lambda(\mathcal{P};\mathcal{D})$ and all $(\mathcal{P}', \mathcal{D}')$ in $\mathcal{U}$.

Let $G_2$ be the map that associates an unbounded packing $(\mathcal{P}';\mathcal{D}')$ to a residual point in $\Lambda(\mathcal{P};\mathcal{D})$ as follows:

$$(\mathcal{P}', \mathcal{D}') \mapsto \phi_{D'}(\infty)$$

where $\phi_{D'}$ is the unique generalized Möbius transformation with the property that $\phi_{D'}(\mathcal{D}') = \mathcal{D}$; here $\mathcal{D}$ is the fixed Descartes configuration of our fixed bounded packing. Such a transformation exists by the fact that there exists a bijection $H_D$ : Möb(2)$\to$ {Set of all Descartes configurations} defined by

$$\phi \mapsto \phi(D) = D'$$

Intuitively speaking, this means that given any fixed Descartes configuration $\mathcal{D}$, one can obtain any other Descartes configuration $\mathcal{D}'$ via an element of Möb(2).

We first show that this map $G_2$ is well-defined, up to the relation of similarity in $\mathcal{U}$. Let $(\mathcal{P}', \mathcal{D}'), (\mathcal{P}'', \mathcal{D}'')$ be any two unbounded packings such that there exists a similarity transformation $\psi$ with $\psi(\mathcal{D}') = \mathcal{D}''$. We must show that the following is true:

$$G_2(\mathcal{P}') = G_2'(\mathcal{P}'') = z_0,$$

we obtain the following identity:

$$\phi_{\mathcal{D}''}(\infty) = \phi_{\mathcal{D}''} \circ \psi(\infty) = \phi_{\mathcal{D}'}(\infty).$$

This follows from $\psi$ being a similarity transformation, which necessarily fixes $\infty$. Thus, $G_2$ is well-defined.

We now show that $G_2$ is indeed the inverse of $F_2$. We first show that $G_2(F_2(z_0)) = z_0$ for all $z_0$ in $\Lambda(\mathcal{P};\mathcal{D})$. Given a $(\mathcal{P}_{z_0}, \mathcal{D}_{z_0})$ such that $F_2(z_0) = (\mathcal{P}_{z_0}, \mathcal{D}_{z_0})$, we may define a generalized Möbius transformation $\phi_{D_{z_0}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with the property that $\phi_{\mathcal{D}_{z_0}}(\mathcal{D}_{z_0}) = (\mathcal{D})$. It follows that $\phi_{D_{z_0}} \circ \phi_{z_0}$ fixes the Descartes configuration $\mathcal{D}$ and is therefore the identity map. Indeed, this shows that $G_2(F_2(z_0)) = z_0$

Finally, we show that $F_2(G_2(\mathcal{P}', \mathcal{D}')) = (\mathcal{P}', \mathcal{D}')$ for all unbounded packings in $\mathcal{U}$. We show that any $(\mathcal{P}', \mathcal{D}')$ is similar to its image $F_2(G_2(\mathcal{P}', \mathcal{D}'))$. It would therefore suffice to show that the map $\phi_{z_0} \circ \phi_{D'}$ is itself, a similarity (where $\phi_{D'}$ is defined as above). This result, however follows from the fact that $\phi_{z_0} \circ \phi_{D'}$ fixes $\infty$. Thus, up to the equivalence of similarity transformations, the map $F_2$ is bijective.

□

## 4. Unbounded Packings and Descartes Quadruples

Letting $\mathcal{U}$ be defined as before, we define the map $F_3 : \mathcal{U} \to \mathbb{P}D$, where $\mathbb{P}D$ denotes the projective Descartes variety, the set of non-zero real Descartes quadruples $\mathbf{q} \in \mathbb{R}^4$ up to scaling by some $\lambda \in \mathbb{R}^*$ by:

$$F_3(\mathcal{P}_u, \mathcal{D}) = \mathbf{q},$$

where $\mathbf{q}$ denotes the Descartes quadruple that is the curvatures of the Descartes configuration $\mathcal{D}$. It is not hard to see that this map is injective. Elements of $\mathcal{U}$ are after all, defined up to similarity, leaving the curvatures of the circles from the packing fixed or dilating them by some $\lambda \in \mathbb{R}^*$. Since an element of $\mathbb{P}D$ is unique up to scaling by a real number, this relation is preserved. Along with the fact that two dissimilar Apollonian packings cannot have the same Descartes quadruple, this fact guarantees injectivity.

### 4.1. The Reduction Algorithm and a Correspondence between $\mathcal{A}_\infty$ and $\mathcal{U}$.

Given the composition of maps $F_2 \circ F_1$, one may ask whether there is a a direct way to obtain unbounded packings in $\mathcal{U}$ from infinite sequences in $\mathcal{A}_\infty$. It can be shown that given a quadruple $\mathbf{q}$ in a bounded packing $\mathcal{P}$, there exists a unique sequence of generators of $\mathcal{A}$ that reduce that quadruple to a root quadruple. From the properties of our maps, we have the following proposition:

**Proposition 7.** *The element $a_\infty$ in $\mathcal{A}_\infty$ is the reduction algorithm for a unique $\boldsymbol{q}$ in $\mathbb{P}D$; in particular, $F_3 \circ F_2 \circ F_1(a_\infty) = \boldsymbol{q}$. This Descartes quadruple corresponds, via $F_3$, to a unique unbounded packing with marked quadruple up to similarity in $\mathcal{U}$.*

*Proof.* Recall that we have the following sequence of maps:

$$\mathcal{A}_\infty \xrightarrow{F_1} \Lambda(\mathcal{P};\mathcal{D}) \xrightarrow{F_2} \mathcal{U} \xrightarrow{F_3} \mathbb{P}D$$

Given some element $a_\infty$ in $\mathcal{A}_\infty$, we can associate to it a sequence of circles $C_1$, $C_2$,... associated with $S_{i_1}S_{i_2}$... approaching a residual point $z_0$ of the packing $\mathcal{P}$. There is a Möbius transformation $\phi_{z_0}$ taking $(\mathcal{P},\mathcal{D})$ to $F_2(z_0)$. Under $\phi_{z_0}$, the residual point $z_0$ is moved to $\infty$. The map $\phi_{z_0}$ also takes the sequence

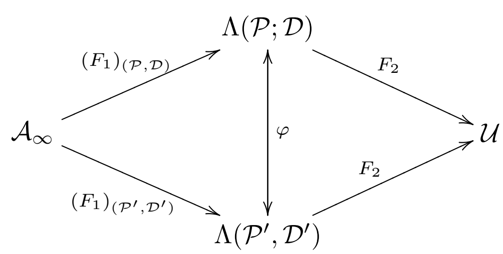
$$C_1, C_2, \ldots \mapsto \phi_{z_0}(C_1), \phi_{z_0}(C_2), \ldots$$

This sequence has to approach the image of $\phi_{z_0}(z_0)$, which as we know is at $\infty$. From this, we note that we have constructed a sequence of generators that approach the point $\infty$ in some unbounded packing in the set $\mathcal{U}$. Since the reduction algorithm for such an unbounded packing approach infinity as well, and since the reduction algorithm for any packing is unique, we conclude that we have arrived at the reduction algorithm for some unbounded packing in $\mathcal{U}$, whose Descartes quadruple will be the image of $F_3 \circ F_2 \circ F_1(a_\infty)$.

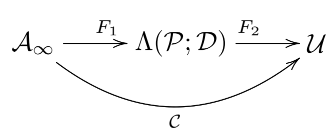
It must also be noted that reduction algorithm in $\mathcal{A}_\infty$ is unique for the Descartes quadruple $F_3 \circ F_2 \circ F_1(a_\infty)$ as long as it does not generate a strip packing. If there exists such a Descarte quadruple in $\mathbb{P}D$ with more than one reduction algorithm, we know from the bijectivity of $F_2 \circ F_1$ that the reduction algorithms corresponding to the image in $\mathbb{P}D$ must be the same.

□

One result of this is that we now know that there must exist some coordinate-free map between elements of $\mathcal{A}_\infty$ and $\mathcal{U}$ that is independent of the Apollonian packing $P$ and Descartes configuration $D$. More specifically, we can associate an element of $\mathcal{A}_\infty$ with the unbounded packing in $\mathcal{U}$ where the sequence in $\mathcal{A}_\infty$ acts as the reduction algorithm on a Descartes quadruple associated with the packing $\mathcal{U}$. This then proves that the following diagram commutes:




And demonstrates a correspondence $\mathcal{C} : \mathcal{A}_\infty \to \mathcal{U}$ independent of some packing $P$ with a Descartes configuration $D$, allowing us to make the following addition to our maps:




Since each unbounded packing corresponds with exactly one reduction algorithm, we claim that this correspondence is bijective on the elements in $\mathcal{A}_\infty$ where at least three generators appear infinitely many times. This can also be demonstrated by noting that $F_1 : \mathcal{A}_\infty \to \Lambda(\mathcal{P};\mathcal{D})$ is bijective on the elements in $\mathcal{A}_\infty$ where at least three generators appear infinitely many times, and that it is precisely the image of those elements under $F_1$ that the also bijective map $F_2 : \Lambda(\mathcal{P};\mathcal{D}) \to \mathcal{U}$ sends to either half-plane or full-plane packings. The composition of these bijective maps is also bijective.

## 5. Action of the Apollonian Group and $\mathcal{A}$-equivariance

There is a right regular action of the Apollonian group $\mathcal{A}$ on the set $\mathcal{A}_\infty$. More precisely, given any Apollonian sequence $...S_{i_3}S_{i_2}S_{i_1}$ in $\mathcal{A}_\infty$, and any $a$ in $\mathcal{A}$ , we let $a(...S_{i_3}S_{i_2}S_{i_1}) = ...S_{i_3}S_{i_2}S_{i_1}a^{-1}$.

Fix an Apollonian packing $\mathcal{P}$ and one of its Descartes configurations $\mathcal{D}$. To each element of $a$ of $\mathcal{A}$, there now corresponds a unique Möbius transformation $\phi_a$. In terms of the generators $S_i$ for $i \in \{1, 2, 3, 4\}$ of the group, the corresponding Möbius transformation is inversion with respect to the dual, i.e. an inversion in the circle passing through the three tangency points of the three circles of $\mathcal{D}$ not including the $i$th.

Letting $z_0 \in \Lambda(\mathcal{P}; \mathcal{D})$ be a residual point, we see that under the identification of $\mathcal{A}$ with a group of Möbius transformations having the above properties, $\mathcal{A}$ will act on $\Lambda(\mathcal{P}; \mathcal{D})$ as follows:

$$a(z_0) = \phi_{a^{-1}}(z_0)$$

Whether or not this satisfies the properties of a group action is not immediately clear.

**Proposition 8.** *The action of $\mathcal{A}$ on $\Lambda(\mathcal{P}; \mathcal{D})$ defined by $a(z_0) = \phi_{a^{-1}}(z_0)$ is a group action.*

*Proof.* Given any $a, b$ in $\mathcal{A}$ and any $z_0$ in $\Lambda(\mathcal{P}; \mathcal{D})$, we show that the defined action satisfies $ab(z_0) = a(b(z_0))$. It will suffice to show the following:

$$ab(z_0) = \phi_{b^{-1}a^{-1}}(z_0)$$
$$= \phi_{a^{-1}} \circ \phi_{b^{-1}}(z_0)$$

This will follow from the fact that given a Descartes quadruple $\mathcal{D}$,

$$\phi_{a^{-1}} \circ \phi_{b^{-1}}(\mathcal{D}) = \phi_{a^{-1}}(b^{-1}\mathcal{D}) = b^{-1}(\phi_{a^{-1}}(\mathcal{D})) = b^{-1}a^{-1}(\mathcal{D})$$
$$= \phi_{b^{-1}a^{-1}}(\mathcal{D})$$

And thus, $\phi_{a^{-1}} \circ \phi_{b^{-1}} = \phi_{b^{-1}a^{-1}}$ □

One can then ask how the group $\mathcal{A}$ acts on the set $\mathcal{U}$. An $a$ in $\mathcal{A}$ takes an unbounded packing to itself, an action which corresponds to the group's aforementioned action on $\Lambda(\mathcal{P}; \mathcal{D})$, but in the process changes the marked configuration $\mathcal{D}'$ to $a\mathcal{D}'$, i.e.

$$a(\mathcal{P}', \mathcal{D}') = (\mathcal{P}', a\mathcal{D}').$$

Finally, we see that $\mathcal{A}$ acts on $\mathbb{P}D$ via matrix multiplication as described in section 1. It permutes theDescartes quadruples $\mathbf{q}$ associated to an unbounded packing via the map $F_3$.

One may ask whether the maps $F_1$, $F_2$ and $F_3$ are $\mathcal{A}$-equivariant, i.e, whether or not they preserve the group action.

**Proposition 9.** *The map $F_1$ is $\mathcal{A}$-equivariant.*

*Proof.* Let $...S_{i_3}S_{i_2}S_{i_1}$ be a sequence in $\mathcal{A}_\infty$. Given a packing $(\mathcal{P}, \mathcal{D})$ the circle associated to some finite sequence of generators $S_{i_n}...S_{i_1}$ is $S_{i_n}...S_{i_1}(\mathcal{D}) \setminus S_{i_n-1}...S_{i_1}(\mathcal{D})$. Then,

$$F_1(a(...S_{i_2}S_{i_1})) = F_1(...S_{i_2}S_{i_1}a^{-1})$$
$$= \lim_{n\to\infty} S_{i_n}...S_{i_1}a^{-1}(\mathcal{D}) \setminus S_{i_n-1}...S_{i_1}a^{-1}(\mathcal{D})$$
$$= \lim_{n\to\infty} S_{i_n}...S_{i_1}\phi_{a^{-1}}(\mathcal{D}) \setminus S_{i_n-1}...S_{i_1}\phi_{a^{-1}}(\mathcal{D})$$
$$= \phi_{a^{-1}}( \lim_{n\to\infty} S_{i_n}...S_{i_1}\mathcal{D} \setminus S_{i_n-1}...S_{i_1}(\mathcal{D})$$
$$= a(F_1(...S_{i_2}S_{i_1}))$$

The fourth equality is due to the fact that if $a$ is in $A$, $\mathcal{D}$ is a fixed Descartes quadruple and if $\phi$ is a Mobius transformation, then $\phi(aD) = a\phi(D)$ and also because of the bijectivity and continuity of $\phi$. Thus, we see that $F_1$ is indeed $A$-equivariant. □

**Proposition 10.** *The map $F_2$ is $\mathcal{A}$-equivariant.*

*Proof.* Given a $z_0$ in $\Lambda(\mathcal{P}; \mathcal{D})$ we show that $F_2(az_0) = aF_2(z_0)$. For notational purposes, set

$$\psi_{z_0}(z) = \frac{1}{z - z_0}$$

Since

$$F_2(az_0) = F_2(\phi_{a^{-1}}(z_0) = (\psi_{\phi_{a^{-1}}(z_0)}(\mathcal{P}), \psi_{\phi_{a^{-1}}(z_0)}(\mathcal{D}))$$

and

$$aF_2(z_0) = a(\psi_{z_0}(\mathcal{P}, \mathcal{D}) = (\psi_{z_0}(\mathcal{P}), a\psi_{z_0}(\mathcal{D})),$$

it will be enough to show that $\psi_{\phi_{a^{-1}}(z_0)}(\mathcal{D}) \sim a\psi_{z_0}(\mathcal{D})$, where $\sim$ denotes similarity.

We now set a generalized Möbius transformation $\phi'_a$ s.t. $a(\psi_{z_0}(\mathcal{D})) = \phi'_a \circ \psi_{z_0}(\mathcal{D})$. Since similarity transformations are exactly the Möbius transformations that preserve $\infty$, the theorem will follow upon the verification that:

$$\infty = \phi'_a \circ \psi_{z_0} \circ \psi^{-1}_{\phi_{a^{-1}}(z_0)}(\infty)$$

but we know that

$$\phi'_a \circ \psi_{z_0} \circ \psi^{-1}_{\phi_{a^{-1}}(z_0)}(\infty) = \phi'_a \circ \psi_{z_0} \circ \phi_{a^{-1}}(z_0)$$

so now it is enough to show that

$$\phi'_a \circ \psi_{z_0} \circ \phi_{a^{-1}} = \psi_{z_0}$$

This follows naturally from the definitions of the above transformations and how both functions act on $\mathcal{D}$. Thus, $\psi_{\phi_{a^{-1}}(z_0)}(a\mathcal{D}) \sim a\psi_{z_0}(\mathcal{D})$ and so, $F_2$ is indeed $A$-equivariant. $\square$

It is an easy consequence of the definition of $F_3 : \mathcal{U} \to \mathbb{P}D$ that this map is $\mathcal{A}$-equivariant as well. In short, $F_3$ associates a quadruple to a Descartes configuration of a given unbounded packing; permuting between Descartes configurations of a packing is equivalent to permuting between their associated Descartes quadruples.

## 6. The image of $\overline{\mathcal{A}}$ in $\mathcal{U}$; Self-similar unbounded packings

Suppose we take the following representation for an element $\mathbf{a}$ in $\overline{\mathcal{A}}$:

$$\mathbf{a} = ...aaab$$

Where $a$ is some finite sequence of the four generating elements of $\mathcal{A}$, and $b$ is a similar finite sequence of generators. It is clear that there exist two elements in $\overline{\mathcal{A}}$ that are fixed by $c = b^{-1}ab$:

$$...aaab \quad \text{and} \quad ...a^{-1}a^{-1}a^{-1}b$$

Following the image of these elements in $\overline{\mathcal{A}}$ through $F_3 \circ F_2 \circ F_1$, we see that these elements correspond to the Descartes quadruples $\mathbf{q}$ of some unbounded packings. Now, by $\mathcal{A}$-equivariance of the associated maps, $cq = q \in \mathbb{P}D$. Thus, $c\mathbf{q} = \lambda\mathbf{q}$ in the set $\mathbb{R}^4$, for some $\lambda$ in $\mathbb{R}^*$. In other words, $\mathbf{q}$ is an eigenvector of $c$ as an element of $\mathcal{A} \subseteq GL_4(\mathbb{Z})$. It is clear that by relating curvature values in the packing via some constant $\lambda$, this unbounded packing generated by $\mathbf{q}$ is self-similar.

We now prove the following converse to the above result.

**Theorem 11.** *Let $(\mathcal{P}', \mathcal{D}')$ denote a self-similar unbounded packing in $\mathcal{U}$. Then, $(\mathcal{P}', \mathcal{D}')$ is the image of some element* $\mathbf{a}$ *in $\overline{A}$ under the map $F_2 \circ F_1$.*

*Proof.* Let $(\mathcal{P}', \mathcal{D}')$ be a self-similar unbounded packing. Notice that the self-similarity of the packing requires the self-similarity to be in the Apollonian group. The operations of self-similarity all correspond to multiplication of the associated Descartes quadruple $\mathbf{q}$ by some fixed scalar $\lambda$. Since $\mathbf{q}$ is taken to be an element of the projective Descartes variety $\mathbb{P}D$, the following holds:

$$aq = q \in \mathbb{P}D,$$

where $a$ is the fixed scalar operating on $\mathbf{q}$ via the self-similarity. Thus, $\mathbf{q}$ may be viewed as the eigenvector in $\mathbb{R}^4$ of $a$ with eigenvalue $\lambda$, which corresponds to $a$ leaving the packing associated to $\mathbf{q}$ fixed. In light of the $\mathcal{A}$-equivariance of the map $F_3 \circ F_2 \circ F_1$ and the surjectivity of $F_2 \circ F_1$, we conclude that this operator corresponds to an element of the Apollonian group leaving $a_\infty$ fixed in $\mathcal{A}_\infty$ where $F_3 \circ F_2 \circ F_1(a_\infty) = \mathbf{q}$. Such an action by an element of the Apollonian group can only exist, however, for those $a_\infty$ belonging to $\overline{\mathcal{A}}$. Otherwise, the $a$ in $\mathcal{A}$ fixing $a_\infty$ would have to be of infinite length, which is impossible. It follows from this that $(P', D')$ is the image of some $a_\infty = \mathbf{a}$ of $\overline{\mathcal{A}}$ with eigenvector $\mathbf{q}$ under $F_2 \circ F_1$. □

Another result of this is with regards to the eigenvectors of elements in $\overline{\mathcal{A}}$. Recall that the sequence of generators $c = b^{-1}ab$ in $\mathcal{A}$ fixes exactly two elements:

$$...aaab \quad \text{or} \quad ...a^{-1}a^{-1}a^{-1}b$$

So that the image of $\mathbf{a}$ in the projective variety $\mathbb{P}D$ is some eigenvector $\mathbf{q}$ that is fixed by $c$. Since the eigenvector corresponds to a self-similar packing, and since the curvatures decrease with respect to the action of $a$ on a Descartes quadruple of the packing, we know that this implies that there exists at least one eigenvalue $\lambda$ such that

$$c\,\mathbf{q} = \lambda \mathbf{q} \quad \text{and} \quad \lambda \in (0,1)$$

In fact, we have the following claim:

**Proposition 12.** *If $\mathbf{w}$ is an eigenvector of $c$ as above with eigenvalue $\nu$, then $\mathbf{w} = F_3 \circ F_2 \circ F_1(...aaab)$ or $\mathbf{w} \notin \mathbb{P}D(\mathbb{R})$ or $\nu \notin (0,1)$.*

*Proof.* Given an eigenvector $\mathbf{w}$ in $\mathbb{P}D(\mathbb{R})$ with eigenvalue $\nu \in (0,1)$, we know that $\mathbf{w}$ corresponds to an Apollonian packing such that

$$c\,\mathbf{w} = \nu\,\mathbf{w}$$

where the curvatures are decreasing at each action of $c$ on $\mathbf{w}$. This tells us that the packing follows the reduction algorithm $...aaab$. Since the reduction algorithm uniquely determines each unbounded packing with a marked configuration, and each Apollonian packing with a Descartes configuration corresponding to $\mathbf{w}$ has the same algorithm, we can conclude that $\mathbf{w} = \mathbf{q}$.

□

## 7. Closing Remarks

During the course of this four-week program, our group managed to successfully construct a correspondence between residual points and unbounded packings. We also arrived at a method with which one can arrive at residual points. We also successfully found how to construct any self-similar unbounded packing by finding one of its quadruples as an eigenvector of an Apollonian group member. Among other things however, we were unable to discover explicit conditions a quadruple can satisfy in order for it to correspond to an unbounded packing. This would have answered one of our initial motivating questions: how will the shape of a packing change depending on the placement of the center of a circle $C_3$ lying tangent to 2 given tangent circles $C_1$ and $C_2$ with curvatures 1 and $\alpha$? Nevertheless, the results in this paper point towards various surprising and potentially deep consequences. For instance there exists a natural topology on the set $\mathcal{A}_\infty$, which is preserved via the (continuous) maps $F_1, F_2$ and $F_3$. Such a topological approach to the topic gives us an especially elegant description and consequence of the action of $\mathcal{A}$ on the above sets; it may also be used for other applications along the lines of Diophantine approximation. Finally, one can potentially say more about the elements of $\overline{\mathcal{A}} \subset \mathcal{A}$. Since all these elements correspond, via the above maps, to eigenvectors of certain $4 \times 4$ matrices, one may look at the eigenvalues acting on them. These eigenvalues, zeros of quartic polynomials bring to the forefront the following potential correspondence:

$$\overline{\mathcal{A}} \leftrightarrow \mathbb{P}D(\mathcal{Q}(\mu))$$

between the elements in $\overline{\mathcal{A}}$ and the image of such elements in the projective Descartes variety with the added condition of being in the extended quartic field of $\mathbb{Q}$. This correspondence alone may hold interesting number-theoretic applications to integral Apollonian circle packings.

## 8. Appendix I: Tables

Included below is a table with the first 13 unique Apollonian group elements ordered by length up to cyclic rotation of the generators and permutation of the indices by the Permutation group $S_4$. By our discussion in section 5 we have shown that each such element has a eigenvector, which is the Descartes quadruple of a self-similar unbounded packing. The type of packing generated by the Apollonian group element has been noted in the table, along with the Descartes quadruple eigenvector and the eigenvalue, which is the self-similarity ratio . Both the eigenvalues and the corresponding eigenvectors belong to a shared quartic extension of $\mathbb{Q}$ and have been approximated for the table.

| Type | Length | Apollonian Group Element | Eigenvalue | Eigenvector |
|---|---|---|---|---|
| Strip | 2 | $S_2S_1$ | 1 | [1,1,0,0] |
| Half-plane | 3 | $S_3S_2S_1$ | 0.055728092 | [6.854101945, 2.617033985,1,0] |
| Half-plane | 4 | $S_2S_3S_2S_1$ | .02943726 | [5.828427077, 2, 1, 0] |
| Full-plane | 4 | $S_4S_3S_2S_1$ | .01433430 | [24.13891204, 8.352410442, 2.890053609,1] |
| Half-plane | 5 | $S_2S_1S_3S_2S_1$ | .01515499 | [6.561552896, 2.438447188, 1, 0] |
| Full-plane | 5 | $S_4S_1S_3S_2S_1$ | .00684964 | [48.99771, 18.29081102, 6.208048046, 1] |
| Full-plane | 6 | $S_4S_2S_1S_3S_2S_1$ | .0022111 | [149.4152903, 55.95385998, 21.26666935, 1] |
| Half-plane | 6 | $S_2S_3S_1S_3S_2S_1$ | .00515475 | [7.464101628, 3, 1, 0] |
| Full-plane | 6 | $S_4S_3S_1S_3S_2S_1$ | .0028021 | [117.3983197, 46.77931940, 14.76462982, 1] |
| Half-plane | 6 | $S_2S_1S_2S_3S_2S_1$ | .01020514 | [6, 2.101020513, 1, 0] |
| Full-plane | 6 | $S_4S_1S_2S_3S_2S_1$ | .0031057 | [107.6656464, 35.88854801, 17.94427419, 1] |
| Full-plane | 6 | $S_4S_3S_2S_3S_2S_1$ | .0038178 | [88.19916404, 27.00126707, 16.26670899, 1] |
| Full-plane | 6 | $S_2S_4S_2S_3S_2S_1$ | .0038611 | [40.63102908, 13.01970752, 6.374248457, 1] |

We hope to expand this table with future research and provide a counting formula to predict the number of unique packings occur given length $n$. Currently, we have a formula for $n$ not divisible by 2 or 3:

$$U(n) = \frac{1}{24n}\sum_{q|n}(3^q - 3) \text{ such that all } q \text{ are proper divisors of } n$$

We have run into difficulties accounting for the symmetries that occur when $n$ is divisible by 2 or 3. For future research, we will also be normalizing the eigenvectors so that the sum of the entries is equal to 1.

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