# APOLLONIAN SUMS

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Abstract. Given any collection of circles $C_i$ with radii $r_i$ in an Apollonian circle packing, we call the function $F(s) = \sum r_i^s$ an Apollonian sum. In this paper, we study the decay of radii in a packing by studying the convergence and special values of Apollonian sums. We first examine sums over sequences of tangent circles inside a packing. We give examples of such sequences where the radii decay quadratically and exponentially, and then construct sequences whose sums converge for $s > r$ and diverge for $s \leq r$, where r is any real number in $[0, 1/2]$. We also examine the collection of all circles tangent to a fixed circle–in this case the Apollonian sum is related to an interesting Dirichlet series. Finally we study generations of circles in the construction of a packing. We find an explicit formula for the Apollonian sum over generation $n$ when $s = -1$, i.e. for the sum of curvatures in generation $n$. We extend this formula to the case of a four-colored packing.

## 1. Introduction

A Descartes configuration is a collection of four mutually tangent circles; that is, any two circles touch tangentially and the points of tangency are distinct. We consider these configurations as lying in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and extend the definition to "generalized circles," meaning circles and lines. We may orient the circles to make their interiors disjoint. A Descartes configuration can be extended to an Apollonian packing by filling in additional circles in the exterior of the configuration. Each new circle fits in a lune between three existing circles, and is tangent to all three. An Apollonian packing is the union of all such circles.

Let us label each circle by its curvature, $\epsilon$. Descartes discovered the relation $(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)^2 = 2(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2)$, where $\epsilon_i$ are the curvatures of four circles in a Descartes configuration. The curvature of a circle is taken to be negative if its interior contains $\infty$, and zero if it is a line. Given three mutually tangent circles in a packing with curvatures $\epsilon_1$,$\epsilon_2$,$\epsilon_3$, there are two possible circles tangent to these three circles. If the curvatures of the two possible circles are $\epsilon_4$ and $\epsilon_4'$, Descartes' equation gives the useful result: $\epsilon_4' = 2(\epsilon_1 + \epsilon_2 + \epsilon_3) - \epsilon_4$. It follows that if the four initial curvatures of a Descartes configuration are integers, then all the curvatures in the packing are integers. Such a packing is called integral.

The collection of curvatures $\epsilon_i$ in a packing (with multiplicity) is an important object of study. There are clearly countably many such curvatures. It seems most natural to index them by a certain Coxeter group called the Apollonian group,

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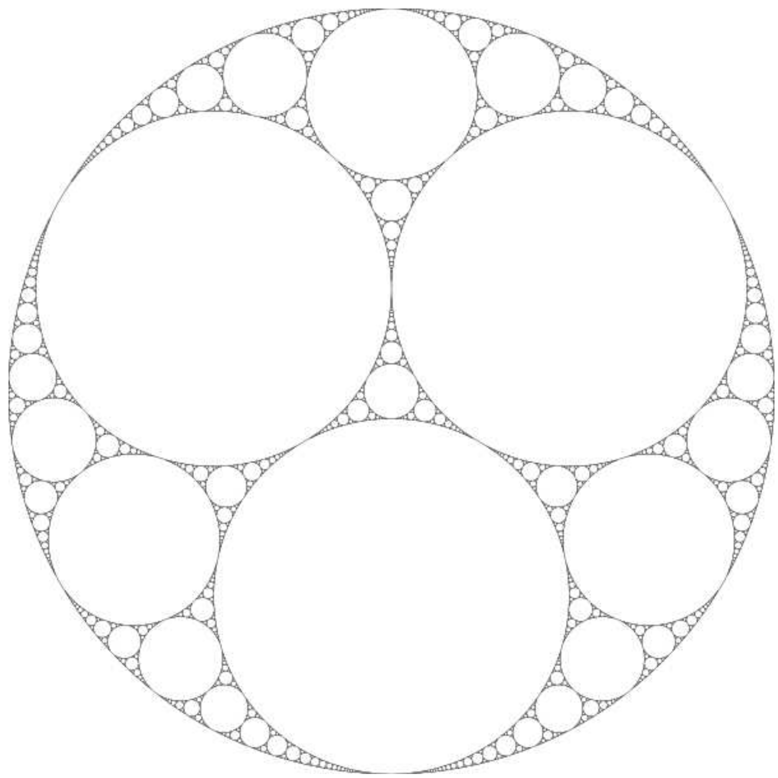

Figure 1. An Apollonian packing

which is discussed below, rather than by the integers. The collection of curvatures in an integral Apollonian packing has interesting arithmetic properties; for example, it is conjectured [6] that all sufficiently large integers satisfying certain congruence conditions appear. In this paper we study the collection of curvatures analytically, without assuming integrality, and hope that the analysis sheds some light on the arithmetic. Most of our results apply to all packings, not just integral ones.

Some theorems of Boyd are especially relevant to us. In [2] he studies the number $N_P(x)$ of circles with curvature at most $x$ in a packing $P$, and proves that $\lim_{n\to\infty} \frac{\log N_P(x)}{\log x} = \delta$ (Kontorovich and Oh have recently improved upon this result, finding the asymptotic formula $N_P(x) \sim c \cdot x^{\delta}$ [8]). In these formulas $\delta$ is the Hausdorff dimension of the residual set of a packing, i.e. the complement of the interiors of all the circles. Since any two Apollonian packings are equivalent under some Mobius transformation of $\hat{\mathbb{C}}$, the residual dimension does not depend on the choice of packing. It is valued at approximately 1.3058 [10], and thought not to have an elementary closed form. Boyd shows that the residual dimension $\delta$ also appears as the minimal exponent of what we will call an Apollonian sum. Define $F_P(s) := \sum \epsilon_i^{-s} = \sum r_i^s$ where the $\epsilon_i$ are the curvatures of all circles in the packing, or equivalently $r_i$ are all the radii. Then the Apollonian sum $F_P(s)$ converges for $s > \delta$ and diverges for $s < \delta$ [1]. This sum has other properties worth noting: $\pi \cdot F_p(2)$ is simply twice the area of the outer circle, since the residual set has measure zero. The divergence of $F_P(1)$ can be proven by elementary Euclidean geometry.

In this paper, we study the Apollonian sum above by slicing and dicing it, looking at $\sum \epsilon_i^{-s}$ where the curvatures $\epsilon_i$ range over particular subsets of circles in a fixed packing $P$. For each Apollonian sum we study, we obtain a minimal exponent between zero and $\delta$. In many cases, we are able to find explicit formulas relating the Apollonian sum to the Riemann zeta function, Hurwitz zeta functions, or Dirichlet series. The special values of these sums frequently have geometric significance; e.g. $s = 1$ is a sum of radii while $s = 2$ is a sum of areas.

In Section 3 we examine sums over sequences of tangent circles within a packing. We study one explicit sequence where the curvatures grow quadratically, so the Apollonian sum, which can be written in terms of Hurwitz zeta functions, has minimal exponent 1/2. We also construct a sequence where the curvatures grow exponentially, producing a minimal exponent of zero, and then show that every real number in $[0, 1/2]$ is obtained as the minimal exponent of some sequence. Section 4 focuses on the Apollonian sum over all circles tangent to a fixed circle in a packing. Here the minimal exponent is one, and, in the special case where the fixed circle is a line, the Apollonian sum is closely related to an interesting Dirichlet series.

Section 5 is somewhat different. It is natural to build Apollonian packings generation-by-generation, with each new generation of circles filling in the interstices of the last. The Apollonian sum over a generation is a finite sum, so convergence is not an issue. Instead, we attempt to understand the relationship between generations. The special value $s = -1$ corresponds to a sum of curvatures, and the Descartes formula gives a recursive relation between generations. Mallows [9] has studied generating functions for curvature sums of this type in Apollonian sphere packings. We give an explicit formula for the curvature sum in generation $n$ of a circle packing. We also partition the circles in a packing into four classes, which we call colors, and give formulas for curvature sums by generation and color. Further study of the Apollonian sums over generations and colors could lead to interesting geometric results.

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## 2. Background

**The Apollonian Group**: The Apollonian group $\mathcal{A}$ is a Coxeter group on 4 generators with the presentation $< S_1, S_2, S_3, S_4 | \, S_i^2 = e >$. We use a particular faithful representation $\mathcal{A} \to GL(4, \mathbb{Z})$, given by:

$$S_1 \mapsto \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$S_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, S_4 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

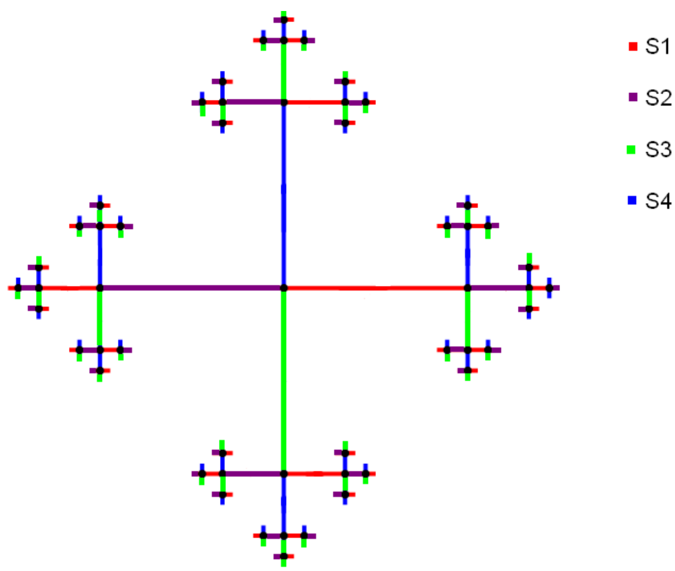


FIGURE 2. Cayley graph of the Apollonian group.

In this representation, $\mathcal{A}$ acts on quadruples $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)^T$ satisfying Descartes' equation. The action of $\mathcal{A}$ takes Descartes configurations to Descartes configurations within the same circle packing. For example, $S_4 \in \mathcal{A}$ takes the Descartes quadruple $\mathbf{v} = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)^T$ to the Descartes quadruple $S_1 \cdot \mathbf{v} = (\epsilon_1, \epsilon_2, \epsilon_3, 2(\epsilon_1 + \epsilon_2 + \epsilon_3) - \epsilon_4)^T$. The circle with curvature $2(\epsilon_1 + \epsilon_2 + \epsilon_3) - \epsilon_4$ is the second circle tangent to the mutually tangent circles with curvatures $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$.

If we fix an initial Descartes configuration in a packing $P$, we may label every circle by the shortest sequence of generators of $\mathcal{A}$ required to reach it. This is almost a one-to-one correspondence between $\mathcal{A}$ and circles in the packing. The only ambiguity is that all four circles of the initial configuration should be labelled with the identity $e \in \mathcal{A}$. We will sometimes abuse notation slightly and refer to subsets of $\mathcal{A}$ and sets of circles in $P$ interchangeably.

**Apollonian Sums**: Let $P$ be an Apollonian packing and $C$ be a collection of circles in $P$. Then the Apollonian sum over $C$ is defined as $F_{C,P}(s) = \sum r_i^s$ where the $r_i$ are the radii of circles in $C$. We may also assume that an initial Descartes quadruple is fixed in $P$ and use the correspondence above to define sums over subsets $\mathcal{B} \subset \mathcal{A}$. We will only be interested in Apollonian sums where the summed circles lie in a bounded region of the complex plane; otherwise the sum will be divergent for all $s \geq 0$.

**Minimal Exponent**: The minimal exponent of an Apollonian sum $F_{C,P}$ is defined as $\inf\{\sigma > 0 : F_{C,P}(\sigma) < \infty\}$.

We note that any two Descartes configurations (and hence any two packings) are equivalent under a Mobius transformation. As long as the circles remain in a bounded region of the complex plane, the minimal exponent of a sum is invariant under Mobius transformations (since Mobius transformations are Lipschitz away from the point which maps to infinity). Thus we have the following:

**Lemma 2.1.** *Given a subset $\mathcal{B} \subset \mathcal{A}$, the minimal exponent of the sum over $\mathcal{B}$ is independent of the choice of packing and initial configuration.*

**Hurwitz Zeta Function**: The Hurwitz zeta function is a generalization of the Riemann zeta function and is given by $\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}$ when $\Re(s) > 1$. When $\alpha$ is rational the Hurwitz zeta function may be written as a linear combination Dirichlet L-series. In general the Hurwitz zeta function satisfies a functional equation and has meromorphic continuation in $s$ to the whole complex plane. As a function of $\alpha$ where $s \neq 1$ is fixed $\zeta(s,\alpha)$ is analytic on $\Re(\alpha) > 0$ [7].

## 3. Sums Over Paths

In this section we take Apollonian sums over sequences of tangent circles inside a packing. We may identify these with sequences of generators in the Apollonian group. Let $\mathcal{A}_\infty$ be the set of all infinite sequences where the entries are the generators of $\mathcal{A}$ and no two adjacent entries are the same. Any truncation of the sequence gives an element of $\mathcal{A}$.

A sequence $\mathbf{x} = \ldots s_{i_3} s_{i_2} s_{i_1} \in \mathcal{A}_\infty$ applied to an initial Descartes configuration gives rise to a sequence of circles $c_i$ with radii $r_i$. Lemma 2.1 implies that the minimal exponent of the sequence is independent of the initial configuration. It follows that the minimal exponent depends only on the tail of $\mathbf{x} \in \mathcal{A}_\infty$ not the first finitely many terms.

We begin with sums over the sequences in $\mathcal{A}_\infty$ generated by 2 of the $S_i$ matrices. We find an explicit formula for all such sums, which clearly demonstrates that the minimal exponent is $\frac{1}{2}$, and which can be put in terms of an infinite series of Hurwitz zeta functions. We note that the radii of these sums grow quadratically, as opposed to exponentially, which makes them the slowest growing sums in any Apollonian packing. By reordering Descartes quadruples we only need consider sums of the form $\ldots S_1 S_2 S_1 S_2$.

**Proposition 3.1.** *The sum over the sequence $\mathbf{x} = \ldots S_1 S_2 S_1 S_2$ is given by $F_{\mathbf{x},p}(s) = (c+d)^{-s} \sum_{n=1}^{\infty} ((n + \frac{a-b+c+d}{2(c+d)})^2 - \frac{cd}{(c+d)^2})^{-s}$ where $(a,b,c,d)$ is the initial quadruple and $c, d$ correspond to the fixed circles.*

*Proof.* We begin by computing $(S_1S_2)^n$ which is given by:

$$\begin{pmatrix} 2n+1 & -2n & 4n^2+2n & 4n^2+2n \\ 2n & 1-2n & 4n^2-2n & 4n^2-2n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Applying this transformation to the Descartes quadruple $(a, b, c, d)$ gives

$$(4n^2(c+d)+2n(a-b+c+d)+a, 4n^2(c+d)+2n(a-b-c-d)+b, c, d). \tag{2}$$

Notice that the first two entries are quadratic in $n$ with discriminant $16cd$ by the Descartes equation. Summing over $n$ gives

(3)

$$F_{\mathbf{x},p}(s) = \sum_{n=1}^{\infty}(4(c+d)(n+\frac{a-b+c+d}{4(c+d)})^2-\frac{cd}{c+d})^{-s}+(4(c+d)(n+\frac{a-b-c-d}{4(c+d)})^2-\frac{cd}{c+d})^{-s}.$$

Simplifying we obtain the final result. □

**Corollary 3.2.** *Sequences on 2 generators always have minimal exponent* $\frac{1}{2}$, *unless the initial Descartes quadruple is of the form* $(1, 1, 0, 0)$. *In fact, this shows that all sequences on 2 generators lie in a bounded region of the complex plane, except for this case.*

**Corollary 3.3.** *When one of the fixed circles is zero, i.e. the quadruple contains a line, the sum simplifies to* $c^{-s}\zeta(2s, \frac{a-b+3c}{2c})$. *If the initial quadruple is* $(1, 0, 1, 0)$, *the sum is* $\zeta(2s) - 1$.

**Corollary 3.4.** *By performing binomial expansion on the sum in Proposition 3.2 we get an alternative form*

$$F_{\mathbf{x},p}(s) = (c+d)^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}(\frac{-cd}{(c+d)^2})^k\zeta(2s+2k, \frac{3}{2}+\frac{a-b}{2(c+d)}) \tag{4}$$

**Corollary 3.5.** *By partial fraction decomposition the sum of the radii is*

(5)

$$F_{\mathbf{x},p}(1) = \frac{1}{2\sqrt{cd}}\sum_{n=1}^{\infty}(n+\frac{a-b+c+d}{2(c+d)}-\frac{\sqrt{cd}}{c+d})^{-1}-(n+\frac{a-b+c+d}{c+d}+\frac{\sqrt{cd}}{c+d})^{-1}.$$

*The sum of the areas is*

$$\pi F_{\mathbf{x},p}(2) = \frac{\pi}{4cd}\sum_{n=1}^{\infty}((\alpha\beta_+)^{-1}-(\alpha\beta_-)^{-1}+(\beta_+)^{-2}+(\beta_-)^{-2}. \tag{6}$$

*where* $\alpha = \frac{\sqrt{cd}}{c+d}$, $\beta_\pm = n + \frac{a-b+c+d}{2(c+d)} \pm \frac{\sqrt{cd}}{c+d}$

We now explore more general sequences in $\mathcal{A}_\infty$. We will primarily be concerned with the minimal exponent of these sequences. For computational convenience, we will work below in the circle packing with initial Descartes configuration $(0, 2, 2, 0)$. This packing has long been studied in connection with the theory of Diophantine approximation. We give some background on this connection in the following definitions.

**The Stern-Brocot Tree** Given 2 fractions $\frac{a}{b}, \frac{c}{d}$ we define the mediant of the fractions to be $\frac{a+c}{b+d}$. Consider the fractions $\frac{0}{1}, \frac{1}{0}$. The mediant is $\frac{1}{1}$, the root of the

Stern-Brocot tree. $\frac{1}{1}$ has two children $\frac{1}{2}$ and $\frac{2}{1}$ formed by taking the mediant of $\frac{1}{1}$ with $\frac{0}{1}$ and $\frac{1}{0}$ respectively. In general each node $\frac{p}{q}$ of the tree has two offspring which result from taking the mediant of $\frac{p}{q}$ with one of its two parents (see figure 3). Each fraction $\frac{p}{q}$ in the tree is in lowest possible terms, i.e. $p, q$ are always relatively prime. In fact the Stern-Brocot tree and positive rational numbers are in one-to-one correspondence. There is also a deep connection between the Stern-Brocot tree and continued fractions. If $[a_0; a_1, \ldots, a_n]$ is the continued fraction representation of a rational number in the tree then its two offspring are $[a_0; a_1, \ldots, a_n + 1]$ and $[a_0; a_1, \ldots, a_n - 1, 2]$. Given any irrational number $x$ there is a unique branch in the Stern-Brocot tree that converges to $x$. [3] [12]

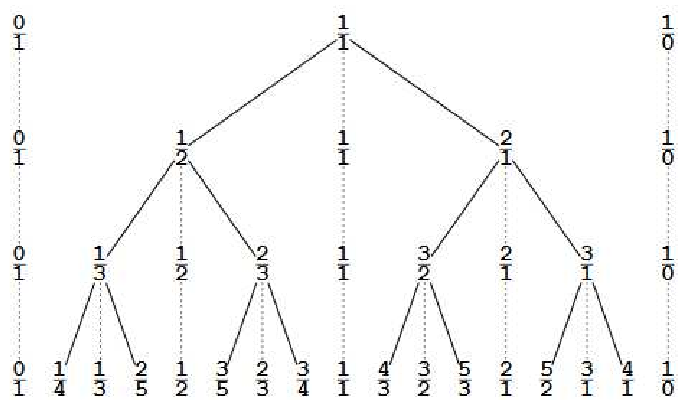

Figure 3. The Stern-Brocot Tree.

Fractions in the Stern-Brocot tree also correspond to certain circles, called Ford circles, in the $(0, 2, 2, 0)$ packing. The basis for this correspondence is explained by the following simple lemma, which follows from Descartes' equation.

**Lemma 3.6.** *Suppose that mutually tangent circles of curvatures* $m^2$ *and* $n^2$ *are also tangent to a line. If we add a new circle between the two circles and the line, it has curvature* $(m+n)^2$.

**Ford Circles**: For any two relatively prime integers $m, n$ the circle with center $(\frac{m}{n}, \frac{1}{2n^2})$ and radius $\frac{1}{2n^2}$ is a Ford circle. Every Ford circle is tangent to the real line and no two Ford circles intersect except for points of tangency [4]. Ford circles are a subset of the $(0, 2, 2, 0)$ circle packing, and are the result of only using the first three generators $S_1, S_2, S_3$. If we identify one circle and one line in the initial configuration as the parents $\frac{0}{1}$ and $\frac{1}{0}$, then Ford circles correspond to fractions in the Stern-Brocot tree. The point of tangency between a Ford circle and the real line is precisely the Stern-Brocot fraction. The curvature of a Ford circle is twice the square of the denominator of the Stern-Brocot fraction.

**Proposition 3.7.** *There exists a sequence in* $\mathcal{A}_\infty$ *with minimal exponent* 0.

*Proof.* Consider the repeating sequence $\ldots S_3 S_2 S_1 S_3 S_2 S_1$ applied to the circle packing $(0, 2, 2, 0)$. This produces a sequence of Ford circles whose radii are $\frac{1}{f_n^2}$

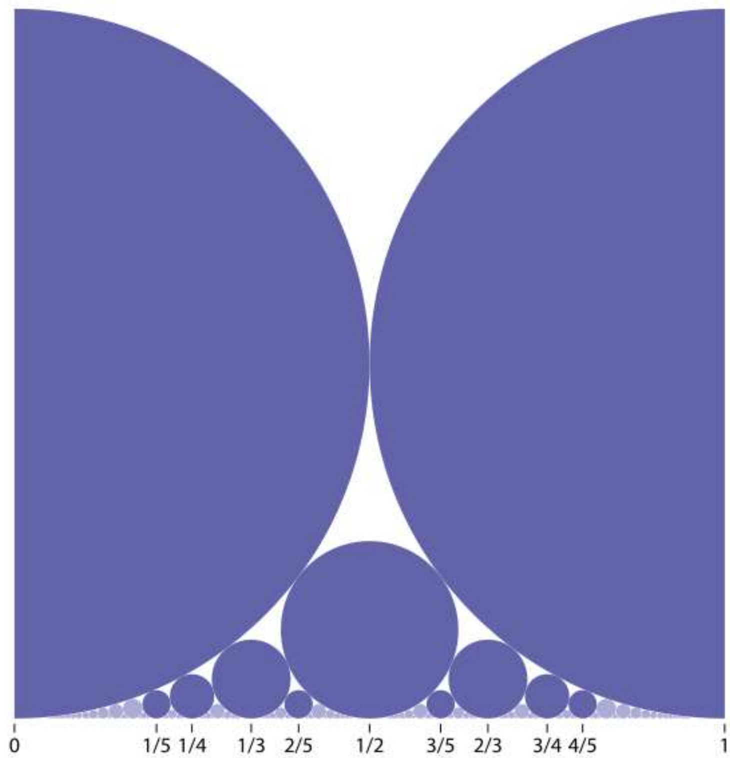


FIGURE 4. Ford Circles.

where $f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$ is the $n^{th}$ Fibonacci number. It follows immediately that the Apollonian sum $\sum_{n=1}^{\infty} f_n^{-2s}$ has minimal exponent $s = 0$. $\square$

**Theorem 3.8** (Graham et al [6]). *When applied to an initial configuration $(a, b, c, d, )$ with $a < 0 < b \leq c \leq d$ and $a + b + c \geq d$, the curvatures from the sequence $\ldots S_4S_3S_2S_1S_4S_3S_2S_1$ grow faster than any other sequence and the curvatures from the sequence $\ldots S_4S_3S_4S_3$ grow slower than any other sequence.*

Along with our above results, Theorem 3.8 shows that the minimal exponent of any sequence must lie in the set $[0, \frac{1}{2}]$. This leads to the natural question: can the minimal exponent of a sequence be any element of $[0, \frac{1}{2}]$? The following theorem answers the question affirmatively.

**Theorem 3.9.** *The minimal exponent map $M : A_\infty \to [0, \frac{1}{2}]$ is surjective.*

We will prove the theorem by constructing sequences in the $(0, 2, 2, 0)$ circle packing that have minimal exponent $\beta \in (0, \frac{1}{2})$. The key will be to consider increasing functions $\varphi : \mathbb{N} \to \mathbb{N}$ with sufficiently large growth rates. First we begin with a technical lemma.

**Lemma 3.10.** *Let $g(x)$ and $h(x)$ be polynomials of degree 2 with positive coefficients. Set*

$$F_\varphi(s) = \sum_{n=1}^{\infty} \frac{1}{\Pi_{m=0}^{n-1} g(\varphi(m))^s} \sum_{k=0}^{\varphi(n)} \frac{1}{h(k)^s}. \tag{7}$$

*If we let $\varphi(n) = \lfloor 2^{\alpha^n} \rfloor$ for $\alpha \in (1, \infty)$ then the minimal exponent of $F_\varphi$ is $\frac{\alpha-1}{2\alpha}$.*

*Proof.* If $s > \frac{1}{2}$ then the inner sum is bounded and the series clearly converges. Thus we need only consider $s \in (0, \frac{1}{2})$. For $x \geq 1$ we have the trivial inequalitiy $\gamma_1 x^2 \leq h(x) \leq \gamma_2 x^2$ where $\gamma_1$ is the degree 2 coefficient of $h$ and $\gamma_2$ is the sum of the coefficients of $h$. Estimating the inner sum with an integral we have the following bounds:

$$\sum_{n=1}^{\infty} \frac{\frac{\gamma_2^{-s}}{1-2s}(\varphi(n)^{1-2s} - 1)}{\Pi_{m=0}^{n-1} g(\varphi(m))^s} \leq F_\varphi(s) \leq \sum_{n=1}^{\infty} \frac{\frac{2}{h(0)^s} + \frac{\gamma_1^{-s}}{1-2s}(\varphi(n)^{1-2s} - 1)}{\Pi_{m=0}^{n-1} g(\varphi(m))^s}. \tag{8}$$

The growth of the denominator and the restriction $s \in (0, \frac{1}{2})$ makes it readily apparent that the upper and lower bounds converge if and only if the series $\sum_{n=1}^{\infty} \frac{\varphi(n)^{1-2s}}{\Pi_{m=0}^{n-1} g(\varphi(m))^s}$ converges. If $\beta_1$ is the degree 2 coefficient of $g$ and $\beta_2$ is the sum of the coefficients of $g$ we get the bound

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{\beta_2^{s(n-1)} \Pi_{m=0}^{n} \varphi(m)^{2s}} \leq \sum_{n=1}^{\infty} \frac{\varphi(n)}{\varphi(n)^{2s} \Pi_{m=0}^{n-1} g(\varphi(m))^s} \leq \sum_{n=1}^{\infty} \frac{\varphi(n)}{\beta_1^{s(n-1)} \Pi_{m=0}^{n} \varphi(m)^{2s}}. \tag{9}$$

The lower bound converges if and only if the upper bound converges and convergence is unaffected by replacing $\varphi(n)$ with $2^{\alpha^n}$. We have now shown that $F_\varphi$ converges if and only if $\sum_{n=1}^{\infty} \beta_2^{-s(n-1)} 2^{\alpha^n(1-2s\sum_{m=0}^{n} \alpha^{m-n})}$ converges. Since $\lim_{n\to\infty} \sum_{m=0}^{n} \alpha^{m-n} = \frac{\alpha}{\alpha-1}$ it follows that $F_\varphi$ converges for $s > \frac{\alpha-1}{2\alpha}$ and diverges for $s < \frac{\alpha-1}{2\alpha}$ as desired. □

**Construction of an intermediate sequence**: For any irrational number $x \in (0, 1)$, the continued fraction of $x$ defines a unique path in the Stern-Brocot tree converging to $x$, which in turn defines a sequence of tangent Ford circles in the $(0, 2, 2, 0)$ packing. By choosing an appropriate irrational number we can get a sequence with minimal exponent $s \in (0, \frac{1}{2})$. Consider the irrational number $x$ with continued fraction $[0; 1, \varphi(1), 1, \varphi(2), \ldots]$ written:

$$x = \cfrac{1}{1 + \cfrac{1}{\varphi(1) + \cfrac{1}{1 + \cfrac{1}{\varphi(2) + \cfrac{1}{\ddots}}}}} \tag{10}$$

The path converging to $x$ in the Stern-Brocot tree is most easily described using continued fractions as follows:

$$[0; 2], [0; 1, 2], [0; 1, 3] \ldots [0; 1, \varphi(1)+1], [0; 1, \varphi(1), 2], [0; 1, \varphi(1), 1, 2] \ldots [0; 1, \varphi(1), 1, \varphi(2)+1] \ldots \tag{11}$$

Let $S_x$ be this sequence of rational numbers. We denote the corresponding sequence of circles by $C(p, q)$ where $\frac{p}{q} \in S_x$ and $p, q$ are relatively prime. The curvature of

$C(p,q)$ is $2q^2$. Thus the Apollonian sum is given by $G_\varphi(s) = 2^{-s}\sum_{\frac{p}{q}\in S_x}\frac{1}{q^{2s}}$. Unfortunately this form is not very useful but we can find recursive formulas to make the sum tractable.

Every continued fraction of the form $[0;1,\varphi(1),\ldots 1,\varphi(m)]$ or of the form $[0;1,\varphi(1),\ldots,\varphi(m)+1]$ is a convergent of $x$. Let $\frac{p_n}{q_n}$ be the fractional representation of the $n^{th}$ convergent with $p_n, q_n$ relatively prime. Then it is a well known result from continued fractions that $q_n = a_n q_{n-1} + q_{n-2}$ where $a_n$ is the $n^{th}$ entry in the continued fraction. In the case of the convergents of $x$, $q_{2n+1} = q_{2n} + q_{2n-1}$ and $q_{2n} = \varphi(n)q_{2n-1} + q_{2n-2}$. Thus $q_{2n+1} = (\varphi(n)+1)q_{2n-1} - q_{2n-3}$. Since the denominators are strictly increasing, it follows that $\varphi(n)q_{2n-1} \le q_{2n+1} \le (\varphi(n)+1)q_{2n-1}$. Therefore $\Pi_{m=1}^{n}\varphi(m) \le q_{2n+1} \le \Pi_{m=1}^{n}(\varphi(m)+1)$. There are $\varphi(n)+1$ circles between $q_{2n-1}$ and $q_{2n+1}$ counting $q_{2n+1}$ but not $q_{2n-1}$. The denominator of the $k^{th}$ circle between $q_{2n-1}$ and $q_{2n+1}$ is given by $(k+1)q_{2n-1} + q_{2n-2}$ which is between $(k+1)q_{2n-1}$ and $(k+2)q_{2n-1}$. Finally we have

$$2^{-s}\sum_{n=1}^{\infty}\frac{1}{\Pi_{m=1}^{n-1}(\varphi(m)+1)^{2s}}\sum_{k=0}^{\varphi(n)}\frac{1}{(k+2)^{2s}} \le G_\varphi(s) \le 2^{-s}\sum_{n=1}^{\infty}\frac{1}{\Pi_{m=1}^{n-1}\varphi(m)^{2s}}\sum_{k=0}^{\varphi(n)}\frac{1}{(k+1)^{2s}}. \tag{12}$$

If we take $\varphi(n) = \lfloor 2^{\alpha^n}\rfloor$ then $G_\varphi$ is bounded by sums satisfying Lemma 3.10. Thus $G_\varphi$ has minimal exponent $\frac{\alpha-1}{2\alpha}$. This completes the proof of Theorem 3.9.

Using a slightly modified sequence we can prove a more precise result that does not rely on the $(0,2,2,0)$ packing or continued fractions.

**Theorem 3.11.** *Let $D \subset A_\infty$ be the set of sequences where all 4 generators appear infinitely many times. Then the minimal exponent map $M : D \to [0,\frac{1}{2}]$ is surjective.*

*Proof.* Consider the sequence:

$$\mathbf{x} = \ldots S_4S_3\underbrace{S_2S_1\ldots S_2S_1}_{\varphi(3)}S_1S_2\underbrace{S_4S_3\ldots S_4S_3}_{\varphi(2)}S_4S_3\underbrace{S_2S_1\ldots S_2S_1}_{\varphi(1)} \in A_\infty \tag{13}$$

Set $a_n$ to be the curvature of circle obtained after the $n^{th}$ repeat is broken. Set $b_n$ to be the curvature of the circle immediately after $a_n$. Let the $n^{th}$ chain be all of the circles including and after $b_n$ and before $a_{n+1}$. Denote the $k^{th}$ circle in the $n^{th}$ chain by $D(n,k)$ where $0 \le k \le \varphi(n)$. For compactness of notation we set $p_n = D(n,\varphi(n))$, $q_n = D(n,\varphi(n)-1)$. By Decartes' equation we get the following formulas:

$$\begin{aligned} a_n &= 2(p_{n-1}+q_{n-1}+p_{n-2}) - q_{n-2} \\ D(n,0) = b_n &= 2(a_n + p_{n-1} + q_{n-1}) - p_{n-2} \\ D(n,1) &= 2(q_{n-1}+b_n+p_{n-1}) - a_n \\ D(n,k) &= 2(p_{n-1}+q_{n-1}+D(n,k-1)) - D(n,k-2) \quad (2\le k\le\varphi(n)). \end{aligned} \tag{14}$$

Iterating the expression for $D(n,k)$ we obtain, for all $(0 \le k \le \varphi(n))$:

$$D(n,k) = (k+2)(k+3)(q_{n-1}+p_{n-1}) + (k+3)p_{n-2} - (k+2)q_{n-2}. \tag{15}$$

We work with a bounded packing so that the curvatures of the circles are eventually strictly increasing. Therefore we have bounds for $D(n,k)$: $g_1(k)p_{n-1} \le D(n,k) \le g_2(k)p_{n-1}$ where $g_1(k) = k^2+5k+6$ and $g_2(k) = 2k^2+11k+15$. Applied inductively, this gives bounds for $p_n$: $\Pi_{m=2}^n g_1(\varphi(m))p_1 \le p_n \le \Pi_{m=2}^n g_2(\varphi(m))p_1$. Hence $g_1(k)\Pi_{m=2}^{n-1} g_1(\varphi(m))p_1 \le D(n,k) \le g_2(k)\Pi_{m=2}^{n-1} g_2(\varphi(m))p_1$. The sum over the sequence is given by $G_\varphi(s) = \sum_{n=1}^{\infty}(\frac{1}{a_n^s} + \sum_{k=0}^{\varphi(n)} \frac{1}{D(n,k)^s})$. The $a_n$ grow exponentially so we may ignore them. The rest of the sum is bounded by sums satisfying Lemma 3.10. □

**Theorem 3.12.** *Every periodic sequence on more than* 2 *generators has minimal exponent* 0*.*

*Proof.* Let $\mathbf{x} = \dots s_{i_m} \dots s_{i_1} s_{i_m} \dots s_{i_1}$ be a periodic sequence and let $H_{\mathbf{x}}(s)$ be the sum over the sequence. Let $h(k,n)$ be the curvature of the $(nm+k)^{th}$ circle, where $1 \le k \le m$ and $n \ge 1$. Since the sequence of circles lies in a bounded region, the curvatures are eventually strictly increasing. Thus we know $h(1,n) \le h(k,n)$ for $n$ sufficiently large. By Descartes' equation we have $h(1,n) = 2(h(m,n-1) + v(n) + u(n)) - w(n)$ where $v, u, w$ correspond to the curvatures of some circles in the packing. Since the sequence is over more than 2 generators we may assume $v(n) = h(c, n-1)$ and $w(n) = h(d, n-1)$ for some $1 \le c, d \le m-1$ and $c \ne d$. Let $\gamma$ be the smallest curvature in the entire packing. Next we may choose $N$ so large that $h(m,N) > 2\gamma$ and $h(1,N) > 0$. Then for $n > N$ we have

$$\begin{aligned} h(1,n) &= 2h(m,n-1) + 2h(c,n-1) + 2u(n) - h(d,n-1) \\ &\ge 2h(m,n-1) + 2h(1,n-1) + 2\gamma - h(m-1,n-1) \\ &\ge h(m,n-1) + 2h(1,n-1) + 2\gamma \\ &\ge 2h(1,n-1) \\ &\quad \dots \\ &\ge 2^{n-N} h(1,N). \end{aligned} \tag{16}$$

Finally,

$$\begin{aligned} H_{\mathbf{x}}(s) &\le \sum_{n=1}^{N}\sum_{k=1}^{m} \frac{1}{h(k,n)^s} + \sum_{n>N} \frac{m}{h(1,n)^s} \\ &\le \sum_{n=1}^{N}\sum_{k=1}^{m} \frac{1}{h(k,n)^s} + h(1,N)^{-s} \sum_{n>N} \frac{m}{2^{(n-N)s}} < \infty \end{aligned} \tag{17}$$

for all $s > 0$. Therefore $H_{\mathbf{x}}(s)$ has minimal exponent 0. □

## 4. Sums Over Subgroups

In this section we examine Apollonian sums over the collection of all circles tangent to a fixed circle in a packing (the collection of Ford circles is one example). In the Apollonian group, this corresponds to a subgroup generated by three of the $S_i$. We compute the sum explicitly in the case where the fixed circle is a line. Of course, the collection of circles tangent to a line is unbounded, so we also fix two tangent bounding circles, of curvatures $m^2$ and $n^2$, and look only at the circles between these and the line. In the simplest case $m = n = 1$, the result is an elegant Dirichlet series. In general it can be written in terms of Hurwitz zeta functions. The minimal exponent of sums of this type is 1.

**Proposition 4.1.** *Fix two tangent circles of curvatures $m^2$ and $n^2$, and a line tangent to both circles at distinct points. Extend this triple to an Apollonian* *packing. Let $C$ be the collection of circles between the two fixed circles and the line, and tangent to the line. The Apollonian sum over this collection is given by $F(s) = \frac{m^{-2s}}{\zeta(2s)} \sum_{k=1}^{\infty} \zeta(2s, \frac{n}{m}k) - (\frac{m}{nk})^{2s}$.*

*Proof.* Each circle in the packing is in one-to-one correspondence with a node in the Stern-Brocot tree. Given $(a, b)$ relatively prime there is a circle of curvature $(am + bn)^2$. Thus

$$\begin{aligned} F(s) &= \sum_{(a,b)=1} \frac{1}{(am+bn)^{2s}} \\ &= \frac{1}{\zeta(2s)} \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{(am+bn)^{2s}} \\ &= \frac{m^{-2s}}{\zeta(2s)} \sum_{k=1}^{\infty} \zeta(2s, \frac{n}{m}k) - (\frac{m}{nk})^{2s}. \end{aligned} \tag{18}$$

□

**Corollary 4.2.** *This sum, and hence general sums over three generators in the Apollonian group, have minimal exponent* 1*.*

A simpler form is possible when $m, n$ are linearly dependent over $\mathbb{Q}$. In this case, we may rescale the packing to assume that $m, n$ are relatively prime integers. We need an elementary lemma on integer combinations of $m$ and $n$.

**Lemma 4.3.** *Given $m, n \in \mathbb{N}$ relatively prime let $f_{m,n}(c) = |\{(a, b) \in \mathbb{N}^2 : am + bn = c\}|$. If $c = kmn + j$ with $1 \leq j \leq mn$ then $f_{m,n}(c) = k + f_{m,n}(j)$. Futhermore $f_{m,n} : \{1, \ldots, mn\} \to \{0, 1\}$.*

*Proof.* Given a solution pair $(x, y)$ every other solution is given by $(x + in, y - im)$ where $i \in \mathbb{Z}$. Now suppose that $c \leq mn$. If there is a solution pair $(x, y)$ satisfying $mx + by = c$ and $x, y > 0$ it is clear that $x \leq n$ and $y \leq m$. Therefore there can be no other solution pairs satisfying the equation and satisfying $x, y > 0$. Therefore when $c < mn$ the answer is either 1 or 0. If the answer is 0 there still exists a solution pair $(x, y)$ where either $x > 0$ and $-m < y \leq 0$ or $-n < x \leq 0$ and $y > 0$ since $c$ is positive. In the general case we may write $c = mnk + j$ where $k \geq 0$ and $1 \leq j \leq mn$. Fixing $j$ and letting $k$ vary it is easy to see that each time $k$ increases by 1 the number of solutions increases by 1. This is equivalent to $f_{m,n}(kmn + j) = k + f_{m,n}(j)$ where $0 < j \leq mn$. □

**Remark**: It should be stressed that $f$ is in general not a group homomorphism, and that $f$ is never an extension of a character on $(\mathbb{Z}/mn\mathbb{Z})^*$. $f$ is a homomorphism only in the special case where either $m = 1$ or $n = 1$ in which case $f$ is the trivial homomorphism.

**Proposition 4.4.** *In the setup of Proposition 4.1, assume that $m$ and $n$ are coprime integers. Then*

$$F(s) = \frac{\zeta(2s-1)}{mn\zeta(2s)} + \frac{(mn)^{-2s}}{\zeta(2s)} \sum_{j=1}^{mn} \zeta(2s, \frac{j}{mn})(f_{m,n}(j) - \frac{j}{mn}). \tag{19}$$

*Proof.* As above, we have $F(s) = \frac{1}{\zeta(2s)} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{(am+bn)^{2s}}$. Now $am + bn$ must be an integer, and we write

$$\begin{aligned} F(s) &= \frac{1}{\zeta(2s)} \sum_{c=1}^{\infty} \sum_{am+bn=c} \frac{1}{c^{2s}} \\ &= \frac{1}{\zeta(2s)} \sum_{k=0}^{\infty} \sum_{j=1}^{mn} \frac{k + f_{m,n}(j)}{(kmn+j)^{2s}} \\ &= \frac{\zeta(2s-1)}{mn\zeta(2s)} + \frac{(mn)^{-2s}}{\zeta(2s)} \sum_{j=1}^{mn} \zeta(2s, \frac{j}{mn})(f_{m,n}(j) - \frac{j}{mn}). \end{aligned} \tag{20}$$

□

**Corollary 4.5.** *In the case* $m = n = 1$*, the sum is* $F(s) = \frac{\zeta(2s-1)}{\zeta(2s)} - 1 = \sum_{n=2}^{\infty} \frac{\phi(n)}{n^{2s}}$ *where* $\phi$ *denotes Euler's totient function.*

The latter equality comes from identifying Ford circles with rational numbers between 0 and 1.

## 5. Sums Over Generations

We now turn to Apollonian sums over generations of circles in a packing. Given an initial Descartes configuration with four mutually tangent circles, the circles filling in the four interstices are said to be of *generation one*. The circles filling in the next twelve available interstices are said to be of a *generation two*, and so on. The Apollonian packing is union of all generations.

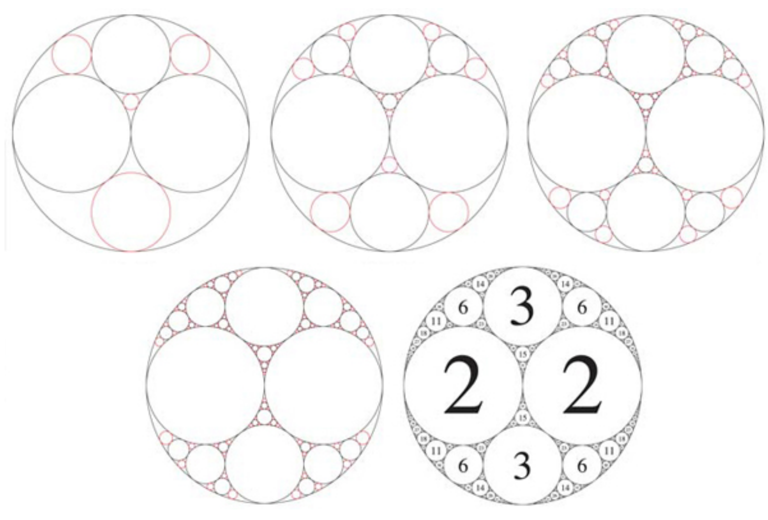


Figure 5. Building a packing generation by generation

Let $G_{n,P}(s)$ denote the Apollonian sum over the $n^{th}$ generation of circles in a packing $P$ with fixed initial Descartes configuration. Each generation is a finite collection of circles, so $G_{n,P}(s)$ converges for all $s$. Thus instead of the minimal exponent, we will focus on special values of the Apollonian sum, using recursive relations between successive generations. The values when $s = 0, -1, -2, \ldots$ can be viewed as the moments structure of the distribution of curvatures in generation $n$. By a noncommutative version of the central limit theorem, this distribution is approximately log-normal for large $n$. [11]

Note that the number of circles in generation $n$ is $4 \cdot 3^{n-1}$, for $n \geq 1$. This is the value of $G_{n,P}(0)$. We will study $G_{n,P}(-1)$, deriving an explicit formula from a recursive relationship between generations. Since $-1$ and $P$ are fixed here, we define $g_n = G_{n,P}(-1)$.

**Theorem 5.1.** *In an Apollonian packing, the sum of the curvatures in generation $n$ is given by $g_n = -\frac{g_0}{9}(\frac{\alpha}{\beta^{n+1}} + \frac{\bar{\alpha}}{\bar{\beta}^{n+1}})$ for $n \geq 1$, where $\alpha = \frac{91+31\sqrt{13}}{26}$, $\bar{\alpha} = \frac{91-31\sqrt{13}}{26}$ and $\beta = \frac{4+\sqrt{13}}{3}$, $\bar{\beta} = \frac{4-\sqrt{13}}{3}$.*

*Proof.* We first find a recursive formula for the sum of the curvatures in generation $n$. The sum of the curvatures in an initial Descartes quadruple, $(a, b, c, d)$, is given by $g_0 = a + b + c + d$.

The four circles of generation one have curvatures $2(b+c+d)-a$, $2(a+c+d)-b$, $2(a+b+d)-c$ and $2(a+b+c)-d$. So, their sum is $g_1 = 5g_0$.

To find the curvature sum for the $4 \cdot 3^{n-1}$ circles in generation n, we view the previous generations as a set of $4 \cdot 3^{n-1}$ overlapping Descartes quadruples. This yields recursive formulas:

$$\begin{aligned}
g_2 &= 5(3g_0 + g_1) - g_0 \\
g_3 &= 5(6g_0 + 3g_1 + g_2) - 3g_0 \\
g_4 &= 5(12g_0 + 6g_1 + 3g_2 + g_3) - (6g_0 + 3g_1) \\
g_5 &= 5(24g_0 + 12g_1 + 6g_2 + 3g_3 + g_4) - (12g_0 + 6g_1 + 3g_2) \\
&\vdots \\
g_n &= 27(\sum_{i=0}^{n-3} 2^{n-(3+i)} g_i) + 15g_{n-2} + 5g_{n-1}, \; for\ all\ n \geq 3 \qquad (21)
\end{aligned}$$

Define a generating function by

$$A(x) = \sum_{n=0}^{\infty} g_n x^n = g_0 + g_1 x + g_2 x^2 + \cdots \qquad (22)$$

The recursive formulas above imply that $A(x) = g_0(\frac{2}{3} + \frac{1}{3} \cdot \frac{7x+1}{3x^2-8x+1})$

To recover the coefficients $g_n$ we decompose $A(x)$ by partial fractions and rewrite it as a sum of geometric series.

$$A(x) = g_0 - \frac{g_0}{9}(\frac{\alpha}{\beta^2} + \frac{\bar{\alpha}}{\bar{\beta}^2})x - \frac{g_0}{9}(\frac{\alpha}{\beta^3} + \frac{\bar{\alpha}}{\bar{\beta}^3})x^2 - \frac{g_0}{9}(\frac{\alpha}{\beta^4} + \frac{\bar{\alpha}}{\bar{\beta}^4})x^3 - \cdots \qquad (23)$$

where $\alpha = \frac{91+31\sqrt{13}}{26}$, $\bar{\alpha} = \frac{91-31\sqrt{13}}{26}$ and $\beta = \frac{4+\sqrt{13}}{3}$, $\bar{\beta} = \frac{4-\sqrt{13}}{3}$.

Comparing the coefficients of (23) and (22), we find the closed form expression for $g_n$.

$$g_n = -\frac{g_0}{9}(\frac{\alpha}{\beta^{n+1}} + \frac{\bar{\alpha}}{\bar{\beta}^{n+1}}), \; for\ all\ n \geq 1. \qquad (24)$$

□

A similar recursive relationship between generations exists for $G_{n,P}(-2)$, the sum of squares of the curvatures. This uses Descartes' equation in a fundamental way, suggesting that higher powers are substantially more difficult. Leaving that topic aside, we now partition the all circles in a packing into four classes, which we call **colors**.

We note that the Apollonian group does not allow for permutations of the entries in a Descartes quadruple $(a, b, c, d)$. In fact, the order of entries naturally divides the set of circles in a packing into four classes. Another approach is to assign four different colors to the circles of an initial Descartes configuration. This induces a coloring of the entire packing, which is in fact the unique four-coloring of an associated planar graph.

Given Apollonian packing $P$ with initial configuration $(a, b, c, d)$, partition the set of circles in the packing into four classes, called colors:
(1) pink: the circle $a$, and circles of reduced form $S_1 S_{i_1} ... S_{i_n}$
(2) blue: the circle $b$, and circles of reduced form $S_2 S_{i_1} ... S_{i_n}$
(3) yellow: the circle $c$, and circles of reduced form $S_3 S_{i_1} ... S_{i_n}$
(4) red: the circle $d$, and circles of reduced form $S_4 S_{i_1} ... S_{i_n}$.
The reduced form of a circle refers to the shortest sequence of generators $S_i$ required to reach it.

We will study blue circles, without loss of generality. Let $G_{n,P,blue}(s)$ denote the Apollonian sum over blue circles of generation $n$. We have $G_{n,P,blue}(0) = 3^n - 1$ for $n \geq 1$. We will now build off Theorem 5.1 to give an explicit formula for $G_{n,P,blue}(-1)$. For convenience, we denote this quantity $b_n$.

**Theorem 5.2.** *The sum of the curvatures of blue circles in generation $n$ is:*

$$b_n = \frac{2g_0}{9}\Big(\big(\frac{\alpha}{\beta^n}\big)\big(\frac{(-3\beta^2)^{\lfloor \frac{n+1}{2} \rfloor} - 1}{3\beta^2 + 1}\big) + \big(\frac{\bar{\alpha}}{\bar{\beta}^n}\big)\big(\frac{(-3\bar{\beta}^2)^{\lfloor \frac{n+1}{2} \rfloor} - 1}{3\bar{\beta}^2 + 1}\big)\Big) + (-3)^{\lfloor \frac{n+1}{2} \rfloor} b_0 \tag{25}$$

*where $\alpha$, $\bar{\alpha}$, $\beta$, and $\bar{\beta}$ are as in the previous theorem. This is solely a function of $n$, $g_0$, and $b_0$, not of the other initial curvatures.*

*Proof.* We first find a recursive formula for $b_n$ using the same technique as in Theorem 5.1. We show the first few generations $b_i$ to demonstrate the derivation of this formula.

$$\begin{aligned} b_0 &= b \\ b_1 &= 2a - b + 2c + 2d = 2(g_0 - b_0) - b_0 \\ b_2 &= 4(g_1 - b_1) + 2(g_0 - b_0) - 3b_0 \\ b_3 &= 8(g_0 - b_0) + 4(g_1 - b_1) + 2(g_2 - b_2) - 6b_0 - 3b_1 \\ &\vdots \end{aligned}$$

$$b_n = \sum_{i=0}^{n-1} 2^{n-i}(g_i - b_i) - 3\sum_{i=0}^{n-2} 2^{n-2-i} b_i \tag{26}$$

Rewriting $b_{n-1}$ according to this expression, substituting, and simplifying, we obtain $b_n = 2g_{n-1} - 3b_{n-2}$ for $n \geq 2$. For $n = 1$, we have $b_1 = 2g_0 - 3b_0$. A

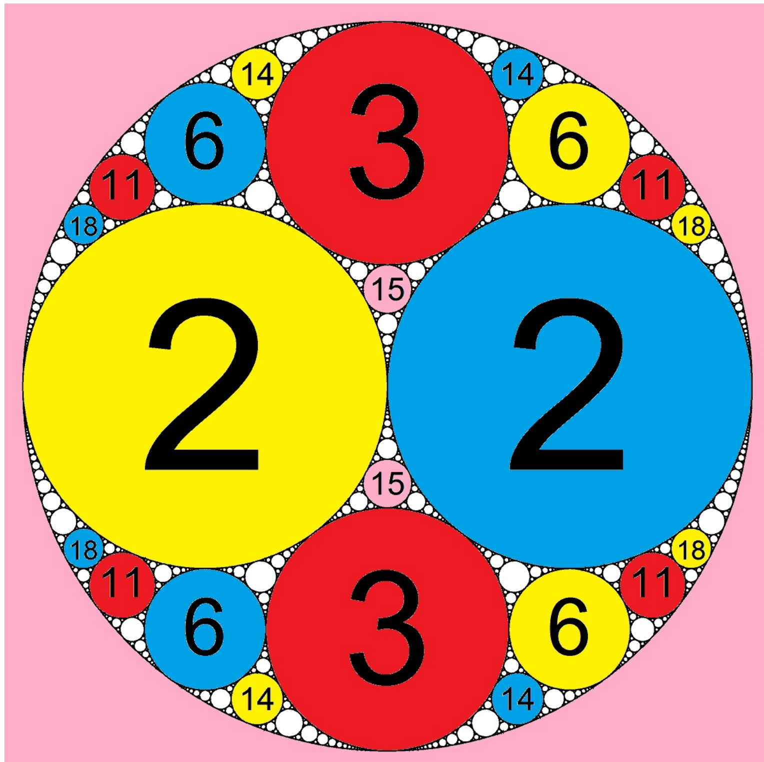


FIGURE 6. A packing with coloring partially illustrated.

continued expansion of the $b_i$ terms yields the sum:

$$b_n = 2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-3)^k g_{n-1-2k} + (-3)^{\lfloor \frac{n+1}{2} \rfloor} b_0. \tag{27}$$

We now use Theorem 5.1 to obtain

$$b_n = \frac{-2g_0}{9} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-3)^k \left(\frac{\alpha}{\beta^{n-2k}} + \frac{\bar{\alpha}}{\bar{\beta}^{n-2k}}\right) + (-3)^{\lfloor \frac{n+1}{2} \rfloor} b_0 \tag{28}$$

where $\alpha$, $\bar{\alpha}$, $\beta$, and $\bar{\beta}$ are as above. Computing the summation and simplifying gives the theorem.

□

## REFERENCES

[1] D. W. Boyd *The residual set dimension of the Apollonian packing*, Mathematika, 20 (1973), pp 170-174.
[2] D. W. Boyd *The sequence of radii in an Apollonian packing*, Mathematics of Computation, 39 (1982), pp 249-254.
[3] A. Brocot, *Calcul des rouages par approximation, nouvelle methode*, Revue Chonometrique, 3, pp 186-194 (French).
[4] L. R. Ford, *Fractions*, The American Mathematical Monthly, 45 No. 9 (1938).

[5] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. R. Wilks and C. H. Yan, *Apollonian circle packings: geometry and group theory I. The Apollonian group*, Discrete and Computational Geometry, 34 No. 4 (2005), pp 547-585.

[6] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. R. Wilks and C. H. Yan, *Apollonian circle packings: number theory*, Journal of Number Theory, 100 Issue 1 (2003), pp 1-45.

[7] A. Hurwitz, *Einige Eigenschaften der Dirichletschen Functionen* $F(s) = \sum(\frac{D}{n})\frac{1}{n}$, *die bei der Bestimmung der Klassenanzahlen binrer quadratischer Formen auftreten*, Zeitschrift fr Math. u. Physik, 27 (1882), pp. 86101 (German).

[8] A. Kontorovich and H. Oh, *Apollonian packings and horospherical flows on hyperbolic 3-manifolds*, Journal of the American Mathematical Society 24 (2011), pp 603-648.

[9] C. Mallows, *Growing Apollonian Packings*, Journal of Integer Sequences, 12 (2009).

[10] B. B. Mandelbrot, The Fractal Geometry of Nature, New York: W.H. Freeman (1983), pp 169-172.

[11] P. Sarnak, *Integral Apollonian Packings*, The American Mathematical Monthly, 118 No. 4 (2011), pp. 291-306.

[12] M. A. Stern, *Ueber eine zahlentheoretische Funktion* Journal fur die reine und angewandte Mathematik, 55, pp 193-220 (German).