MATHEMATICAL THEORY OF GENERAL RELATIVITY

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General Relativity is a domain of extraordinary interest from a variety of points of view, ranging from the purely theoretical need to understand the mathematical structure of this very rich and fascinating non-linear theory, to the computational one of being able to design sophisticated numerical algorithms which could be used to detect gravitational waves and thus help provide a new and extremely powerful window into the physical universe, to the speculative efforts of theoretical physicists to develop a quantum theory of gravity. Like in the case of many other established physical theories the role of mathematicians is to uncover the main consequences of the theory and, in the process, better understand its structure. At the heart of the mathematical theory of General Relativity are a number of very deep and challenging conjectures whose resolution requires the development of completely new geometric and analytic ideas. In my lectures I will try to illustrate the current sense of excitement in the field by describing some of the most interesting recent developments starting with the nonlinear stability of the Minkowski space, formation of trapped surfaces, breakdown criteria, bounded L^2 curvature conjecture, rigidity and stability of black holes.

Short hysterical sketch. The formulation of General Relativity has initially attracted the interest of the leading mathematicians at the time, such as D. Hilbert, E. Cartan, H. Weyl, A. Lichnerowicz, J. Leray etc and, somewhat incidentally, has given an important boost to the development of Riemannian Geometry. Soon thereafter, due to the complexity and lack of mathematical tools to deal with the new theory, few mathematicians continued to work on it and the subject was developed mainly by physicists, through the discovery and analysis of important special solutions. The exceptions were few but very important; A. Lichnerowicz, J. Leray and Y. Choquet- Bruhat in France were the first to recognize the central role of the Cauchy problem, and stress its intimate connection to the theories of nonlinear elliptic and hyperbolic partial differential equations, which were being developed, by leading analysts, at around the same time (1930-1950).

The period (1960-1980) saw the beginning of a well defined mathematical theory with well formulated problems and first outstanding general results. This was prompted by the theoretical discovery of black holes a monumental scientific discovery in which the mathematicians J.L. Synge and M. Kruskal played a very important role. Then followed, in rapid order, the famous singularity theorems of R. Penrose and S. Hawking and the deep results on uniqueness of black holes by W. Israel, D. Robinson, B. Carter and S. Hawking. This was also the period, when some of the main goals of General Relativity were formulated, among them are the Cosmic Censorship Conjectures and the Final State Conjecture, whose mathematical depth and broad significance is on par with the deepest problems within mathematics.

The larger mathematical community was somewhat slow to take note of the opportunities opened by these remarkable developments, partly because there was still a large disconnect between General Relativity and the traditional branches of mathematics, such as Differential Geometry and PDE, which are by nature the fields of mathematics most needed in the development of General Relativity. This situation changed dramatically in the following period (1979-1993) through two dramatic mathematical achievements; The proof of the positive mass theorem showed how the methods of Riemannian geometric analysis can be used to settle an outstanding theoretical issue of General Relativity while the proof of the global stability of the Minkowski space has shown how to fuse geometric ideas with hyperbolic PDE methods to deal with global issues in the problem of evolution. Both results have introduced powerful mathematical techniques into the subject and thus paved the way for the explosion in interest, development and results of the last 20 years.

The present period (1993-2009) is characterized by spectacular progress, in which mathematicians have taken the leading role. Among the recent achievements in the field here are a few for illustrations: D. Christodoulou's work on the Weak Cosmic Censorship Conjecture in the spherically symmetric case, G. Huisken-T. Ilmanen's and H. Bray's proofs of the Penrose inequality, R. Schoen and J. Corvino's gluing techniques for constraint equations, the work of H. Bahouri-J. Y. Chemin, H. Smith-D. Tataru and S. Klainerman-I. Rodnianski on the optimal regularity and breakdown criteria of the evolution problem, M. Dafermos' and H. Ringström's progress towards the Strong Cosmic Censorship Conjecture for certain symmetry reduced models, the work of V. Moncrief and collaborators on global solutions for a variety of cosmological models, H. Lindblad-I. Rodnianski's new proof of a weak stability of Minkowski space, G. Galloway's splitting theorems, S. Alexakis- A. Ionescu-S. Klainerman's work on the unconditional black hole uniqueness theorem, M. Dafermos-I. Rodnianski's and others' results on linear stability of the Kerr spacetimes , D. Christodoulou's proof of the evolutionary formation of black holes and the recently announced proof of the bounded L^2 curvature conjecture. But not only have remarkable results been proved on problems of General Relativity using geometric and PDE methods, the reverse is also true. The positive mass theorem, for example, is now routinely used in Geometric Analysis. Theoretical interest in finding an appropriate notion of quasi local mass has led to interesting extensions of the classical Weyl embedding problem problem. The entire field of nonlinear hyperbolic equations is driven, to a large extent, by ideas, methods and problems which originate and are motivated by General Relativity. As a dramatic illustration of this fact, D. Christodoulou has analyzed the formation of shocks in 3-dimensional fluids using methods directly inspired from the work on the stability of the Minkowski space As another a case in point, one of the most impressive recent results in hyperbolic PDE's, the global regularity result for the critical wave maps, due to T. Tao, J. Sterbenz-D. Tataru and J. Krieger-W. Schlag, originates in the works of D. Christodoulou-S. Tahvildar-Zadeh and S. Klainerman-M. Machedon and has direct roots in General Relativity. More broadly, the current interest in optimal well-posedness for nonlinear, geometric, wave equations, is ultimately motivated by General Relativity i.e. the cosmic censorship conjectures.

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