This examination booklet contains 6 problems plus an extra credit problem. There are 11 sheets of paper including the front cover and formula sheet at the back. This is a closed book exam. Do all of your work on the pages of this exam booklet. Show all your computations and justify/explain your answers (except for the true/false problems). Cross out anything you do not want graded. Calculators are NOT allowed.

You have 75 minutes to complete the exam. Do not begin until instructed to do so. When time is up, stop working and close your test booklet. Cell phones, headphones, laptops and other electronic devices are not allowed.

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<th>Problem</th>
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Grades will be posted on CourseWorks.
1. Let \( \vec{r}(t) \) be the vector-valued function defined by
\[
\vec{r}(t) = \langle t^3, \cos t, e^t \rangle.
\]

(a) (3 pts.) Find the derivative of \( \vec{r}(t) \).

(b) (4 pts.) Compute the definite integral
\[
\int_0^1 \vec{r}(t) dt.
\]
2. Define a function \( f(x, y) \) by
\[
f(x, y) = e^x \cos(y).
\]
(a) (4 pts.) Find an equation for the tangent plane to the graph \( z = f(x, y) \) at the point \((\ln 2, \frac{\pi}{2}, 0)\).

(b) (1 pt.) What is the linearization \( L(x, y) \) of \( f(x, y) \) at \((\ln 2, \frac{\pi}{2}, 0)\)?

(c) (3 pts.) Show that \( f(x, y) \) satisfies the Laplace differential equation
\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.
\]
3. Circle the word True or False to indicate your answer. (No explanation needed.)

(a) (1 pt.) The curvature of a circle with radius 3 is equal to 3 at every point.
True    False

(b) (1 pt.) Let \( \mathbf{u}(t) \) be any vector valued function. Then we have the identity
\[
\frac{d}{dt}(|\mathbf{u}(t)|) = |\mathbf{u}'(t)|,
\]
where ‘’ denotes the derivative.
True    False

(c) (1 pt.) Suppose that \( f(x, y) \) is a function that is differentiable at \((a, b)\). Then the tangent plane to the graph \( z = f(x, y) \) at \((a, b, f(a, b))\) contains at least two distinct lines that are tangent to the graph at \((a, b, f(a, b))\).
True    False

(d) (1 pt.) If the curvature of a smooth curve \( C \) is 0 at every point of \( C \), then the curve \( C \) is a straight line.
{
\textbf{Hint:} You may want to look at the formula sheet for the definition of curvature.
True    False

(e) (1 pt.) There exists a function \( f(x, y) \) with continuous second-order partial derivatives such that \( f_x(x, y) = x^2 + y \) and \( f_y(x, y) = x^2 - y \).
{
\textbf{Hint:} Why have we required \( f \) to have continuous second-order partial derivatives?
True    False

(f) (1 pt.) Suppose \( F(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz \) for real constants \( a, b, c, d, e, f \). Then all of the third partial derivatives of \( F \) are equal to 0.
True    False
(g) (3 pts.) Circle the Roman numeral of the graph matching the given equations. (No explanation needed.)

\[ f(x, y) = \frac{1}{1 + x^2 + y^2} \quad I \quad II \quad III \quad IV \quad V \quad VI \]

\[ f(x, y) = \frac{1}{1 + x^2 y^2} \quad I \quad II \quad III \quad IV \quad V \quad VI \]

\[ f(x, y) = \ln(x^2 + y^2) \quad I \quad II \quad III \quad IV \quad V \quad VI \]

\[ f(x, y) = |xy| \quad I \quad II \quad III \quad IV \quad V \quad VI \]

\[ f(x, y) = \cos(x^2 + y^2) \quad I \quad II \quad III \quad IV \quad V \quad VI \]

\[ f(x, y) = \cos(xy) \quad I \quad II \quad III \quad IV \quad V \quad VI \]
4. Evaluate the limit or show that it does not exist.

(a) (3 pts.) Find

\[
\lim_{(x,y)\to (0,0)} \frac{x^2 - y^2}{x + y}.
\]

(b) (3 pts.) Find

\[
\lim_{(x,y)\to (0,0)} \frac{xy^3}{x^2 + y^6}.
\]
(c) (4 pts.) Find

\[ \lim_{(x,y) \to (0,0)} \frac{x^2 y^3}{x^2 + y^6} \]
5. (a) (4 pts.) Reparametrize the vector valued function \( \vec{r}(t) = (3t, \cos t, \sin t) \) with respect to arc length starting at the point \((0, 1, 0)\).

(b) (4 pts.) Find a vector valued function that parametrizes the curve of intersection of
\[
x^2 + y^2 = 1 \text{ and } x + y + z = 1.
\]
6. (a) (6 pts.) Suppose that $f(x, y)$ is a differentiable function of $x$ and $y$, and that $g(r, s)$ is a function of two variables $r, s$ that is defined by the equation

$$g(r, s) = f(2 - rs, 1 - r + s).$$

Moreover, suppose that we have the following table of values of $f, g, f_x,$ and $f_y$ at the points $(2,1)$ and $(0,0)$.

<table>
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<tr>
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<th>$f$</th>
<th>$g$</th>
<th>$f_x$</th>
<th>$f_y$</th>
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<tbody>
<tr>
<td>(0,0)</td>
<td>1</td>
<td>2</td>
<td>3</td>
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<td>(2,1)</td>
<td>2</td>
<td>1</td>
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Compute $g_r(2,1)$ and $g_s(2,1)$.

*Hint:* Think of the inputs to $f(x, y)$ as being two functions $x(r, s) = 2 - rs$ and $y(r, s) = 1 - r + s$ and apply the chain rule. You may not need all the information given.

(b) (2 pts.) Suppose that $y$ is implicitly defined as a function of $x$ by the equation

$$F(x, y) = 0.$$ 

In terms of the partial derivatives $F_x$ and $F_y$, what is $\frac{dy}{dx}$?
Extra credit. (5 pts.) Find the osculating circle of the parabola $y = x^2$ at the point $(0,0)$. 
Some useful formulas

Feel free to tear this sheet off from your exam booklet.

**Arc length and curvature**

\[
\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}, \quad \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}, \quad \text{and} \quad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \tag{1}
\]

\[
\kappa(t) = \left| \frac{d\vec{T}}{ds} \right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{T}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}, \quad \kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{\frac{3}{2}}} \tag{2}
\]

**Motion in space**

\[
\vec{\ddot{a}}(t) = \vec{\ddot{v}}(t) = \vec{r}''(t) \tag{3}
\]

\[
\vec{\ddot{v}}(t) = v(t)\vec{T}(t) \tag{4}
\]

\[
\vec{\ddot{a}}(t) = v'(t)\vec{T}(t) + \kappa(t)v^2(t)\vec{N}(t) = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|} \vec{T}(t) + \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|} \vec{N}(t) \tag{5}
\]

**Trigonometry**

\[
\frac{d}{dx}(\tan x) = \sec^2 x, \quad \frac{d}{dx}(\csc x) = - \csc x \cot x, \quad \frac{d}{dx}(\sec x) = \sec x \tan x, \quad \frac{d}{dx}(\cot x) = - \csc^2 x \tag{6}
\]

\[
\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}, \quad \frac{d}{dx}(\cos^{-1} x) = - \frac{1}{\sqrt{1 - x^2}}, \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2} \tag{7}
\]

\[
\frac{d}{dx}(\csc^{-1} x) = - \frac{1}{x\sqrt{x^2 - 1}}, \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}, \quad \frac{d}{dx}(\cot^{-1} x) = - \frac{1}{1 + x^2} \tag{8}
\]