1. (Exercise 12) Find the maximum and minimum of

\[ f(x, y, z) = x^4 + y^4 + z^4 \]

subject to the constraint

\[ x^2 + y^2 + z^2 = 1. \]

**Solution:** We have

\[ \nabla f(x, y, z) = (4x^3, 4y^3, 4z^3) = (2\lambda x, 2\lambda y, 2\lambda z) = \lambda \nabla g(x, y, z). \]

**Case 1:** If all of \( x, y, z \neq 0 \), we can divide \( 4x^3 = 2\lambda x \), \( 4y^3 = 2\lambda y \), \( 4z^3 = 2\lambda z \) by \( 4x, 4y, \) and \( 4z \) to get \( x^2 = y^2 = z^2 = \frac{\lambda}{2} \). Since \( x^2 + y^2 + z^2 = \frac{3}{2}\lambda = 1 \), we get \( \lambda = \frac{2}{3} \) and thus each of \( x, y, z \) is \( \pm \frac{1}{\sqrt{3}} \).

For all possible signs the value of \( f(x, y, z) = \frac{1}{3} \).

**Case 2:** If exactly one of \( x, y, z \) is 0, say \( x \), then by the same argument we get \( y^2 = z^2 = \frac{1}{2} \). Then \( y^2 + z^2 = \lambda = 1 \), so each of \( y, z \) is \( \pm \frac{1}{\sqrt{2}} \) and \( f(x, y, z) = \frac{1}{2} \).

**Case 3:** If exactly two of \( x, y, z \) is 0, say \( x \) and \( y \), then using \( x^2 + y^2 + z^2 = z^2 = 1 \) we have \( z = \pm 1 \), so \( f(x, y, z) = 1 \).

So the maximum is 1 and minimum is \( \frac{1}{3} \).

2. (Exercise 17) Find the minimum/maximum of \( f(x, y, z) = x + y + z \) subject to \( x^2 + z^2 = 2 \), \( x + y = 1 \).

**Solution using Lagrange multipliers:** The gradient equation gives

\[ 1 = 2\lambda x + \mu, 1 = \mu, 1 = 2\lambda z. \]

The third equation forces \( \lambda \neq 0 \). Plugging \( \mu = 1 \) into the first equation, we get \( 1 = 2\lambda x + 1 \) or \( 0 = 2x \) and deduce that since \( \lambda \neq 0 \), we must have \( x = 0 \). Then the constraint \( x + y = 1 \) gives \( y = 1 \) and the constraint \( x^2 + z^2 = 2 \) gives \( z = \pm \sqrt{2} \). The values of \( f(x, y, z) \) on these two points are \( 1 \pm \sqrt{2} \).

**Alternative solution:** Parametrize the intersection by \( \vec{r}(t) = (\sqrt{2}\cos t, 1 - \sqrt{2}\cos t, \sqrt{2}\sin t) \). Then \( f(\vec{r}(t)) = \sqrt{2}\cos t + (1 - \sqrt{2}\cos t) + \sqrt{2}\sin t = 1 + \sqrt{2}\sin t \). It follows that the maximum is \( 1 + \sqrt{2} \) and the minimum is \( 1 - \sqrt{2} \).

3. (Exercise 20) Find the minimum/maximum of \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to \( x - y = 1 \) and \( y^2 - z^2 = 1 \).

**Solution:** The gradient equation gives

\[ 2x = \lambda, 2y = -\lambda + 2\mu y, 2z = -2\mu z. \]

**Case 1:** If \( z = 0, y^2 - z^2 = 1 \) implies \( y = \pm 1 \) and from \( x - y = 1 \) we get the points \( (2, 1, 0) \) and \( (0, -1, 0) \) with values \( f(x, y, z) = 5 \) and 1.

**Case 2:** If \( z \neq 0 \), the third equation gives \( \mu = -1 \). Then the second equation gives \( 2y = -\lambda - 2y \) or \( y = -\frac{\lambda}{4} \). Then plugging \( x = \frac{\lambda}{2} \) into \( x - y = 1 \) gives \( \frac{\lambda}{2} + \frac{\lambda}{4} = 1 \) or \( \lambda = \frac{4}{3} \). This gives \( x = \frac{2}{3}, y = -\frac{1}{3} \) But this value of \( y \) is inconsistent with \( y^2 - z^2 = 1 \), so we have no additional points to consider.

So the maximum is 5 and minimum is 1.
4. (Exercise 22) Find the minimum/maximum of \( f(x, y) = 2x^2 + 3y^2 - 4x - 5 \) when \( x^2 + y^2 \leq 16 \).

We can look for extrema separately when \( x^2 + y^2 < 16 \) and \( x^2 + y^2 = 16 \). For the former, we have \( f_x(x, y) = 4x - 4 \) and \( f_y(x, y) = 6y \), so the only critical point is \((1, 0)\) with value \( f(1, 0) = -7 \). For the latter we use Lagrange multipliers with the constraint \( x^2 + y^2 = 16 \). We get the equations

\[
4x - 4 = 2\lambda x, 6y = 2\lambda y.
\]

**Case 1:** If \( y = 0 \), then \( x = \pm 4 \), which gives values \( f(4, 0) = 11 \) and \( f(-4, 0) = 43 \).

**Case 2:** If \( y \neq 0 \), it follows from \( 6y = 2\lambda y \) that \( \lambda = 3 \). Then \( 4x - 4 = 6x \), so \( x = -2 \) and \( y = \pm \sqrt{12} \).

We have \( f(-2, \pm \sqrt{12}) = 47 \).

So the minimum is \(-7\) and the maximum is \(47\).

5. Find the minimum possible distance from the point \((4, 0, 0)\) to a point on the surface \( x^2 + y^2 - z^2 = 1 \).

**Solution:** We can just minimize the squared distance \( f(x, y, z) = (x - 4)^2 + y^2 + z^2 \) subject to the constraint \( g(x, y, z) = x^2 + y^2 - z^2 = 1 \) and then take the square root.

We have

\[
\nabla f(x, y, z) = (2(x - 4), 2y, 2z) = (2\lambda x, 2\lambda y, -2\lambda z) = \lambda \nabla g(x, y, z).
\]

Notice that the second component gives \( 2y = 2\lambda y \). So it is natural to break into cases based on whether \( \lambda = 1 \) or not.

**Case 1:** If \( \lambda = 1 \), then the first component gives \( 2(x - 4) = 2x \), which leads to \(-8 = 0\), a contradiction, so this case is not possible.

**Case 2:** If \( \lambda \neq 1 \), the equation \( 2y = 2\lambda y \) forces \( y = 0 \). Now consider the third component, \( 2z = -2\lambda z \). This equation makes it natural to consider cases based on whether \( \lambda = -1 \) or not.

**Subcase 2a:** If \( \lambda = -1 \), then the first component gives \( 2(x - 4) = -2x \) or \( x = 2 \). The constraint \( g(x, y, z) = 2^2 + 0^2 - z^2 = 1 \), so \( z = \pm \sqrt{3} \). This yields the points \((2, 0, \pm 3)\), which have a distance of \(\sqrt{7} \) from \((4, 0, 0)\).

**Subcase 2b:** If \( \lambda \neq -1 \), then the equation \( 2z = -2\lambda z \) forces \( z = 0 \). So \( g(x, y, z) = x^2 + 0^2 - 0^2 = 1 \), so \( x = \pm 1 \). The distances of \((1, 0, 0)\) and \((-1, 0, 0)\) from \((4, 0, 0)\) are respectively 3 and 5, which are larger than \(\sqrt{7} \).

So the distance is \(\sqrt{7} \).