

# Dynamic and Static Hedging of Exotic Equity Options

Presentation for Course  
at Columbia University

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Initial version: February 10, 1999

File reference: overheads99.tex

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# Part I

## Review of Dynamic Hedging

# I Review of Dynamic Hedging of Path-Independent Derivatives

- Let  $S_t$  denote the dollar price at time  $t$  of an underlying stock.
- We focus attention on derivative securities which have a specified final dollar payout  $f(S_T)$  paid at a fixed time  $T$ , and which also have a specified intermediate dollar payout  $i(S_t, t)$  paid at every  $t \in [0, T]$ .
- Consider the problem of dynamically hedging the sale of such a claim under the following assumptions:

1. Frictionless markets
2. No arbitrage
3. Constant interest rate  $r$
4. Underlying pays a constant proportional dividend continuously over time:

$$\frac{\text{\$ amount of dividend over } [t, t + dt]}{dt} = \delta S_t,$$

where  $\delta$  is a non-negative constant.

5. Continuous spot price process:

$$\frac{dS_t}{S_t} = m_t dt + \sigma_t dW_t, \quad t \in [0, T],$$

where the mean growth rate process  $m_t$  is adapted and the volatility process  $\sigma_t$  is a function of  $S_t$  and  $t$  only, i.e., there is a function  $\sigma$  such that:

$$\sigma_t = \sigma(S_t, t).$$

- We also assume that  $m_t$  and  $\sigma(S, t)$  are chosen so as to prevent negative prices and explosions.

## I-A Representing the Payoffs

- Itô's lemma applied to the function  $V(S_t, t)e^{r(T-t)}$  gives:

$$V(S_T, T) = V(S_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) dS_t + \int_0^T e^{r(T-t)} \left[ \frac{\sigma^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) - rV(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right] dt.$$

- The 1st term is a constant, while the 2nd is a stochastic integral. Thus, the 1st term can be created by depositing  $V(S_0, 0)$  in the bank and the 2nd term accumulates gains on  $\frac{\partial V}{\partial S}(S_t, t)$  shares held at each  $t \in [0, T]$ .
- However, long positions in stock are costly. If we borrow to finance the stock position, then gains from the stock are reduced by the carrying cost as follows:

$$V(S_T, T) = V(S_0, 0)e^{r(T-t)} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt] + \int_0^T e^{r(T-t)} \left[ \frac{\sigma^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) + (r - \delta)S_t \frac{\partial V}{\partial S}(S_t, t) - rV(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right] dt.$$

- Now, by choosing  $V(S, t)$  to solve the following fundamental PDE:

$$\frac{\sigma^2(S, t) S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + (r - \delta)S \frac{\partial V}{\partial S}(S, t) - rV(S, t) + \frac{\partial V}{\partial t}(S, t) = -i(S, t),$$

with:

$$V(S, T) = f(S),$$

we get  $f(S_T) + \int_0^T e^{r(T-t)} i(S_t, t) dt$

$$= V(S_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt].$$

## Representing the Payoffs (con'd)

- Recall the representation of the final and intermediate payoffs:

$$\begin{aligned} f(S_T) + \int_0^T e^{r(T-t)} i(S_t, t) dt \\ = V(S_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt]. \end{aligned}$$

- Thus the final and intermediate payoffs are the sum of:
  1. the future value of the initial investment  $V(S_0, 0)$  and
  2. the accumulated gains from holding  $\frac{\partial V}{\partial S}(S_t, t)$  shares, where all purchases are financed by borrowing and all sales are invested in the bank.
- It follows that the fair value of the payoff is  $V(S_0, 0)$ .
- The initial lending is  $V(S_0, 0) - \frac{\partial V}{\partial S}(S_0, 0)S_0$  (which may be negative). At any time  $t$ , the lending must be  $V(S_t, t) - \frac{\partial V}{\partial S}(S_t, t)S_t$ . This is proven for the special case of the Black Scholes model with zero intermediate payouts in Appendix 1.



## I-B Examples

- All of our examples will assume constant volatility:

$$\sigma^2(S, t) = \sigma^2.$$

Thus, we are assuming the validity of the Black-Scholes model. If the drift were also constant, then the stock price would follow geometric Brownian motion.

### Example 1: Butterfly Spread

- Assume no intermediate payouts and that the final payoff is:

$$f(S) = \delta(S - K),$$

where  $\delta(\cdot)$  is Dirac's delta function.

- In this case, the fair value  $V(S, t)$  must solve:

$$\frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + (r - \delta)S \frac{\partial V}{\partial S}(S, t) - rV(S, t) + \frac{\partial V}{\partial t}(S, t) = 0,$$

with the terminal condition:

$$V(S, T) = \delta(S - K).$$

- Let  $\hat{V}$  be the forward price of the derivative:

$$\hat{V}(S, T) = e^{r(T-t)}V(S, t).$$

- Then  $\hat{V}$  must satisfy:

$$\frac{\sigma^2 S^2}{2} \frac{\partial^2 \hat{V}}{\partial S^2}(S, t) + (r - \delta)S \frac{\partial \hat{V}}{\partial S}(S, t) + \frac{\partial \hat{V}}{\partial t}(S, t) = 0,$$

with the same terminal condition

$$\hat{V}(S, T) = \delta(S - K).$$

Notice that we have eliminated the potential term  $-rV(S, t)$  from the PDE.

### Butterfly Spread Valuation (con'd)

- Recall the PDE for the forward price of the butterfly spread:

$$\frac{\sigma^2 S^2}{2} \frac{\partial^2 \hat{V}}{\partial S^2}(S, t) + (r - \delta) S \frac{\partial \hat{V}}{\partial S}(S, t) + \frac{\partial \hat{V}}{\partial t}(S, t) = 0,$$

and the terminal condition:

$$\hat{V}(S, T) = \delta(S - K).$$

- Now let us change the spatial independent variable as:

$$\begin{aligned} x &= \ln S \\ u(x, t) &= \hat{V}(S, t). \end{aligned}$$

- Then the PDE for  $u(x, t)$  is:

$$\frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}(x, t) + \mu \frac{\partial u}{\partial x}(x, t) + \frac{\partial u}{\partial t}(x, t) = 0,$$

where  $\mu \equiv r - \delta - \frac{\sigma^2}{2}$ , and its terminal condition is:

$$u(x, T) = \frac{1}{K} \delta(x - \ln K),$$

where the division by  $K$  is a consequence of the requirement that the delta function in  $x$  integrates to 1.

### Butterfly Spread Valuation (con'd)

- Recall that the PDE for the forward price of the butterfly spread written as a function of  $x = \ln S$  is:

$$\frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}(x, t) + \mu \frac{\partial u}{\partial x}(x, t) + \frac{\partial u}{\partial t}(x, t) = 0,$$

where  $\mu \equiv r - \delta - \frac{\sigma^2}{2}$ , and its terminal condition is:

$$u(x, T) = \frac{1}{K} \delta(x - \ln K).$$

- Recognizing this PDE as the Kolmogorov backward equation for arithmetic Brownian motion with constant drift rate  $\mu$  and constant diffusion rate  $\sigma$ , we can write the solution as:

$$u(x, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}K} \exp \left\{ -\frac{1}{2} \left[ \frac{\ln K - (x + \mu(T-t))}{\sigma\sqrt{T-t}} \right]^2 \right\}.$$

- Then:

$$\hat{V}(S, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}K} \exp \left\{ -\frac{1}{2} \left[ \frac{\ln K - (\ln S + \mu(T-t))}{\sigma\sqrt{T-t}} \right]^2 \right\}$$

$$V(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}K} \exp \left\{ -\frac{1}{2} \left[ \frac{\ln K - (\ln S + \mu(T-t))}{\sigma\sqrt{T-t}} \right]^2 \right\}.$$

- Notice that  $\hat{V}(S, t)$  is a lognormal density function and that  $V(S, t)$  is the Green's function for the fundamental PDE, which we started with.

**Example 2: Binary Call**

- A binary call has no intermediate payouts and has a final payoff which can be written as:

$$f(S) = 1(S > K) = \int_K^\infty \delta(S - L)dL.$$

- From the integral representation of the indicator function and the previous example, we easily get the price of the binary call:

$$V(S, t) = \int_K^\infty \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}L} \exp\left(-\frac{1}{2}\left[\frac{\ln L - (\ln S + \mu(T-t))}{\sigma\sqrt{T-t}}\right]^2\right)dL,$$

where  $\mu \equiv r - \delta - \frac{\sigma^2}{2}$ .

- Let:

$$d_2(L) \equiv \frac{\ln\left(\frac{S}{L}\right) + \mu(T-t)}{\sigma\sqrt{T-t}}$$

be a standardizing transformation. Then the fair value of the binary call becomes:

$$V(S, t) = e^{-r(T-t)}N(d_2(K)),$$

where  $N(x)$  is the distribution function of a standard normal random variable.

**Example 3: Plain Vanilla Call**

- A plain vanilla call has no intermediate payouts and has a final payoff which can be written as the integral of the binary call:

$$f(S) = (S - K)^+ = \int_K^\infty 1(S - L)dL.$$

- Using integration by parts,

$$\begin{aligned} V(S, t) &= \int_K^\infty e^{-r(T-t)} N(d_2(L)) dL \\ &= e^{-r(T-t)} \left[ LN(d_2(L)) \Big|_K^\infty - \int_K^\infty LN'(d_2(L)) dL \right] \\ &= e^{-r(T-t)} \left[ -KN(d_2(K)) - \int_K^\infty Se^{(r-\delta)(T-t)} N'(d_1(L)) dL \right], \end{aligned}$$

since by completing the square  $LN'(d_2(L)) = Se^{(r-\delta)(T-t)} N'(d_1(L))$ , where:

$$d_1(L) = d_2(L) + \sigma\sqrt{T-t}.$$

- Rewriting the last integral in terms of the standard normal distribution function, we get the (award-winning!) Black-Scholes formula:

$$V(S, t) = -Ke^{-r(T-t)} N(d_2(K)) + Se^{-\delta(T-t)} N(d_1(K)).$$

## I-C The Martingale Measure

- Recall that the cost of creating  $\int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt]$  paid at  $T$  was zero, given that  $V$  satisfied the fundamental PDE.
- Viewed as a process in  $T$ , the absence of arbitrage clearly requires that this stochastic integral have realizations on both sides of zero for all  $T$  (or else be zero).
- Consequently, one can define a measure  $Q^\$$  such that the integral has zero mean under  $Q^\$$  for all  $T$ .
- Since the integral is a  $Q^\$$ -martingale by the definition of  $Q^\$$ ,  $Q^\$$  is called a martingale measure.
- Given our assumptions and given the initial prices of the stock and bond, this martingale measure is uniquely determined.
- Under  $Q^\$$ , the integrator  $dS_t - (r - \delta)S_t dt$  has zero mean and has variance  $\sigma^2 S_t^2 dt$ . Consequently, there exists a unique  $Q^\$$ -Brownian motion  $W_t^\$$  such that:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t^\$, \quad t \in [0, T], \text{ where } S_0 = S.$$

## Risk-Neutral Stock Price Process

- Recall that the absence of arbitrage has allowed us to define a unique martingale measure  $Q^\$$  and a unique standard Brownian motion  $\{W_t^\$; t \in [0, T]\}$  such that:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t^\$, \quad t \in [0, T], \text{ where } S_0 = S.$$

- The drift of this process is simply the cost of carrying the underlying and has no greater significance. By sheer coincidence, it is also the drift which would arise in equilibrium if all investors were risk-neutral, and for this reason, the process is also called the “risk-neutral” process. The martingale measure is also called the risk-neutral measure. These are terribly mis-leading terms, since we are definitely *not* assuming that investors are risk-neutral.
- The volatility of the risk-neutral process is the same as the volatility of the assumed process. This arises whenever one has continuous sample paths, but does *not* necessarily follow when prices can jump.
- The risk-neutral process simply tells us the market’s unique arbitrage-free forward price for delta function type payoffs defined over various path bundles.

## Risk-Neutral Valuation

- The examples hopefully made it clear that the key to valuing derivatives with no intermediate payouts and any final (path-independent) payoff was to find the value of a butterfly spread.
- The forward price of a butterfly spread with payoff  $\delta(S_T - K)$  at  $T$  is the risk-neutral density of the terminal stock price. Thus:

$$V(S, t) = e^{-r(T-t)} \int_0^\infty f(K) Q^\$ \{S_T \in dK | S_t = S\}.$$

- Note that the payoff  $f(K)$  is unitless, while  $Q^\$$  is measured in time  $T$  dollars.
- When the payoffs are possibly path-dependent (eg. for barrier options), we can write:

$$V_t = e^{-r(T-t)} E^{Q^\$} \{V_T | \mathcal{F}_t\}.$$

- This says that to value a path-dependent derivative, we first determine the forward price of each path from  $Q^\$$  and then we determine the payout along each path from  $V_T$ . The value is given by multiplying the payout along each path by its price and then summing (integrating) over paths.
- For example, given that we are at time  $t$  with the stock price at  $S$ , the forward price of the security paying  $\delta(S_T - K)$  at  $T$  is simply the sum(integral) of the forward prices of all securities paying off if a given path occurs, where each such path starts at  $(t, S)$  and ends at  $(T, K)$ . The total measure of this path bundle is well-known to be:

$$Q\{S_T \in dK | S_t = S\} = \frac{dK}{\sqrt{2\pi\sigma^2(T-t)}K} \exp \left\{ -\frac{1}{2} \left[ \frac{\ln(K/S) - \mu(T-t)}{\sigma\sqrt{T-t}} \right]^2 \right\}$$



## II Finding Payoffs for Given Value Functions

- The last section showed how to hedge and value a derivative given the intermediate and final payoffs. In this section, we turn things around and find the payoffs corresponding to a given value function  $V(S, t)$  and corresponding hedge  $\frac{\partial V}{\partial S}(S, t)$ .
- Suppose we are given the value function  $V(S, t)$  corresponding to a European-style claim with zero intermediate payoff  $i(S, t)$  and an unknown final payoff  $f(S)$  paid at  $T$ .
- To recover the final payoff, recall that  $V(S, T) = f(S)$ . Thus, all we need do is set  $t = T$  in the value function and the final payoff emerges.
- For example, if we are given that  $V(S, t) = e^{-r(T-t)}N(d_2(K))$  is the value function, then by setting  $t = T$ , we see that the corresponding final payoff is  $1(S > K)$ .

## II-A Stationary Value Functions

- Now suppose that the given value function depends only on the stock price  $V(S, t) = V(S)$ . What can we conclude about its payoffs?
- Since when  $t = T$ ,  $V(S, T) = V(S) = f(S)$ , we conclude that the final payoff must be the same function of the stock price as its value.
- To find the intermediate payoff, we apply Itô's lemma to the function  $f(S_t)e^{r(T-t)}$ :

$$f(S_T) = f(S_0)e^{rT} + \int_0^T e^{r(T-t)} f'(S_t) dS_t + \int_0^T e^{r(T-t)} \left[ \frac{\sigma^2(S_t, t) S_t^2}{2} f''(S_t) - r f(S_t) \right] dt.$$

- Once again, subtracting and adding the carrying cost of the underlying gives:

$$f(S_T) = f(S_0)e^{rT} + \int_0^T e^{r(T-t)} f'(S_t) [dS_t - (r - \delta) S_t dt] + \int_0^T e^{r(T-t)} \left[ \frac{\sigma^2(S_t, t) S_t^2}{2} f''(S_t) + (r - \delta) S_t f'(S_t) - r f(S_t) \right] dt.$$

- Finally, bringing the last term to the LHS implies that a derivative with intermediate payoffs of  $r f(S_t) - (r - \delta) S_t f'(S_t) - \frac{\sigma^2(S_t, t) S_t^2}{2} f''(S_t)$  at each  $t \in (0, T)$  and a final cash flow of  $f(S_T)$  at  $T$  can be dynamically hedged:

$$f(S_T) + \int_0^T e^{r(T-t)} \left[ r f(S_t) - (r - \delta) S_t f'(S_t) - \frac{\sigma^2(S_t, t) S_t^2}{2} f''(S_t) \right] dt = f(S_0)e^{rT} + \int_0^T e^{r(T-t)} f'(S_t) [dS_t - (r - \delta) S_t] dt.$$

## II-B Examples of Stationary Securities

- Both of our examples will again assume constant volatility:

$$\sigma^2(S, t) = \sigma^2.$$

### Example 1: Logger

- Consider a derivative security whose value  $X_t = \ln S_t, t \in [0, T]$ .
  - The logger has initial value  $X_0 = \ln S_0$ .
  - The intermediate payoffs are linear in the value:

$$rf(S_t) - (r - \delta)S_t f'(S_t) - \frac{\sigma^2 S_t^2}{2} f''(S_t) = r \ln S_t - (r - \delta - \sigma^2/2).$$

- The final payoff is  $X_T = \ln S_T$ .
- Note that since the intermediate and final payoffs can be negative, the initial value can also be negative.

## Examples of Stationary Securities(con'd)

### Example 2: Power Plays

- Consider the family of derivative securities with value  $\Pi_t(p) = S_t^p, t \in [0, T]$ , where  $p$  is any real number.
  - Power plays have initial value  $\Pi_0(p) = S_0^p$ .
  - The intermediate payoffs are proportional to the value:

$$rf(S_t) - (r - \delta)S_t f'(S_t) - \frac{\sigma^2 S_t^2}{2} f''(S_t) = q(p)S^p,$$

$$\text{where } q(p) \equiv r - (r - \delta - \sigma^2/2)p - \frac{\sigma^2 p^2}{2}.$$

- The final payoff is  $\Pi_T(p) = S_T^p$ .
- When  $p = 0$ , a power play is a par bond,  $\Pi_t(0) = 1$ , while when  $p = 1$ , a power play is a stock,  $\Pi_t(1) = S_t$ .
- For all  $p$ , the intermediate and final payoffs are non-negative, and thus so is the value. The delta  $\frac{\partial \Pi_t(p)}{\partial S}$  has the same sign as  $p$ , while gamma  $\frac{\partial^2 \Pi_t(p)}{\partial S^2}$  has the same sign as  $p(p - 1)$ .

## II-C The Risk-Neutral Process of a Stationary Security

- Recall that the risk-neutral process for a stock is:

$$dS_t = (rS_t - \delta S_t)dt + \sigma(S_t, t)S_t dW_t^\$, \quad t \in [0, T], \text{ where } S_0 = S.$$

- The risk-neutral drift grows by  $rS_t$  because the stock position is financed by borrowing at the riskless rate. The risk-neutral drift drops by  $\delta S_t$  because these dividends reduce the carrying cost.
- More generally, the risk-neutral process for a stationary security is:

$$\begin{aligned} df(S_t) &= \left\{ rf(S_t) - \left[ rf(S_t) - (r - \delta)S_t f'(S_t) - \frac{\sigma^2(S_t, t)S_t^2}{2} f''(S_t) \right] \right\} dt \\ &\quad + f'(S) \sigma(S_t, t) S_t dW_t^\$, \\ &= \left[ (r - \delta)S_t f'(S_t) + \frac{\sigma^2(S_t, t)S_t^2}{2} f''(S_t) \right] dt + f'(S) \sigma(S_t, t) S_t dW_t^\$. \end{aligned}$$

- For example, in the Black Scholes model, the risk-neutral process for the logger  $X_t = \ln S_t$  is:

$$dX_t = (r - \delta - \sigma^2/2)dt + \sigma dW_t^\$, \quad t \in [0, T],$$

which has constant absolute drift and volatility.

- To take another example, in the Black Scholes model, the risk-neutral process for a power play  $\Pi_t(p) = S_t^p$  is:

$$\frac{d\Pi_t(p)}{\Pi_t(p)} = \left[ (r - \delta - \sigma^2/2)p + \frac{\sigma^2 p^2}{2} \right] dt + p\sigma dW_t^\$, \quad t \in [0, T],$$

which has constant relative drift and volatility.

### III Review of Dynamic Hedging of Quanto Derivatives

- We generalize the previous analysis fairly significantly by now considering the case where the payoff currency of the derivative is different from the currency describing the price  $S_t$  of the underlying stock.
- For example, we will consider the case where the payoff currency of the derivative is in pounds, while the underlying stock is denominated in dollars.

#### III-A Assumptions

1. Frictionless markets
2. No arbitrage
3. Constant interest rates  $r_{\pounds}$  and  $r_{\$}$
4. Underlying stock has a constant dividend yield  $\delta$
5. Continuous underlying stock price process  $S_t$  and (spot) exchange rate process  $R_t$  (in dollars per pound):

$$\begin{aligned}\frac{dS_t}{S_t} &= m_t^s dt + \sigma_s(S_t, t) dW_{1t} \\ \frac{dR_t}{R_t} &= m_t^r dt + \sigma_r(t) [\rho(S_t, t) dW_{1t} + \sqrt{1 - \rho^2(S_t, t)} dW_{2t}],\end{aligned}$$

where  $W_1$  and  $W_2$  are independent standard Brownian motions.

### III-B Representing the Payoff

- Apply Itô's lemma to  $V(S_t, t)e^{r\ell(T-t)}$  to get:

$$\begin{aligned} V(S_T, T) &= V(S_0, 0)e^{r\ell T} + \int_0^T e^{r\ell(T-t)} \frac{\partial V}{\partial S}(S_t, t) dS_t \\ &+ \int_0^T e^{r\ell(T-t)} \left[ \frac{\sigma_s^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) - r\ell V(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right] dt. \end{aligned}$$

- Since the stock trades only in dollars, the gain in pounds from holding one share of the stock over  $[t, t + dt]$  is:

$$\begin{aligned} \frac{dS_t}{R_{t+dt}} &= \frac{dS_t}{R_t + dR_t} \\ &= \frac{1}{R_t} \frac{1}{1 + \frac{dR_t}{R_t}} dS_t \\ &\approx \frac{1}{R_t} \left( 1 - \frac{dR_t}{R_t} \right) dS_t \\ &= \frac{1}{R_t} dS_t - \frac{1}{R_t} \sigma_{rs}(S_t, t) S_t dt, \end{aligned}$$

where  $\sigma_{rs}(S_t, t)$  is the covariance of  $dR/R$  and  $dS/S$ .

- Substituting  $dS_t = R_t \frac{dS_t}{R_{t+dt}} + \sigma_{rs}(S_t, t) S_t dt$  in the top equation, we get:

$$\begin{aligned} V(S_T, T) &= V(S_0, 0)e^{r\ell T} \\ &+ \int_0^T e^{r\ell(T-t)} \frac{\partial V}{\partial S}(S_t, t) R_t \frac{dS_t}{R_{t+dt}} + \\ &\int_0^T e^{r\ell(T-t)} \left\{ \frac{\sigma_s^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) + \sigma_{rs}(S_t, t) S_t \frac{\partial V}{\partial S}(S_t, t) \right. \\ &\left. - r\ell V(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right\} dt. \end{aligned}$$

- Recall:

$$\begin{aligned}
V(S_T, T) &= V(S_0, 0)e^{r_{\pounds}T} \\
&+ \int_0^T e^{r_{\pounds}(T-t)} \frac{\partial V}{\partial S}(S_t, t) R_t \frac{dS_t}{R_{t+dt}} + \\
&\int_0^T e^{r_{\pounds}(T-t)} \left\{ \frac{\sigma_s^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) + \sigma_{rs}(S_t, t) S_t \frac{\partial V}{\partial S}(S_t, t) \right. \\
&\left. - r_{\pounds} V(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right\} dt.
\end{aligned}$$

- Make a further adjustment so that the second term represents the gains from a zero-cost self-financing strategy:

$$\begin{aligned}
V(S_T, T) &= V(S_0, 0)e^{r_{\pounds}T} \\
&+ \int_0^T e^{r_{\pounds}(T-t)} \frac{\partial V}{\partial S}(S_t, t) R_t \left[ \frac{dS_t}{R_{t+dt}} - (r_{\$} - \delta) \frac{S_t}{R_t} dt \right] \\
&+ \int_0^T e^{r_{\pounds}(T-t)} \left\{ \frac{\sigma_s^2(S_t, t) S_t^2}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) + [r_{\$} - \delta + \sigma_{rs}(S_t, t)] S_t \frac{\partial V}{\partial S}(S_t, t) \right. \\
&\left. - r_{\pounds} V(S_t, t) + \frac{\partial V}{\partial t}(S_t, t) \right\} dt.
\end{aligned}$$

- Choose  $V(S, t)$  to solve the following generalized fundamental PDE:

$$\begin{aligned}
&\frac{\sigma_s^2(S, t) S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + [r_{\$} - \delta + \sigma_{rs}(S, t)] S \frac{\partial V}{\partial S}(S, t) - r_{\pounds} V(S, t) \\
&+ \frac{\partial V}{\partial t}(S, t) = -i(S, t), \text{ with } V(S, T) = f(S).
\end{aligned}$$

- Then we get  $f(S_T) + \int_0^T e^{r_{\pounds}(T-t)} i(S_t, t) dt$

$$= V(S_0, 0)e^{r_{\pounds}T} + \int_0^T e^{r_{\pounds}(T-t)} \frac{\partial V}{\partial S}(S_t, t) R_t \left[ \frac{dS_t}{R_{t+dt}} - (r_{\$} - \delta) \frac{S_t}{R_t} dt \right].$$

- In this case, the dynamic strategy is to hold  $\frac{\partial V}{\partial S}(S_t, t) R_t$  shares of the underlying stock at each  $t \in [0, T]$ , financed by borrowing in dollars and with gains converted into pounds.



### III-C Examples

- Again we assume that the volatilities and the covariance are constant.

**Example 1: Butterfly Spread**

- Consider no intermediate payoffs and the Dirac final payoff, where the underlying stock is denominated in dollars and the payoff is quantosed into pounds:

$$f(S) = \delta(S - K).$$

- Compare the PDE with no quantoing:

$$\frac{\sigma_s^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + (r_{\$} - \delta)S \frac{\partial V}{\partial S}(S, t) - r_{\$}V(S, t) + \frac{\partial V}{\partial t}(S, t) = 0,$$

to the PDE with quantoing:

$$\frac{\sigma_s^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + (r_{\$} - \delta + \sigma_{rs})S \frac{\partial V}{\partial S}(S, t) - r_{\pounds}V(S, t) + \frac{\partial V}{\partial t}(S, t) = 0.$$

- If we used “risk-neutral valuation”, we would have an additional term  $\sigma_{rs}$  in the drift and we would have a new discount rate  $r_{\pounds}$ .
- Recall the solution for the non-quantosed butterfly spread:

$$BS(S, t) = \frac{e^{-r_{\$}(T-t)}}{\sqrt{2\pi\sigma_s^2(T-t)}K} \exp \left\{ -\frac{1}{2} \left[ \frac{\ln(K/S) - \mu(T-t)}{\sigma_s\sqrt{T-t}} \right]^2 \right\},$$

where  $\mu \equiv (r_{\$} - \delta - \frac{\sigma_s^2}{2})$ .

- Thus, the quantosed butterfly spread value is:

$$QBS(S, t) = \frac{e^{-r_{\pounds}(T-t)}}{\sqrt{2\pi\sigma_s^2(T-t)}K} \exp \left\{ \frac{-1}{2} \left[ \frac{\ln(K/S) - (\mu + \sigma_{rs})(T-t)}{\sigma_s\sqrt{T-t}} \right]^2 \right\}.$$

**Example 2: Quanto Call**

- Consider no intermediate payoff and a final payoff given by a vanilla call on a US stock quantoed into pounds:

$$f(S) = (S - K)^+.$$

- Note that  $f(S)$  is measured in pounds and not dollars.
- The Black Scholes formula for a call whose payoff and underlying are both denominated in dollars can be written as:

$$C(S, t) = e^{-r_{\$}(T-t)} [S e^{(r_{\$} - \delta)(T-t)} N(d_1(K)) - K N(d_2(K))],$$

where:

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + (r_{\$} - \delta - \frac{\sigma_s^2}{2})(T-t)}{\sigma_s \sqrt{T-t}}, \quad d_1 = d_2 + \sigma_s \sqrt{T-t}.$$

- Adjusting the drift and the discount rate appropriately gives the quanto call value:

$$\begin{aligned} QC(S, t) &= e^{-r_{\pounds}(T-t)} [S e^{(r_{\$} - \delta + \sigma_{rs})(T-t)} N(\tilde{d}_1(K)) - K N(\tilde{d}_2(K))] \\ &= S e^{(r_{\$} - r_{\pounds} - \delta + \sigma_{rs})(T-t)} N(\tilde{d}_1(K)) - K e^{-r_{\pounds}(T-t)} N(\tilde{d}_2(K)), \end{aligned}$$

where:

$$\tilde{d}_2 = \frac{\ln\left(\frac{S}{K}\right) + (r_{\$} - \delta + \sigma_{rs} - \frac{\sigma_s^2}{2})(T-t)}{\sigma_s \sqrt{T-t}} \quad \tilde{d}_1 = \tilde{d}_2 + \sigma_s \sqrt{T-t}.$$

- This quantoed call value could also have been obtained by integrating the quantoed butterfly spread value twice with respect to strike.
- Note that the standard call formula is obtained from the more general quantoed call formula by setting the exchange rate  $R = 1$  and  $r_{\pounds} = r_{\$}$ .

### III-D Risk-Neutral Valuation of Quantoed Derivatives

- Recall the risk-neutral valuation formula for a path-independent derivative, when the payoff and underlying are both denominated in dollars:

$$V(S, t) = e^{-r_{\$}(T-t)} E^{Q^{\$}}[f(S_T)|S_t = S] \equiv e^{-r_{\$}(T-t)} E_{S,t}^{Q^{\$}} f(S_T),$$

where under the risk-neutral measure  $Q^{\$}$ , the risk-neutral stock price process is:

$$\frac{dS_u}{S_u} = (r_{\$} - \delta)du + \sigma(S_u, u)dW_u^{\$}, \quad u \in [t, T], \text{ where } S_t = S.$$

- Also recall that  $f$  is unitless, while  $Q^{\$}$  is measured in time  $T$  dollars.
- If the payoff  $f$  is quantoed into pounds, the dollar value of the derivative changes. If we wish to continue using American forward prices of paths (i.e.  $Q^{\$}$ ), then the change in dollar value arises from a change in the magnitude of the payoff:

$$QV(S, t)R_0 = e^{-r_{\$}(T-t)} E_{S,t}^{Q^{\$}} R_T f(S_T).$$

Here,  $QV$  is in pounds,  $R_T f(S_T)$  is unitless, while  $Q^{\$}$  continues to be measured in time  $T$  dollars.

- Alternatively, one can use British forward prices of the dollar denominated stock price paths. Denoting these forward prices by  $Q_{\$}^{\pounds}$ , quantoed values are obtained by:

$$QV(S, t) = e^{-r_{\pounds}(T-t)} E_{S,t}^{Q_{\$}^{\pounds}} f(S_T),$$

where under our new risk-neutral measure  $Q_{\$}^{\pounds}$ , the risk-neutral stock price process is:

$$\frac{dS_u}{S_u} = (r_{\$} - \delta + \sigma_{rs})du + \sigma(S_u, u)dW_{\$,u}^{\pounds}, \quad u \in [t, T], \text{ where } S_t = S.$$

## Risk-Neutral Valuation of Quantoed Derivatives(con'd)

- Recall the two approaches for valuing a quantoed derivative:

$$QV(S, t) = e^{-r_{\$}(T-t)} E_{S,t}^{Q_{\$}} \frac{R_T}{R_0} f(S_T),$$

$$QV(S, t) = e^{-r_{\pounds}(T-t)} E_{S,t}^{Q_{\pounds}} f(S_T).$$

- Thus, the change in the magnitude of the payoff of the *derivative* from  $f(S_T)$  to  $\frac{R_T}{R_0} f(S_T)$  is handled by keeping the payoff fixed at  $f(S_T)$  and simply changing the growth rate of the *underlying*, and changing the currency in which the premium is financed.
- The intuition for this result arises from the property of any continuous model that at any time  $t \in [0, T)$ , the dollar value of a standard derivative at  $t + dt$  is known to be *linear* in the time  $t + dt$  dollar prices of the bond and stock.

### III-E Interpreting the Standard European Call Black Scholes Formula

- First, we re-write the final payoff of the standard European call as:

$$\begin{aligned} C_T &= (S_T - K)^+ \\ &= S_T 1(S_T > K) - K 1(S_T > K). \end{aligned}$$

- In the last expression, the second term is  $K$  binary calls and its value at time 0 is  $Ke^{-r_{\$}T}N(d_2(K))$  dollars.
- To interpret the first term, we quanto the payoff into shares and adjust the magnitude of the payoff to preserve value.
- If the new payoff currency is to be shares, then the old magnitude of  $S_T 1(S_T > K)$  relevant for dollars must be changed to  $1(S_T > K)$ , since a payoff of  $S_T 1(S_T > K)$  dollars is clearly equivalent to a payoff of  $1(S_T > K)$  shares.
- Since we are now using shares instead of pounds as the “currency” we are quantoing into,  $R_t \equiv S_t$  and  $r_{\pounds} \equiv \delta$ . This makes two changes to the standard binary call valuation:
  1. New drift: since  $\sigma_{rs} = \sigma_s^2$  in this case, we add  $\sigma_s^2$  to  $r_{\$} - \delta - \frac{\sigma_s^2}{2}$ . Then  $\tilde{d}_2(K)$  turns out to be  $d_1(K)$ .
  2. New discount rate: use  $\delta$  instead of  $r_{\$}$  to discount.
- Making these changes, the value of the binary call quantoed into shares becomes  $e^{-\delta T}N(d_1(K))$  shares. Therefore, its value in dollars is  $S_0 e^{-\delta T}N(d_1(K))$ , which is the same as the first term in the Black-Scholes formula.

### III-F Quantoing the Logger in the Black Scholes Model

- Just as quantoing the payoff of a binary call into shares simplifies valuation of a vanilla call in the Black Scholes model, quantoing the payoffs of a logger into a power play will simplify valuation of a barrier option in this model.
- Setting  $\sigma(S, t) = \sigma$ , recall that the risk-neutral processes under  $Q^\$$  of the logger  $X_t = \ln S_t$  and a power play  $\Pi_t(p) = S_t^p$  are:

$$\begin{aligned} dX_t &= (r - \delta - \sigma^2/2)dt + \sigma dW_t^\$, \\ \frac{d\Pi_t(p)}{\Pi_t(p)} &= \left[ (r - \delta - \sigma^2/2)p + \frac{\sigma^2 p^2}{2} \right] dt + p\sigma dW_t^\$. \end{aligned}$$

- Consider quantoing the intermediate and final payoffs of a logger into a power play. The relative drift correction is  $\sigma_{x\pi} \equiv \text{Cov}\left(\frac{dX_t}{X_t}, \frac{d\Pi_t(p)}{\Pi_t(p)}\right)$ , so the risk-neutral price process for the logger under  $Q_\$^{\Pi(p)}$  is:

$$\begin{aligned} dX_t &= \left[ r - \delta - \sigma^2/2 + \text{Cov}\left(dX_t, \frac{d\Pi_t(p)}{\Pi_t(p)}\right) \right] dt + \sigma dW_{\$,t}^{\Pi(p)} \\ &= [r - \delta + (p - 1/2)\sigma^2]dt + \sigma dW_{\$,t}^{\Pi(p)}. \end{aligned}$$

- The power  $p$  can be chosen so that the drift vanishes:

$$r - \delta + (p - 1/2)\sigma^2 = 0 \iff p = \frac{1}{2} - \frac{r - \delta}{\sigma^2} \equiv \gamma.$$

- Thus, under  $Q^\gamma \equiv Q_\$^{\Pi(\gamma)}$ , the risk-neutral price process for the logger is arithmetic Brownian motion with no drift:

$$dX_t = \sigma dW_t^\gamma, \quad t \in [0, T], \text{ where } W_t^\gamma \equiv W_{\$,t}^{\Pi(\gamma)}.$$

- Furthermore, the discount rate under  $Q^\gamma$  is the dividend yield on the power play  $r - (r - \delta - \sigma^2/2)\gamma - \frac{\sigma^2\gamma^2}{2} = r + \frac{\sigma^2\gamma^2}{2} = \frac{\sigma^2\epsilon^2}{2}$ , where  $\epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}$ .
- We will use these results to apply the reflection principle shortly.

## Part II

# Dynamic Hedging of Barrier Options

## IV Introduction

- In general, path-dependence introduces more state variables into the PDE to be solved.
- For simplicity, I only cover up-and-in options with no intermediate payoffs, although it would be straightforward to deal with down-options, out-options, intermediate payoffs, and quantoed barrier options.
- Thus, we retain the assumptions of the first section dealing with path-independent derivatives on spot, but we now suppose that the payoff occurs only if an upper barrier has been touched before maturity.
- The additional variable we will need to keep track of in the PDE is the indicator function describing whether or not this barrier has already been hit.



## V Representing the Payoff

- Let  $V^b(S, t)$ ,  $S \in [0, H]$  and  $V^a(S, t)$ ,  $S > 0$  be functions of the stock price  $S$  and time  $t \in [0, T]$ , which will respectively give the barrier option value before and after hitting the barrier.
- Let  $\tau_S^H$  denote the first passage time of the spot  $S$  to  $H$  and let  $f_{ui}(S_T)$  denote the desired payoff at time  $T$  if  $\tau_S^H < T$ .
- For  $t > \tau_S^H$ , the barrier has already been hit, and so the valuation problem is identical to the path-independent problem:

$$V^a(S, t) = e^{-r(T-t)} E_{S,t}^{Q^{\$}} f_{ui}(S_T).$$

- Let  $\tau \equiv \tau_S^H \wedge T$  denote the smaller of the first passage time and maturity.
- Since  $\tau$  is a bounded stopping time, we can use Itô's lemma on  $V^b(S, t)e^{r(\tau-t)}$ :

$$\begin{aligned} V^b(S_\tau, \tau) &= V^b(S_0, 0)e^{r\tau} + \int_0^\tau e^{r(\tau-t)} \frac{\partial V^b}{\partial S}(S_t, t) dS_t + \\ &\int_0^\tau e^{r(\tau-t)} \left[ \frac{\sigma^2(S_t, t) S_t^2}{2} \frac{\partial^2 V^b}{\partial S^2}(S_t, t) - rV^b(S_t, t) + \frac{\partial V^b}{\partial t}(S_t, t) \right] dt. \end{aligned}$$

- If we borrow to finance long positions in the stock, then gains from the stock are reduced by the carrying cost, so:

$$\begin{aligned} V^b(S_\tau, \tau) &= V^b(S_0, 0)e^{r(\tau-t)} + \int_0^\tau e^{r(\tau-t)} \frac{\partial V^b}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt] \\ &+ \int_0^\tau e^{r(\tau-t)} \left[ \frac{\sigma^2(S_t, t) S_t^2}{2} \frac{\partial^2 V^b}{\partial S^2}(S_t, t) + (r - \delta)S_t \frac{\partial V^b}{\partial S}(S_t, t) - rV^b(S_t, t) \right. \\ &\left. + \frac{\partial V^b}{\partial t}(S_t, t) \right] dt. \end{aligned}$$

## Representing the Payoff (con'd)

- Recall:

$$\begin{aligned}
 V^b(S_\tau, \tau) &= V^b(S_0, 0)e^{r(\tau-t)} + \int_0^\tau e^{r(\tau-t)} \frac{\partial V^b}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt] \\
 &+ \int_0^\tau e^{r(\tau-t)} \left[ \frac{\sigma^2(S_t, t)S_t^2}{2} \frac{\partial^2 V^b}{\partial S^2}(S_t, t) + (r - \delta)S_t \frac{\partial V^b}{\partial S}(S_t, t) - rV^b(S_t, t) \right. \\
 &\left. + \frac{\partial V^b}{\partial t}(S_t, t) \right] dt.
 \end{aligned}$$

- Suppose we again choose  $V^b(S, t)$  to satisfy the fundamental PDE.:

$$\frac{\sigma^2(S, t)S^2}{2} \frac{\partial^2 V^b}{\partial S^2}(S, t) + (r - \delta)S \frac{\partial V^b}{\partial S}(S, t) - rV^b(S, t) + \frac{\partial V^b}{\partial t}(S, t) = 0,$$

for  $S \in (0, H), t \in (0, \tau)$  with:

$$V^b(S, T) = 0,$$

and:

$$\lim_{S \uparrow H} V^b(S, t) = h(t) \equiv e^{-r(T-t)} E_{S,t}^{Q^S} f_{ui}(S_T).$$

Then for  $\tau_S^H < T$ , we get:

$$h(\tau_h) = V^b(S_0, 0)e^{r\tau_S^H} + \int_0^{\tau_S^H} e^{r(\tau_S^H-t)} \frac{\partial V^b}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt],$$

while for  $\tau_S^H \geq T$ , we get:

$$0 = V^b(S_0, 0)e^{rT} + \int_0^T e^{r(\tau_h-t)} \frac{\partial V^b}{\partial S}(S_t, t) [dS_t - (r - \delta)S_t dt].$$

- Thus, in either case the payoff is achieved by investing  $V^b(S_0, 0)$  initially and holding  $\frac{\partial V^b}{\partial S}(S_t, t)$  shares until time  $\tau$ .

## VI A Canonical Problem

- The last section showed that valuing an up-and-in derivative with payoff  $1(\tau_S^H < T)f_{ui}(S_T)$  at  $T$  is equivalent to valuing a claim which pays  $h(u) \equiv e^{-r(T-u)}E_{h,u}^{Q^{\$}}f_{ui}(S_T)$  at  $\tau_S^H$  if  $\tau_S^H < T$  and zero otherwise.
- This section shows that we can restrict attention to the case  $h(u) = 1$ . (In the PDE literature, this is known as Duhamel's principle).
- Let  $ABC(S, t; T)$  denote the value at  $t$  of an American binary call, which pays one dollar at the first passage time to  $H$ , if this occurs before  $T$ , and pays zero otherwise. Let  $\phi(S, t; T) \equiv \frac{\partial}{\partial T}ABC(S, t; T)$  denote the value of a claim paying  $\delta(\tau_S^H - T)$  at  $T$ .
- Then the value of an up-and-in claim is given by:

$$V_{ui}(S, t) = \int_t^T \phi(S, t; u)h(u)du.$$

- Integrating by parts gives:

$$V_{ui}(S, t) = ABC(t, S; T)f(H) - \int_t^T ABC_t(u)h'(u)du,$$

since  $h(T) = f(H)$  and  $ABC(t, S; t) = 0$  for  $S < H$ .

- We next focus on valuing American binary calls and up-and-in claims in the Black-Scholes model.

## VI-A Valuing American Binary Calls in the Black-Scholes Model

- We now assume constant volatility  $\sigma^2(S, t) = \sigma^2$ .
- Recall that power plays were securities with value  $\Pi_t(p) = S_t^p, t \in [0, T], p \in \mathfrak{R}$ , and constant dividend yield:

$$q(p) \equiv r - (r - \delta - \sigma^2/2)p - \frac{\sigma^2 p^2}{2}.$$

- Setting  $q(p) = 0$  and solving for  $p$  yields

$$p_{\pm} \equiv \gamma \pm \epsilon,$$

where recall:

$$\gamma \equiv \frac{1}{2} - \frac{r - \delta}{\sigma^2} \quad \epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}.$$

- Thus, the two power plays with no intermediate payoffs and with final payoffs given by  $S_T^{\gamma+\epsilon}$  and  $S_T^{\gamma-\epsilon}$  respectively have time  $t$  values given by  $S_t^{\gamma+\epsilon}$  and  $S_t^{\gamma-\epsilon}$  respectively.
- It follows that these plays have constant value along any flat barrier  $H$ .
- As shown by the figure on the next page, the play paying  $S_T^{\gamma+\epsilon}$  has zero value at  $S = 0$ , while its cousin has infinite value. Since the ABC has zero value in the BS model, we focus on the play paying  $S_T^{\gamma+\epsilon}$  at  $T$ .
- Normalizing the payoff, the play paying  $\left(\frac{S_T}{H}\right)^{\gamma+\epsilon}$  at  $T$  has time  $t$  value  $\left(\frac{S_t}{H}\right)^{\gamma+\epsilon}$ , which vanishes when  $S_t = 0$  and is unity when  $S_t = H$ .

## Valuing American Binary Calls in the Black-Scholes Model(con'd)

- Recall that the power play with time  $t$  value  $\left(\frac{S_t}{H}\right)^{\gamma+\epsilon}$  vanishes when  $S_t = 0$  and is unity when  $S_t = H$ .
- Unfortunately, this play has value  $\left(\frac{S_T}{H}\right)^{\gamma+\epsilon} > 0$  at  $T$ , while the ABC has zero value.
- To solve this problem, we will reflect the portion of the play's payoff which is below  $H$  to the region above  $H$ . In reflecting the payoff, we must preserve the value along  $S = 0$  and  $S = H$ . Since the stock price absorbs at the origin, the first requirement is automatic. The second requirement entails finding a reflected payoff  $f^r(S)$  with the property that:

$$E_{H,u}^{Q^\$} 1(S_T < H) \left(\frac{S_T}{H}\right)^{\gamma+\epsilon} = E_{H,u}^{Q^\$} 1(S_T > H) f^r(S_T),$$

for all times  $u \in [t, T]$ .

- To find  $f^r(S)$ , it will be useful to think of the underlying as the logger,  $X_t = \ln S_t$ , which has normally distributed terminal values (under  $Q^\$$ ).
- In our new spatial variable, the requirement on  $f^r(\cdot)$  is:

$$E_{h,u}^{Q^\$} 1(X_T < h) e^{(\gamma+\epsilon)(X_T-h)} = E_{h,u}^{Q^\$} 1(X_T > h) f^r(e^{X_T}), \forall u \in [t, T],$$

where  $h \equiv \ln H$ .

## Valuing American Binary Calls in the Black-Scholes Model(con'd)

- Recall that  $f^r(\cdot)$  solves:

$$e^{-r(T-u)} E_{h,u}^{Q^\$} 1(X_T < h) e^{(\gamma+\epsilon)(X_T-h)} = e^{-r(T-u)} E_{h,u}^{Q^\$} 1(X_T > h) f^r(e^{X_T}),$$

$\forall u \in [t, T]$ , where  $h \equiv \ln H$  and under  $Q^\$$ :

$$dX_v = \left( r - \delta - \frac{\sigma^2}{2} \right) dv + \sigma dW_v^\$, \quad v \in [u, T], X_u = h.$$

- The Left Hand Side (LHS) is the value of a normalized binary play along the barrier, which we denote by  $B(h, u)$ .
- Now consider the LHS as arising from quantoing a payoff of  $1(X_T < h) e^{\epsilon(X_T-h)}$  into a power play with normalized payoff  $\frac{\Pi_T(\gamma)}{H^\gamma} = e^{\gamma(X_T-h)}$ .

$$B(h, u) = e^{-r(T-u)} E_{h,u}^{Q^\$} \left[ e^{\gamma(X_T-h)} 1(X_T < h) e^{\epsilon(X_T-h)} \right],$$

- Along the barrier,  $\frac{\Pi_t(\gamma)}{H^\gamma} = 1$ , so the value of the normalized binary play is also given by:

$$B(h, u) = e^{-\frac{\sigma^2 \epsilon^2}{2}(T-u)} E_{h,u}^{Q^\gamma} \left[ 1(X_T < h) e^{\epsilon(X_T-h)} \right],$$

where recall:

$$\gamma \equiv \frac{1}{2} - \frac{r - \delta}{\sigma^2} \quad \epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}},$$

and where under  $Q^\gamma$ , the loger has no drift:

$$dX_v = \sigma dW_v^\gamma, \quad v \in [u, T], X_u = h,$$

or equivalently:

$$X_T = h + \sigma(W_T^\gamma - W_u^\gamma).$$

## Valuing American Binary Calls in the Black-Scholes Model(con'd)

- Recall that given  $X - u = h$ , then under  $Q^\gamma$ :

$$X_T = h + \sigma(W_T^\gamma - W_u^\gamma).$$

and the value of the normalized binary play along the barrier is:

$$\begin{aligned} B(h, u) &= e^{-\frac{\sigma^2 \epsilon^2}{2}(T-u)} E_{h,u}^{Q^\gamma} \left[ 1(X_T < h) e^{\epsilon(X_T-h)} \right] \\ &= e^{-\frac{\sigma^2 \epsilon^2}{2}(T-u)} E_{h,u}^{Q^\gamma} \left[ 1(X_T < h) e^{-\epsilon(X_T-h)} \right], \end{aligned}$$

by the symmetry of the normal distribution.

- Reverting to the original risk-neutral measure:

$$e^{-r(T-u)} E_{h,u}^{Q^\$} \left[ 1(X_T < h) e^{(\gamma+\epsilon)(X_T-h)} \right] = e^{-r(T-u)} E_{h,u}^{Q^\$} \left[ 1(X_T > h) e^{(\gamma-\epsilon)(X_T-h)} \right].$$

- Recalling  $X_T = \ln S_T$  and  $h \equiv \ln H$ ,

$$e^{-r(T-u)} E_{H,u}^{Q^\$} \left[ 1(S_T < H) \left( \frac{S_T}{H} \right)^{\gamma+\epsilon} \right] = e^{-r(T-u)} E_{H,u}^{Q^\$} \left[ 1(S_T > H) \left( \frac{S_T}{H} \right)^{\gamma-\epsilon} \right],$$

so the reflected payoff must be:

$$f^r(S_T) = 1(S_T > H) \left( \frac{S_T}{H} \right)^{\gamma-\epsilon}.$$

- We define the adjusted payoff  $f(S)$  as the payoff of a European-style claim with the same value as the ABC for  $S < H$ . Then:

$$f(S) = 1(S > H) \left[ \left( \frac{S}{H} \right)^{\gamma+\epsilon} + \left( \frac{S}{H} \right)^{\gamma-\epsilon} \right].$$

- Letting  $\tau \equiv T - t$ , the American binary call is valued by:

$$\begin{aligned} ABC(S, t) &= e^{-r\tau} E_{S,t}^{Q^\$} f(S_T) \\ &= \left( \frac{S}{H} \right)^{\gamma+\epsilon} N \left( \frac{\ln \left( \frac{S}{H} \right) + \epsilon \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) + \left( \frac{S}{H} \right)^{\gamma-\epsilon} N \left( \frac{\ln \left( \frac{S}{H} \right) - \epsilon \sigma^2 \tau}{\sigma \sqrt{\tau}} \right), \end{aligned}$$

for  $S \in (0, H)$ , where recall  $\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2}$ ,  $\epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}$ .

## VII Valuing Up-And-In Claims in the Black-Scholes Model

- Recall that the value of an up-and-in derivative with final payoff  $1(\tau_S^H < T)f_{ui}(S_T)$  is obtained by solving:

$$\frac{\sigma^2 S^2}{2} \frac{\partial^2 V^b}{\partial S^2}(S, t) + (r - \delta)S \frac{\partial V^b}{\partial S}(S, t) - rV^b(S, t) + \frac{\partial V^b}{\partial t}(S, t) = 0,$$

for  $S \in (0, H), t \in (0, T)$  with:

$$V^b(S, T) = 0, \text{ and } V^b(H, t) = h(t) \equiv e^{-r(T-t)} E_{H,t}^{Q^\$} f_{ui}(S_T),$$

where under  $Q^\$$ , the risk-neutral stock price process is:

$$\frac{dS_u}{S_u} = (r - \delta)du + \sigma dW_u^\$, \quad u \in [t, T].$$

- To solve this problem, it is helpful to use risk-neutral valuation:

$$\begin{aligned} V^b(S, t) &= e^{-r(T-t)} E_{S,t}^{Q^\$} 1(\tau_S^H < T) f(S_T) \\ &= e^{-r(T-t)} E_{S,t}^{Q^\$} 1(\tau_S^H < T) E_{H,\tau_S^H}^{Q^\$} f(S_T), \end{aligned}$$

by the double expectations theorem.

- As with the ABC, suppose that we can find an “adjusted payoff”  $f(S), S > H$  with the property that:

$$E_{H,u}^{Q^\$} f_{ui}(S_T) = E_{H,u}^{Q^\$} 1(S_T > H) f(S_T), \quad \forall u \in [t, T].$$

- Then valuation is easy since:

$$\begin{aligned} V^b(S, t) &= e^{-r(T-t)} E_{S,t}^{Q^\$} 1(\tau_S^H < T) E_{H,\tau_S^H}^{Q^\$} 1(S_T > H) f(S_T) \\ &= e^{-r(T-t)} E_{S,t}^{Q^\$} 1(S_T > H) f(S_T), \end{aligned}$$

which is path-independent.



## VII-A Finding the Adjusted Payoff

- Recall that the adjusted payoff  $f(S)$ ,  $S > H$  has the property that:

$$E_{H,u}^{Q^\$} f_{ui}(S_T) = E_{H,u}^{Q^\$} 1(S_T > H) f(S_T),$$

for all times  $u \in [t, T]$ .

- Note that the component of  $f_{ui}$  with support above the barrier has the above property trivially. Thus, the adjusted payoff has the form:

$$f(S) = 1(S > H)[f_{ui}(S) + f^r(S)],$$

where  $f^r(S)$  is a reflection of the payoff  $f_{ui}$  from below the barrier to the region above it, i.e. the reflected payoff  $f^r(S)$ ,  $S > H$  solves:

$$E_{H,u}^{Q^\$} 1(S_T < H) f_{ui}(S_T) = E_{H,u}^{Q^\$} 1(S_T > H) f^r(S_T),$$

for all times  $u \in [t, T]$ .

- Expressing our spatial state variable in terms of the logprice  $X_t = \ln S_t$ , the reflected payoff also solves:

$$e^{-r(T-u)} E_{h,u}^{Q^\$} 1(X_T < h) f_{ui}(e^{X_T}) = e^{-r(T-u)} E_{h,u}^{Q^\$} 1(X_T > h) f^r(e^{X_T}),$$

for all times  $u \in [t, T]$ , where  $h \equiv \ln H$  and recall that under  $Q^\$$ :

$$dX_v = \left( r - \delta - \frac{\sigma^2}{2} \right) dv + \sigma dW_v^\$, \quad v \in [u, T], X_u = h.$$

## Finding the Adjusted Payoff(con'd)

- Recall that we are seeking the reflected payoff solving:

$$e^{-r(T-u)} E_{h,u}^{Q^\$} 1(X_T < h) f_{ui}(e^{X_T}) = e^{-r(T-u)} E_{h,u}^{Q^\$} 1(X_T > h) f^r(e^{X_T}).$$

- Let  $U^b(h, u)$  denote the LHS, which is the value at the barrier of a claim with payoff  $1(S_T < H) f_{ui}(S_T)$  at  $T$ .
- Now consider quantoing the payoff of this claim into the play paying  $S_T^\gamma = e^{\gamma X_T}$  at  $T$ , and changing the magnitude of this payoff to conserve value:

$$U^b(h, u) = e^{-r(T-u)} E_{h,u}^{Q^\$} [e^{\gamma X_T} 1(X_T < h) f_{ui}(e^{X_T}) e^{-\gamma X_T}].$$

The value in dollars is given by:

$$U^b(h, u) = e^{\gamma h} e^{-\frac{\sigma^2 \epsilon^2}{2}(T-u)} E_{h,u}^{Q^\gamma} [1(X_T < h) f_{ui}(e^{X_T}) e^{-\gamma X_T}],$$

where recall:

$$\gamma \equiv \frac{1}{2} - \frac{r - \delta}{\sigma^2} \quad \epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}},$$

and under  $Q^\gamma$ , the logprice  $X$  has no drift:

$$dX_v = \sigma dW_v^\gamma, \quad v \in [u, T], X_u = h,$$

or equivalently:

$$X_T = h + \sigma(W_T^\gamma - W_u^\gamma).$$

## Finding the Adjusted Payoff(con'd)

- Recall that given  $X_u = h$ , then under  $Q^\gamma$ ,  $X_T = h + \sigma(W_T^\gamma - W_u^\gamma)$ , and:

$$\begin{aligned}
 U^b(h, u) &= e^{\gamma h} e^{-\frac{\sigma^2 \epsilon^2}{2}(T-u)} E_{h,u}^{Q^\gamma} \left[ 1(X_T < h) f_{ui}(e^{X_T}) e^{-\gamma X_T} \right] \\
 &= e^{\gamma h} \int_{-\infty}^h f_{ui}(e^k) e^{-\gamma k} e^{-\frac{\sigma^2 \epsilon^2}{2}(T-u)} E_{h,u}^{Q^\gamma} \delta(X_T - k) dk \\
 &= e^{\gamma h} \int_{-\infty}^h f_{ui}(e^k) e^{-\gamma k} e^{-\frac{\sigma^2 \epsilon^2}{2}(T-u)} E_{h,u}^{Q^\gamma} \delta(X_T - (2h - k)) dk.
 \end{aligned}$$

since by symmetry:

$$E_{h,u}^{Q^\gamma} \delta(X_T - k) = E_{h,u}^{Q^\gamma} \delta(X_T - (2h - k)).$$

- Letting  $\ell = 2h - k$  be a change of variables:

$$\begin{aligned}
 U^b(h, t) e^{\gamma h} &= e^{\gamma h} \int_h^\infty f_{ui}(e^{2h-\ell}) e^{-\gamma(2h-\ell)} e^{-\frac{\sigma^2 \epsilon^2}{2}(T-u)} E_{h,u}^{Q^\gamma} \delta(X_T - \ell) d\ell \\
 &= e^{\gamma h} \int_h^\infty f_{ui}(e^{2h-\ell}) e^{2\gamma(\ell-h)} e^{-\frac{\sigma^2 \epsilon^2}{2}(T-u)} E_{h,u}^{Q^\gamma} \delta(X_T - \ell) e^{-\gamma \ell} d\ell.
 \end{aligned}$$

- Reverting to the original risk-neutral measure:

$$\begin{aligned}
 U^b(h, t) &= \int_h^\infty f_{ui}(e^{2h-\ell}) e^{2\gamma(\ell-h)} e^{-r(T-u)} E_{h,u}^{Q^\$} \delta(X_T - \ell) d\ell \\
 &= e^{-r(T-u)} E_{h,u}^{Q^\$} 1(X_T > h) f_{ui}(e^{2h-X_T}) e^{2\gamma(X_T-h)}.
 \end{aligned}$$

- Recalling  $X_T = \ln S_T$  and  $h \equiv \ln H$ , the reflected payoff must be:

$$f^r(S_T) = 1(S_T > H) f_{ui} \left( \frac{H^2}{S_T} \right) \left( \frac{S_T}{H} \right)^{2\gamma}.$$

- Thus, the adjusted payoff is:

$$f(S) = 1(S > H) \left[ f_{ui}(S) + f_{ui} \left( \frac{H^2}{S} \right) \left( \frac{S}{H} \right)^{2\gamma} \right].$$

## VII-B Adjusted Payoff for Out and Down Claims

- Recall that the adjusted payoff for an up-and-in claim is:

$$f(S_T) \equiv \begin{cases} f_{ui}(S_T) + \left(\frac{S_T}{H}\right)^{2\gamma} f_{ui}\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H; \\ 0 & \text{if } S_T > H. \end{cases}$$

- The in-out parity relationship implies:

$$V_{uo}(S, t) = V(S, t) - V_{ui}(S, t).$$

- By in-out parity, the adjusted payoff for an up-and-out claim is:

$$f(S_T) \equiv \begin{cases} -\left(\frac{S_T}{H}\right)^{2\gamma} f_{uo}\left(\frac{H^2}{S_T}\right) & \text{if } S_T > H; \\ f_{uo}(S_T) & \text{if } S_T < H. \end{cases}$$

- Similarly, the adjusted payoff for a down-and-in claim is:

$$f(S_T) \equiv \begin{cases} 0 & \text{if } S_T > H, \\ f_{di}(S_T) + \left(\frac{S_T}{H}\right)^{2\gamma} f_{di}\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H. \end{cases}$$

- For a down-and-out security, in-out parity implies that the adjusted payoff is:

$$f(S_T) \equiv \begin{cases} f_{do}(S_T) & \text{if } S_T > H, \\ -\left(\frac{S_T}{H}\right)^{2\gamma} f_{do}\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H. \end{cases}$$

Barrier Security	Adjusted Payoff $f(S)$	
No-touch binary put	1	for $S_T > H$
	$-(S_T/H)^{2\gamma}$	for $S_T < H$
One-touch binary put (European)	0	for $S_T > H$
	$1 + (S_T/H)^{2\gamma}$	for $S_T < H$
Down-and-out call	$(S_T - K_c)^+$	for $S_T > H$
	$-(S_T/H)^{2\gamma} \left(\frac{H^2}{S_T} - K_c\right)^+$	for $S_T < H$
Down-and-out put	$(K_p - S_T)^+$	for $S_T > H$
	$-(S_T/H)^{2\gamma} \left(K_p - \frac{H^2}{S_T}\right)^+$	for $S_T < H$

Table 0.1: Adjusted Payoffs for Down Securities. ( $\gamma = \frac{1}{2} - \frac{r-\delta}{\sigma^2}$ )

### VII-C Examples of Adjusted Payoffs

- The table above and the figures on the next page show the adjusted payoff for some common down securities.
- Note that the payoffs are “close to” piecewise linear, suggesting static replication using options.

## Part III

# Introduction to Static Hedging

## VIII Introduction to Static Hedging of a Path-Independent Payoff

- Consider a final payoff  $f(S_T)$  paid at  $T$ , which is a twice differentiable function of the stock price  $S$ .
- Let  $F_0$  be the initial forward price for delivery at  $T$ .
- Using the fundamental theorem of calculus twice, Appendix 2 proves the following “spectral decomposition”:

$$\begin{aligned} f(S_T) &= f(F_0) + f'(F_0)[S_T - F_0] \\ &\quad + \int_0^{F_0} f''(K)(K - S_T)^+ dK + \int_{F_0}^{\infty} f''(K)(S_T - K)^+ dK. \end{aligned}$$

- This may be interpreted as a Taylor series expansion with remainder of the final payoff  $f(\cdot)$  about the forward price  $F_0$ .
- The first two terms give the tangent to the payoff at  $F_0$ ; the last two terms bend the tangent so as to conform to the payoff.
- The payoff of an arbitrary claim has been decomposed into the payoff from  $f(F_0)$  bonds,  $f'(F_0)$  forward contracts, and the spectrum of out-of-the-money forward options.
- The figures on the next page illustrate this result for the square root payoff  $f(S) = \sqrt{S}$ .

## VIII-A Payoffs and Prices

- Recall the “spectral decomposition” of the final payoff  $f(S_T)$  into payoffs from bonds, forward contracts, and options:

$$f(S_T) = f(F_0) + f'(F_0)[S_T - F_0] + \int_0^{F_0} f''(K)(K - S_T)^+ dK + \int_{F_0}^{\infty} f''(K)(S_T - K)^+ dK.$$

- The initial value  $V_0^f$  of the final payoff  $f(\cdot)$  can be expressed in terms of the initial prices of bonds  $B_0$ , calls  $C_0(K)$ , and puts  $P_0(K)$  respectively:

$$V_0^f = f(F_0)B_0 + \int_0^{F_0} f''(K)P_0(K)dK + \int_{F_0}^{\infty} f''(K)C_0(K)dK.$$

- Note that no term is required for the forward contracts because they are initially costless.



## VIII-B Examples of Static Hedges

- Recall that the value of an arbitrary claim can be decomposed into  $f(F_0)$  bonds,  $f'(F_0)$  costless forward contracts, and the spectrum of out-of-the-money options:

$$V_0^f = f(F_0)B_0 + \int_0^{F_0} f''(K)P_0(K)dK + \int_{F_0}^{\infty} f''(K)C_0(K)dK.$$

- We next illustrate this fundamental decomposition with some examples.
- One of the simplest examples is a claim which pays  $S_T$  at maturity, i.e.  $f(S_T) = S_T$ . Letting  $S_0$  denote the initial spot price of this claim, the above equation implies:

$$S_0 = F_0 B_0,$$

which is the well known cost-of-carry relationship.

- We can relax the assumption that  $f$  be twice differentiable by working with *generalized* functions (a.k.a. distributions). For example, suppose we are interested in decomposing an in-the-money European call, i.e.  $f(S) = (S - K_c)^+$ ,  $K_c < F_0$ . Formally using the above decomposition gives:

$$C_0(K_c) = (F_0 - K_c)B_0 + P_0(K_c),$$

which is Put Call Parity.

## VIII-C Intrinsic and Time Value

- Again recall that the value of an arbitrary claim can be decomposed into  $f(F_0)$  bonds,  $f'(F_0)$  costless forward contracts, and the spectrum of out-of-the-money options:

$$V_0^f = f(F_0)B_0 + \int_0^{F_0} f''(K)P_0(K)dK + \int_{F_0}^{\infty} f''(K)C_0(K)dK.$$

- This fundamental decomposition may be used to provide general definitions of intrinsic value and time value.
- We take  $f(F_0)B_0$  as our definition of the intrinsic value of a claim with an arbitrary continuous payoff  $f(\cdot)$ :

$$IV_0^f \equiv f(F_0)B_0.$$

- We define the time value of this claim as the sum of the last two terms in the above decomposition:

$$TV_0^f \equiv \int_0^{F_0} f''(K)P_0(K)dK + \int_{F_0}^{\infty} f''(K)C_0(K)dK.$$

- Since the options used in the time value definition are out-of-the-money, the definition expresses the time value of an arbitrary claim in terms of the time values of vanilla options.
- Note that if the final payoff function  $f(\cdot)$  is linear, then  $f''(K) = 0$  for all  $K$  and there is no time value. Conversely, if the payoff is globally convex, then  $f''(K) \geq 0$  for all  $K$ , and the time value is positive.

## IX Introduction to Static Hedging of Barrier Options in the Black-Scholes Model

- Recall that in the Black Scholes model, we were able to find adjusted payoffs defined on the whole domain, which had the property that any European-style claim with this payoff had the same value as the barrier option.
- Since this payoff can be created with a static position in options, barrier options can be statically hedged.
- To illustrate with a simple example, recall that the adjusted payoff for a down-and-out call was:

$$f(S) = \begin{cases} (S_T - K_c)^+ & \text{for } S_T > H; \\ -(S_T/H)^{2\gamma} \left(\frac{H^2}{S_T} - K_c\right)^+ & \text{for } S_T < H. \end{cases}$$

where recall  $\gamma = \frac{1}{2} - \frac{r-\delta}{\sigma^2}$ .

- For simplicity, suppose  $K_c > H$  and  $r = \delta$ , which would be the case if the underlying were a futures price. Then  $\gamma = \frac{1}{2}$  and so the adjusted payoff for a down-and-out call simplifies to:

$$f(S) = \begin{cases} (S_T - K_c)^+ & \text{for } S_T > H; \\ -\frac{K_c}{H} \left(\frac{H^2}{K_c} - S_T\right)^+ & \text{for } S_T < H. \end{cases}$$

- Thus, to hedge the sale of a down-and-out call, buy a vanilla call with the same strike and sell  $\frac{K_c}{H}$  vanilla puts struck at  $\frac{H^2}{K_c}$ . All options have the same maturity.
- If the underlying never hits the barrier, then the vanilla call covers the liability on the down-and-out. When the underlying hits the barrier, sell the puts and buy the call. If the Black Scholes model still holds, the switch will be self-financing.

## IX-A Using Dynamic Hedging Results to Uncover Static Hedges

- The valuation formulas for barrier options developed using dynamic replication can be used to uncover the adjusted payoff, and the static hedge.
- For example, assuming we know the formula  $U(S, \tau)$  for an up barrier security as a function of the current stock price  $S$  and time  $t$ , the first step is to find the value of the replicating option portfolio for any stock price by simply removing the restriction that stock prices be below the barrier:

$$V(S, t) = U(S, t), \quad \forall S > 0.$$

- The second step is to obtain the payoff which gives rise to this value. Since values converge to their payoff at maturity, simply take the limit of the value as the current time approaches maturity:

$$f(S) = \lim_{t \uparrow T} V(S, \tau), \quad S > 0.$$

- The third step is to use the static representation of a path-independent payoff to uncover the requisite static position in bonds, forward contracts, and vanilla options.

## IX-B Example 1: American Binary Call

- Recall that the valuation formula for an American binary call (ABC) is:

$$ABC(S, t) = \left(\frac{S}{H}\right)^{\gamma+\epsilon} N\left(\frac{\ln\left(\frac{S}{H}\right) + \epsilon\sigma^2\tau}{\sigma\sqrt{\tau}}\right) + \left(\frac{S}{H}\right)^{\gamma-\epsilon} N\left(\frac{\ln\left(\frac{S}{H}\right) - \epsilon\sigma^2\tau}{\sigma\sqrt{\tau}}\right),$$

for  $S \in (0, H)$ , where  $\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2}$ ,  $\epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}$ .

- Removing the requirement that  $S < H$  and letting  $t \uparrow T$  gives the adjusted payoff as:

$$f(S) = \lim_{t \uparrow T} V(S, t) = \left(\frac{S}{H}\right)^{\gamma+\epsilon} 1(S > H) + \left(\frac{S}{H}\right)^{\gamma-\epsilon} 1(S > H).$$

- The payoff from each of the 2 path-independent calls can be statically replicated with a portfolio of vanilla options, with each portfolio paying off  $S_T^{\gamma \pm \epsilon} 1(S_T > H)$  at  $T$ .
- From the spectral decomposition with  $F_0 < H$ , each payoff can be replicated by a static portfolio consisting of  $H^{\gamma \pm \epsilon}$  bond-or-nothing calls struck at  $H$ ,  $(\gamma \pm \epsilon)H^{\gamma \pm \epsilon - 1}$  vanilla calls struck at  $H$ , and the infinitesimal position  $(\gamma \pm \epsilon)(\gamma \pm \epsilon - 1)dK$  in all vanilla calls struck above  $H$ :

$$\begin{aligned} S_T^{\gamma \pm \epsilon} 1(S_T > H) &= H^{\gamma \pm \epsilon} 1(S_T > H) + (\gamma \pm \epsilon)H^{\gamma \pm \epsilon - 1}(S_T - H)^+ \\ &\quad + \int_H^\infty (\gamma \pm \epsilon)(\gamma \pm \epsilon - 1)K^{\gamma \pm \epsilon - 2}(S_T - K)^+ dK. \end{aligned}$$

- Since the bond-or-nothing call is a vertical spread of vanilla calls, the value at  $t$  of an ABC can be expressed in terms of the contemporaneous prices of vanilla calls:

$$ABC_t(T) = 2B \vee N C_t(T) + \frac{2\gamma}{H} C_t(H, T) + \int_H^\infty n_c(K) C_t(K, T) dK,$$

where:

$$n_c(K) \equiv \left[ (\gamma + \epsilon)(\gamma + \epsilon - 1) \left(\frac{K}{H}\right)^{\gamma + \epsilon - 2} + (\gamma - \epsilon)(\gamma - \epsilon - 1) \left(\frac{K}{H}\right)^{\gamma - \epsilon - 2} \right] \frac{1}{H^2}.$$

**Example 1: American Binary Call(con'd)**

- Recall that the value at  $t$  of an ABC was expressed in terms of the contemporaneous prices of vanilla calls:

$$ABC_t(T) = 2B\vee NC_t(T) + \frac{2\gamma}{H}C_t(H, T) + \int_H^\infty n_c(K)C_t(K, T)dK,$$

for  $S \in (0, H)$ , where  $\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2}$ ,  $\epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}$ , and where:

$$n_c(K) \equiv \left[ (\gamma + \epsilon)(\gamma + \epsilon - 1) \left(\frac{K}{H}\right)^{\gamma+\epsilon-2} + (\gamma - \epsilon)(\gamma - \epsilon - 1) \left(\frac{K}{H}\right)^{\gamma-\epsilon-2} \right] \frac{1}{H^2}.$$

- We note that if  $r = \delta$ , then  $n_c$  simplifies to:

$$n_c(K) = \frac{2r}{\sigma^2 H^2} \left[ \left(\frac{K}{H}\right)^{\gamma+\epsilon-2} + \left(\frac{K}{H}\right)^{\gamma-\epsilon-2} \right].$$

- Furthermore, if  $r = \delta = 0$ , then  $n_c = 0$  and the American binary call value is simply given by:

$$ABC_t(T) = 2B\vee NC_t(T) + \frac{1}{H}C_t(H, T).$$

## Part IV

# A Comparison of Static with Dynamic Hedging

## X Ex Ante Analysis

- When used in the Black Scholes model, static and dynamic hedging both work perfectly in theory and give rise to the same model value.
- In practice, one approach may work better than another:
  - Static hedging generally requires many strikes. Similarly, dynamic hedging requires many trading opportunities.
  - Static hedging may require large positions in options; dynamic hedging may require large positions in the stock. Both may require short positions or excessive borrowing.
  - Static hedging exposes the hedger once to options transactions costs at every strike; dynamic hedging exposes the hedger to underlying transactions costs on every trade.
  - Static hedging with options carries with it a sensitivity to changes in the parameters governing the volatility process. Dynamic hedging is relatively immune to volatility changes.
  - Static hedging may hurt or help vis-a-vis dynamic hedging when jumps are taken into account. If the static hedge has positive gamma near the barrier, then jumps favor the static hedge.



## XI Simulating Static vs. Dynamic Hedging of an American Binary Call

- We simulated the dynamic and static hedge for the sale of an American binary call with strike \$100, maturity 1 year, and an initial spot price of \$90 when  $r = .1$ ,  $\delta = 0$ , and  $\sigma = .2$ .
- The dynamic hedge assumed that the hedger could trade every  $\Delta t \equiv \frac{T}{N}$  years, where  $T = 1$  and  $N$  varied from 4 to 1024.
- The static hedge assumed that strikes were available in increments of  $\Delta K$ , where  $\Delta K$  varied between \$1 and \$10.
- Proportional transactions costs of 0.1% were used for both hedging strategies. The static hedge was also conducted under transactions costs of 0.5%.
- Chart 1 shows that for the same transactions costs, one can always find a static hedge with a higher mean P&L and a lower P&L vol than the “best” dynamic hedge. This result also holds when the static hedge transactions costs are raised 5-fold.
- Chart 2 shows that this mean-variance domination also holds when the stock vol is  $\sigma = .5$ .
- Chart 3 shows that static hedging does not first order stochastically dominate dynamic hedging.
- However, Chart 4 shows that it does second order stochastically dominate. Thus, any investor with increasing utility and positive risk aversion would prefer static hedging to dynamic hedging.
- An American binary call has positive gamma up to the barrier. Consequently, the imposition of jumps in the process would favor the static hedge in this case.

## XII Conclusions

- The concept of an adjusted payoff is useful for understanding the valuation and hedging of barrier options.
- Once this payoff is known, barrier options can be hedged either by dynamically trading in the underlying or statically positioning in options.
- Although static hedging outperformed dynamic hedging for American binary calls, other barrier options (eg. up-and-out calls) might result in the opposite conclusion.
- In general, the best hedge will involve a combination of static and dynamic hedging.
- Dynamic trading in options will also improve hedging in more general situations, although the risk reduction comes at a high price in terms of transactions costs.
- More research needs to be done concerning the impacts of alternative processes (ideally incorporating stochastic vol and jumps) and frictions (especially discrete trading, transactions costs, and position limits).

# Appendix 1: An Analysis of Borrowing when Delta-Hedging in the Black Scholes Model

We assume the standard Black Scholes model of frictionless markets, no arbitrage, a constant riskfree rate  $r$ , a constant continuous dividend yield  $\delta$  and a geometric Brownian motion for the stock price  $S$  over the option's life  $[0, T]$ . Let  $V(S, t)$  be the unique  $C^{2,1}$  function solving the Black Scholes p.d.e.:

$$\frac{\partial}{\partial t}V(S, t) + \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2}V(S, t) + (r - \delta)S \frac{\partial}{\partial S}V(S, t) - rV(S, t) = 0, \quad (0.1)$$

subject to the terminal condition:

$$\lim_{t \uparrow T} V(S, t) = f(S), \quad (0.2)$$

where  $f$  should be continuous, but need not be differentiable everywhere.

Suppose that at time 0, a trader sells a European-style claim paying  $f(S_T)$  at  $T$  for the Black Scholes model value  $V(S_0, 0)$ . We assume that the claim is sold for the implied volatility  $\sigma$  which will be realized over the claim's life. Let  $V(S, t)$  be the value function describing the fair value of the claim. Let  $N_t$  denote the number of shares held by the trader at time  $t \in [0, T]$ . To hedge the sale of the claim, the trader initially buys  $N_0 = \frac{\partial V}{\partial S}(S_0, 0)$  shares, each at price  $S_0$ . Let  $\beta_t \geq 0$  denote cumulative borrowing at time  $t \in [0, T]$ . The initial borrowing is the difference between the cost of setting up the initial stock hedge and the proceeds from the sale of the claim:

$$\beta_0 = \frac{\partial V}{\partial S}(S_0, 0)S_0 - V(S_0, 0). \quad (0.3)$$

We now assume that the trader follows an equity trading strategy where all stock purchases are financed by borrowing and all stock sales are used to reduce cumulative borrowing. We next note that cumulative borrowing is also affected by carrying costs. The cumulative borrowings at  $t$ , denoted  $\beta_t$  will grow at the riskfree rate  $r$  over time. Furthermore, the stock position pays dividends which reduces cumulative borrowings. Thus at each  $t$ , the change in cumulative borrowings can be expressed as:

$$d\beta_t = \underbrace{dN_t(S_t + dS_t)}_{\text{dollar cost of buying shares}} + \underbrace{r\beta_t dt}_{\text{additional interest}} - \underbrace{N_t \delta S_t dt}_{\text{dividends received}} .$$

The first term can be represented as the change in the dollar value of the stock position, *less* the portion of that change due to capital gains on the shares held:

$$d\beta_t = \underbrace{d(N_t S_t)}_{\text{change in \$ value of stock position}} - \underbrace{N_t dS_t}_{\text{capital gains in stock}} - \underbrace{N_t \delta S_t dt}_{\text{dividends received}} + \underbrace{r\beta_t dt}_{\text{additional interest}} . \quad (0.4)$$

Suppose that the trader holds the number of shares warranted by the Black Scholes model

$$N_t = \frac{\partial V}{\partial S}(S_t, t). \quad (0.5)$$

Now, by Itô's Lemma:

$$\begin{aligned} dV_t &= \left[ \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right] dt + \frac{\partial V}{\partial S} dS_t \\ &= r \left[ V - \frac{\partial V}{\partial S} S \right] dt + \delta S \frac{\partial V}{\partial S} dt + \frac{\partial V}{\partial S} dS_t \end{aligned}$$

from the Black Scholes p.d.e. (0.1). Solving for the last two terms and substituting this result and (0.5) in (0.4) gives:

$$d\beta_t = d \left[ \frac{\partial V}{\partial S}(S_t, t) S_t \right] - dV(S_t, t) - r \left[ V(S_t, t) - \frac{\partial V}{\partial S}(S_t, t) S_t \right] dt + r\beta_t dt.$$

The solution to this equation is:

$$\beta_t = S_t \frac{\partial V}{\partial S}(S_t, t) - V(S_t, t).$$

## Appendix 2: Proof of Spectral Decomposition

The fundamental theorem of calculus implies that for any fixed  $F$ :

$$\begin{aligned} f(S) &= f(F) + 1_{S>F} \int_F^S f'(u) du - 1_{S<F} \int_S^F f'(u) du \\ &= f(F) + 1_{S>F} \int_F^S \left[ f'(F) + \int_F^u f''(v) dv \right] du \\ &\quad - 1_{S<F} \int_S^F \left[ f'(F) - \int_u^F f''(v) dv \right] du. \end{aligned}$$

Noting that  $f'(F)$  does not depend on  $u$  and applying Fubini's theorem:

$$f(S) = f(F) + f'(F)(S - F) + 1_{S>F} \int_F^S \int_v^S f''(v) du dv + 1_{S<F} \int_S^F \int_S^v f''(v) du dv.$$

Performing the integral over  $u$  yields:

$$\begin{aligned} f(S) &= f(F) + f'(F)(S - F) + 1_{S>F} \int_F^S f''(v)(S - v) dv + 1_{S<F} \int_S^F f''(v)(v - S) dv \\ &= f(F) + f'(F)(S - F) + \int_F^\infty f''(v)(S - v)^+ dv + \int_0^F f''(v)(v - S)^+ dv. \end{aligned} \tag{0.6}$$

Setting  $F = S_0$ , the initial stock price, gives Theorem 1. Note that if  $F = 0$ , the replication involves only bonds, stocks, and calls:

$$f(S) = f(0) + f'(0)S + \int_0^\infty f''(v)(S - v)^+ dv,$$

provided the terms on the right hand side are all finite. Similarly, for claims with  $\lim_{F \uparrow \infty} f(F)$  and  $\lim_{F \uparrow \infty} f'(F)F$  both finite, we may also replicate using only bonds, stocks, and puts:

$$f(S) = \lim_{F \uparrow \infty} f(F) + \lim_{F \uparrow \infty} f'(F)(S - F) + \int_0^\infty f''(v)(v - S)^+ dv.$$