Perturbative Analysis of Volatility Smiles

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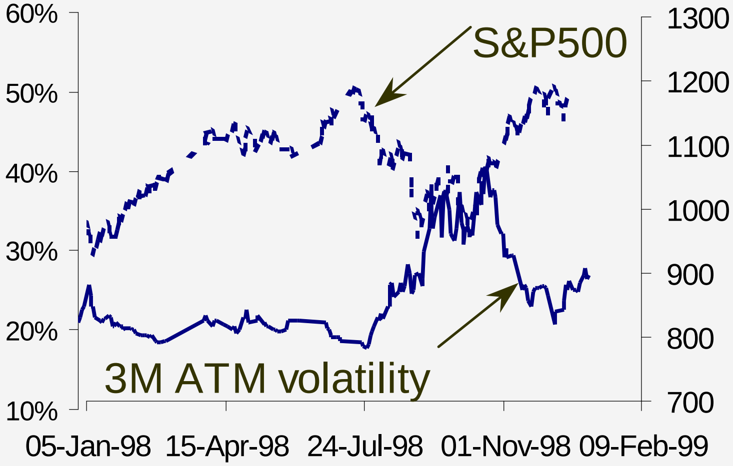
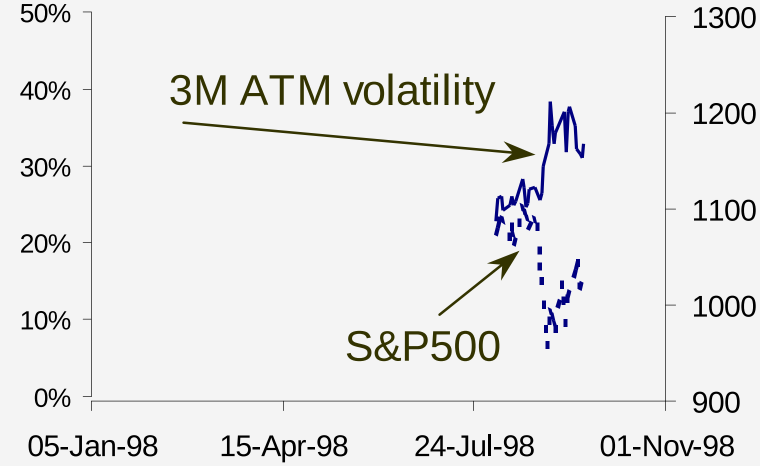
Main Risks in Options Markets

Volatility changes in time



Markets jump



- **Index volatility is mean-reverting**
- **It is negatively correlated with the price**
- **A jump in price often entails a volatility jump**

Most models ignore at least one of these risks

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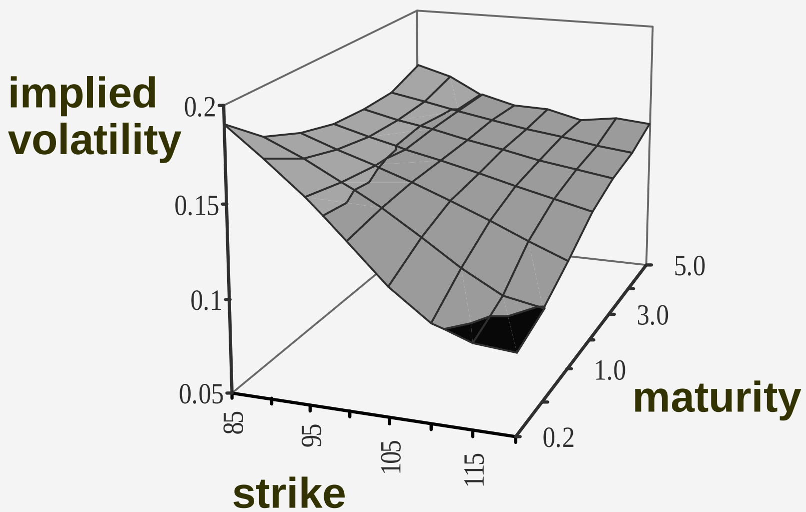
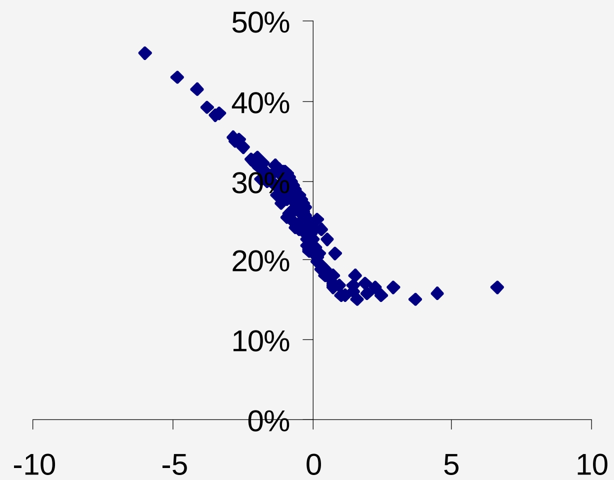
What Is the Shape of Smile?

S&P500 volatility surface on January 11, 1996



Implied volatility vs. standard deviation



- Implied volatility decreases with strike price
- The skew slope is the greatest for short maturities

What underlying processes produce such skews?

Is the Skew Due to Jumps?

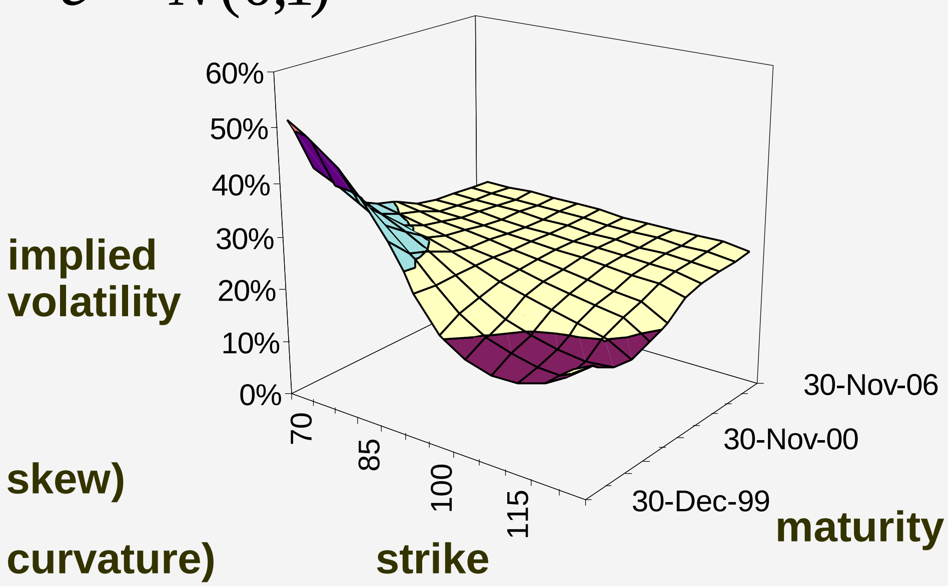
- **Jump Diffusion model**
  - **between jumps** $dS_t / S_t = m^* dt + s\ dz(t)$
  - **in a jump** $S_t \to S_t e^{\bar{g} + de} \qquad e \propto N(0,1)$
  - **jumps arrive with rate** l

$$p_n = \frac{(lT)^n}{n!} e^{-lT}$$

- **For S&P 500**

$$\begin{cases} l \propto 0.5 & \\ g \propto -0.2 & \text{(creates skew)} \\ d \propto 0.05 \div 0.15 & \text{(creates curvature)} \end{cases}$$

Volatility surface from jump diffusion



The jump diffusion model works well for short maturities

What Happens at Longer Maturities?

- **Stochastic volatility**

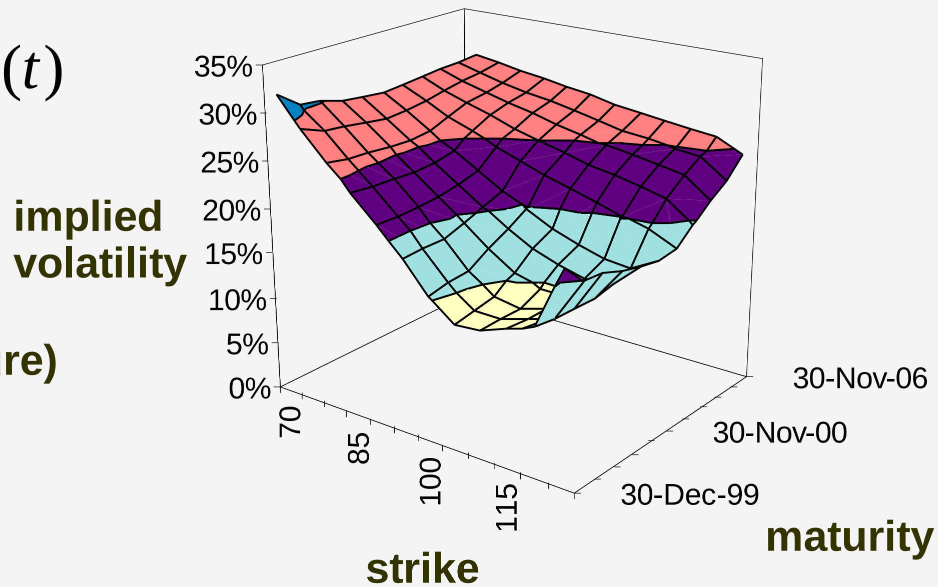
$$\begin{cases} dS_t / S_t = m\,dt + \sqrt{v_t}\, dz_s(t) \\ dv_t = k(q - v_t) + s\sqrt{v_t}\, dz_v(t) \end{cases}$$

$$Corr(dz_s, dz_v) = r < 0$$

- **For S&P 500**

$$\begin{cases} s \propto 0.8 & \text{(creates curvature)} \\ r \propto -0.7 & \text{(creates skew)} \\ k \propto 1 \div 3 \end{cases}$$

Volatility surface from a stochastic volatility model



Stochastic volatility models work well for long maturities

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How to Combine Stochastic Volatility and Jump Diffusion ?

- **between jumps** $\begin{cases} dS/S = m\,dt + \sqrt{v}\,dz_1 \\ dv = k(q - v)dt + s\sqrt{v}\,dz_2 \end{cases}$ $\quad Corr(dz_1, dz_2) = r$

- **market crashes form a Poisson process with rate** l

$$\begin{cases} \log S \to \log S + g_s + d_s e & e \propto N(0,1) \\ v \to v + g_v \end{cases}$$

- **the option price obeys the equation**

$$\frac{\partial f}{\partial t} + m^* S \frac{\partial f}{\partial S} + k(q - v)\frac{\partial f}{\partial v} + \frac{1}{2} v \left\{ S^2 \frac{\partial^2 f}{\partial S^2} + s^2 \frac{\partial^2 f}{\partial v^2} + 2 r s S \frac{\partial^2 f}{\partial S \partial v} \right\}$$

$$+ l E^* \left[f(Se^{g_s + d_s e}, v + g_v) - f(S, v) \right] = rf$$

European option prices can be computed analytically

What Is the Distribution of Stock Prices?

- **Call prices equal** $C = S\,P_1 - K\,e^{-rT}\,P_0$
- **Find the characteristic functional**

$$f(t,f) = E^{*}\left[e^{i f \ln(S/F)}\right] = \text{Fourier Transform of } P_0'$$

- **Use the affine ansatz** $\hat{P}_n = e^{C(T-t,j)+D(T-t,j)v}$ **to derive**

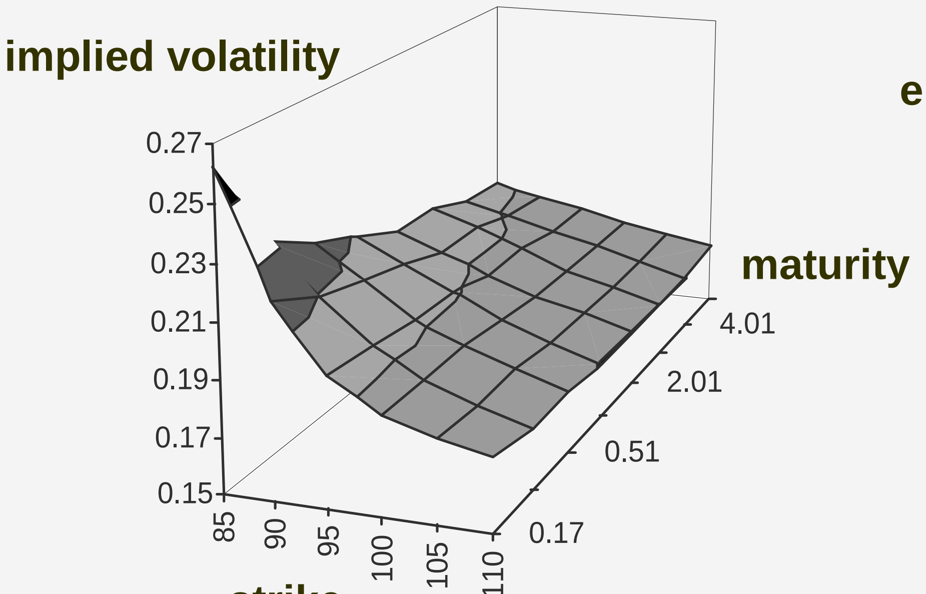
$$\begin{cases} C(t,j) = C_H(t,j) + l t \left[e^{ij g_s - j^2 d_s^2/2} I(t) - 1\right] \\ D(t,j) = D_H(t,j) \end{cases} \qquad p_{\pm} = \frac{g_v}{s^2}(b - r s j i \pm d)$$

$$I(t) = \frac{1}{t}\int_0^t e^{g_v D(t,j)}\,dt = -\frac{2 g_v}{p_+ p_-}\int_0^{-g_v D(t,j)} \frac{e^{-z}\,dz}{(1+z/p_+)(1+z/p_-)}$$

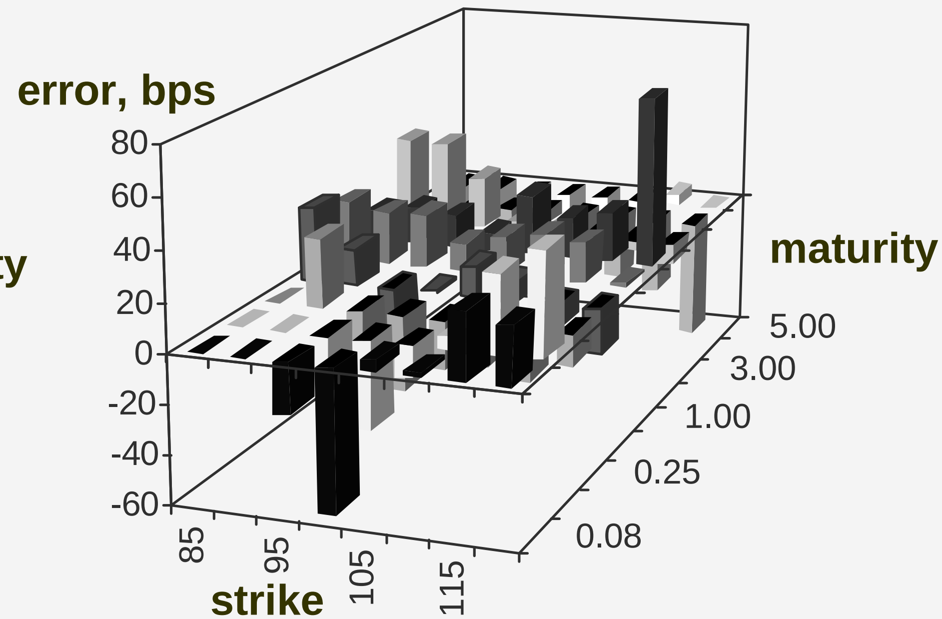
This model accounts for the main risks of options markets

Does the Model Fit the Smile?

S&P500 volatility surface on June 11, 1997



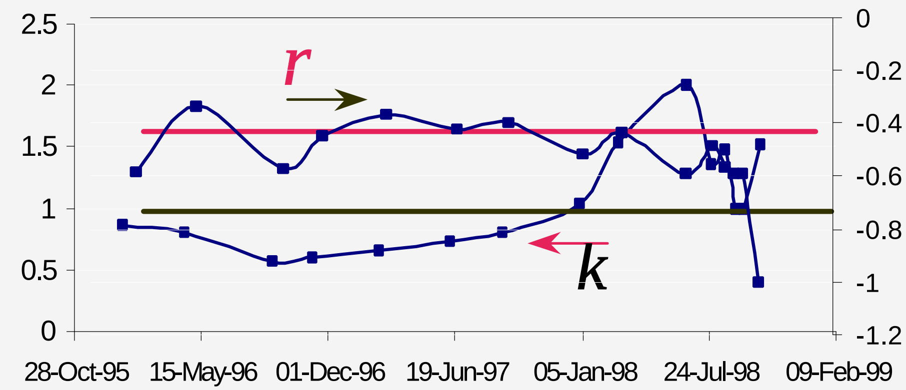
Calibration errors

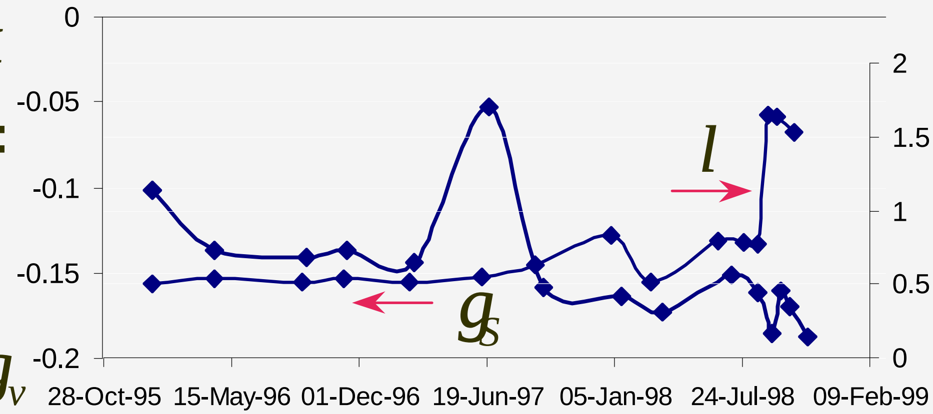


The whole volatility surface is described by one set of constant parameters

Are Smile Parameters Stable Over Time?

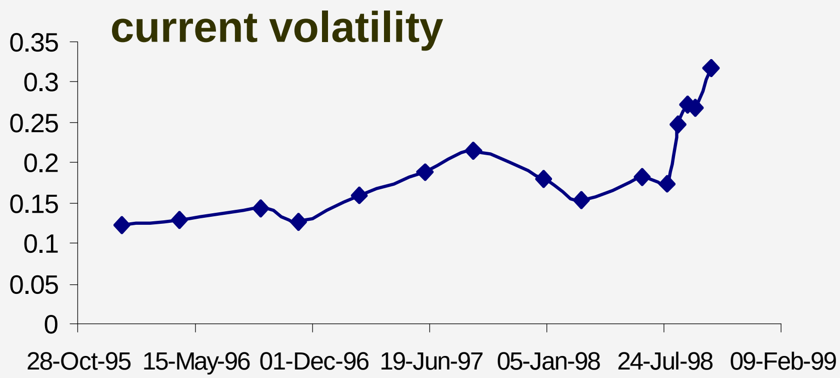
- **Volatility parameters:**
  - current volatility $\sqrt{v}$
  - correlation r
  - vol of vol s
  - long run volatility $\sqrt{q}$
  - mean reversion rate k
- **Market crash parameters:**
  - crash rate l
  - crash magnitude g_s
  - vol jump magnitude g_v

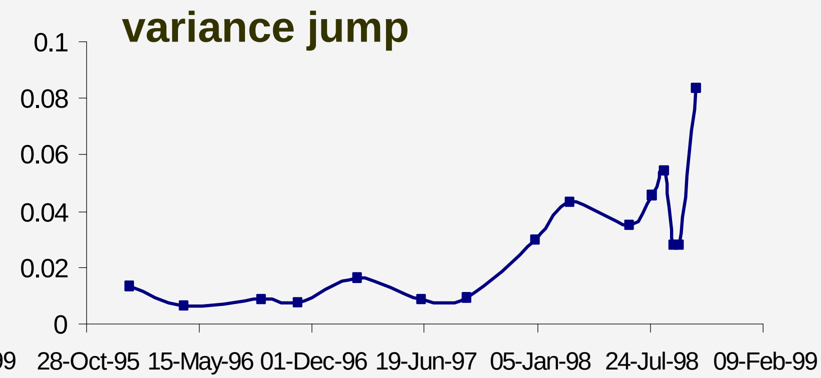


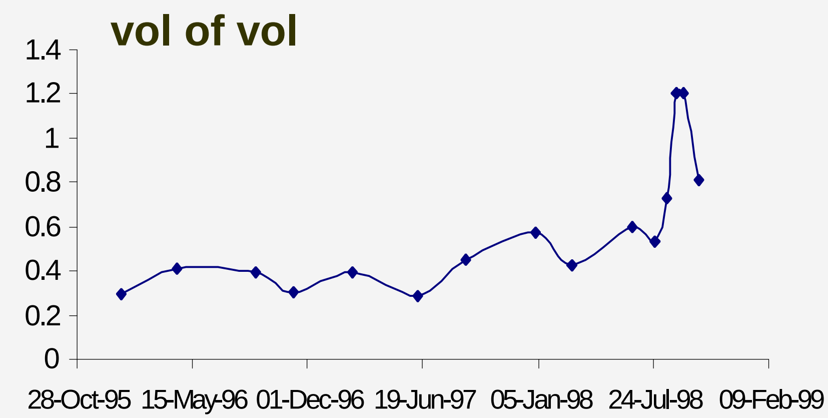


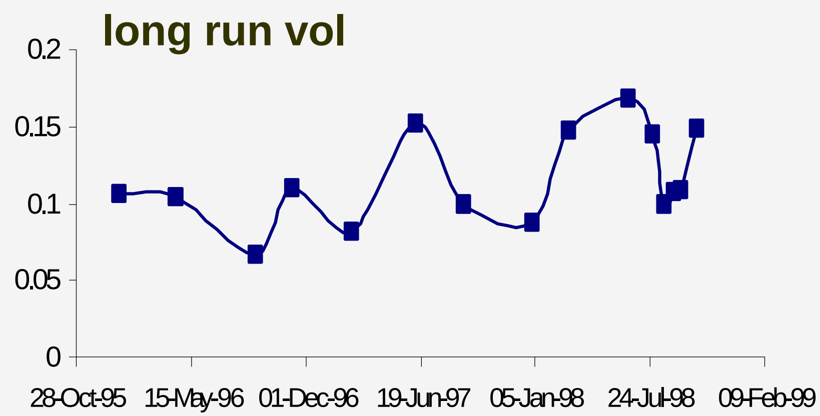
Mean reversion, correlation and crash size are constant

Patterns in Stochastic Volatility Parameters









Long run diffusion volatility is relatively stable

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What Is the Intuition?

- How does each source of risk affect the smile slope
  - at long maturities
  - at short maturities
- What is its effect on
  - ATM volatility
  - smile curvature
- For many models, the “weak smile expansion” is a good guide.
- However, the natural expansion is for the characteristic functional, not the implied volatilities.

How to construct the weak smile expansion?

Linking Characteristic Functionals to Implied Volatilities

- **The characteristic functional**

$$F_t(h) = \int_0^{\infty} p(K)\, e^{ih \ln(K/F)}\, dK$$

The probability distribution

$$p(K) = e^{rT} \frac{\partial^2 C}{\partial K^2}$$

- **Introduce implied standard deviation** $j = \bar{s}(K,T)\sqrt{T}$

Parametrize $j = j(z)$

where

$$z \equiv d_2 = \frac{\ln(F/K)}{j} - \frac{j}{2} = \ln(M/K), \qquad M = Fe^{-j^2/2}$$

- **Then**

$$p(K)\, dK = N'(z)\, dz \left\{ -1 + \frac{z}{j/\dot{j} + z + j} - \frac{\partial}{\partial z}\left(\frac{1}{j/\dot{j} + z + j} \right) \right\} \qquad \dot{j} \equiv \frac{\partial j}{\partial z}$$

Linking Characteristic Functionals to Implied Volatilities

- **Changing the integration variable to z and integrating by parts**

$$F(h) = \int_{-\infty}^{+\infty} dz\, N'(z)\, e^{-ihj\left(\frac{1}{2}j+z\right)} \left(1 + ih\ddot{j}\right)$$

- **In terms of** $w \equiv z + ihj(z)$

$$F(h) = \int_{-\infty}^{+\infty} dw\, N'(w)\, e^{-\frac{1}{2}h(h+i)j^{2}(w)}$$

$$w = \frac{\ln(F/K)}{j(w)} + \left(ih - \frac{1}{2}\right) j(w)$$

$F(h)$ is related to the analytic continuation of j

What Is the First Order Perturbation?

- **Assume** $j^{2}(w) \cong j_{0}^{2} + y_{1}(h, w)$

  with j_{0} **independent of** w**.**

- **Then** $F(h) = e^{-\frac{1}{2} j_{0}^{2} h(h+i)} \left\{ 1 + F_{1}(h) \right\}$

$$\int_{-\infty}^{+\infty} dw\, N'(w) y_{1}(h, w) = -\frac{2}{h(h+i)} F_{1}(h)$$

$$w = -\frac{x}{j_{0}} + j_{0}\left(ih - \frac{1}{2} \right) + O(y_{1})$$

- **As a result**

$$y_{1}(x) = -\frac{2j_{0}}{\sqrt{2p}} e^{\frac{x^{2}}{2j_{0}^{2}}} \int_{-\infty}^{+\infty} \frac{F_{1}(h^{*})\, dh^{*}}{h^{*2} + \frac{1}{4}} e^{-\frac{1}{2} j_{0}^{2} h^{*2} - ixh^{*}} \qquad h^{*} = h + i/2$$

The smile slope is a simple integral of $F_{1}(h^{*})$

What Is the Effect of Price Jumps?

- **In the Merton model**

  The ATM smile slope

$$\left.\frac{\partial y_1}{\partial x}\right|_{x=0} = 2\, lT(e^{g}-1) \qquad \Rightarrow \qquad \left.\frac{d\bar{s}}{dx}\right|_{x=0} = \frac{l(e^{g}-1)}{s_0}$$

  - **stable calibration of expected loss** $l(e^{g}-1)$
  - **more noise in** l **and** g

- **The smile curvature** $(g=0)$

$$\left.\frac{\partial^2 y_1}{\partial x^2}\right|_{x=0} = \frac{lT d^4}{4 j_0^2} \qquad \bar{s} \approx s_0 + \frac{l}{16}\frac{d^4 x^2}{s_0^3 T}$$

  - **at small** d**, very straight skews**
  - **strong dependence on** d **when** d **is large**

What Is the Effect of Stochastic Volatility?

- **Black-Scholes variance** $j_0^2 \equiv qT + \frac{v-q}{k}(1-e^{-kT})$
- **As** $T \rightarrow 0$**,** $y_1'(x) = \frac{1}{2} r s T$

  Hence the smile slope (in stdev space)

$$\frac{1}{\bar{s}_0}\frac{d\bar{s}}{dz} = \sqrt{T}\,\frac{d\bar{s}}{dx} = \frac{r s}{4\sqrt{v_0}}\sqrt{T} \propto 0.16$$

- **As** $T \rightarrow \infty$**,** $y_1'(x) = \frac{r s}{k}$

$$\frac{1}{\bar{s}_0}\frac{d\bar{s}}{dz} = \frac{r s}{2k\sqrt{qT}} \propto 0.06$$

  - **calibration of** $r s$ **more stable**
  - **long run skew often too flat**

The long run Heston smile is often too flat

How Jumps in Volatility Change the Picture?

- **If $g = d = 0$, only the change in volatility level**

$$y_1(x) = -(lT)(g_v T)\frac{1-kT-e^{-kT}}{(kT)^2} \quad \to \frac{1}{2}(lT)(g_v T) \quad \text{as } T \to 0$$

$$\to \frac{l g_v}{k} T \quad \text{as } T \to \infty$$

- **Interaction of vol jumps with price jumps**

$$y_1(x) = \frac{(lT)(g_v T)}{2}\,\frac{g}{j_0^2}\,\frac{1-kT-e^{-kT}}{(kT)^2}$$

–as $T \to 0,\quad \bar{s}_x' = \frac{l g}{s_0} \to (1+a_0)\frac{l g}{s_0} \qquad a_0 = \frac{g_v}{4 s_0^2}$

for the jump from 15% to 35% $\quad g_v \propto 0.10 \Rightarrow a_0 \propto 1.0$

–as $T \to \infty,\quad y_1'(x) = \frac{l g g_v}{kq} \ \text{vs.}\ \frac{r s}{k} \qquad \Rightarrow a_\infty = \frac{l g g_v}{r s q} \propto 0.9$

Volatility jumps significantly affect the skew

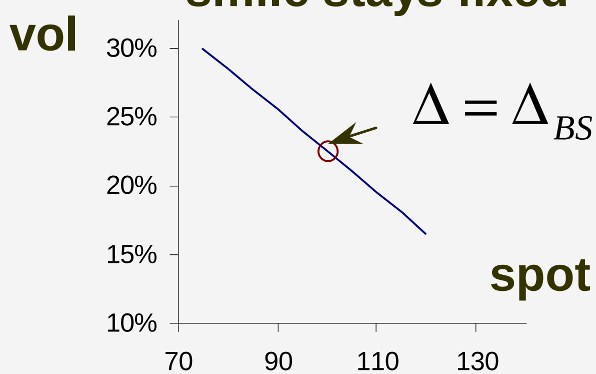
What Is the Delta?

- **When the spot moves, the smile can move too**

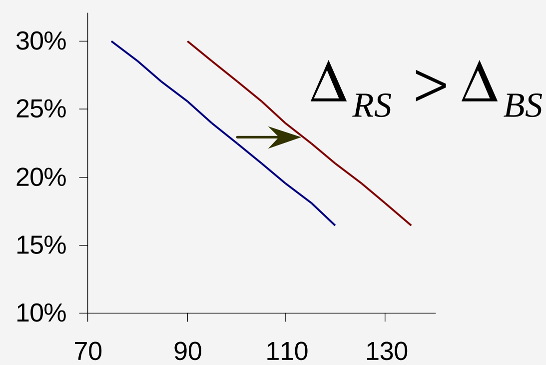
Thus $$\Delta = \frac{dC}{dS} = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial \bar{s}}\frac{d\bar{s}}{dS}$$

- **Three regimes**

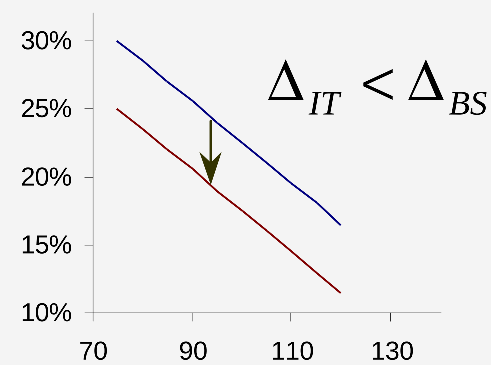
Black-Scholes
smile stays fixed



Relative Smile
smile floats with spot



Fixed Implied Tree
$s(S,t)$ unchanged



Stochastic volatility and jump diffusion yield relative smiles

How to Minimize the P(L) Variance?

- **Given** dS **and** dv**,** $$dC = \Delta\, dS + \Lambda\, dv$$

  Hedge with y **shares:** $$P/L = dC - y\, dS$$

- **In a stochastic volatility model**

  $$\mathrm{Var}(P/L) = (\Delta - y)^2 \mathrm{Var}(dS) + 2(\Delta - y)\Lambda \mathrm{Cov}(dS, dv) + \Lambda^2 \mathrm{Var}(dv)$$

  Minimize with respect to y**:** $$y = \Delta + r\, s \Lambda / S \quad < \Delta$$

- **Since** $\Delta = \Delta_{BS} - \Lambda \left(\overline{s}^{\,2}(x)\right)' / S$

$$y = \Delta_{BS} + r\, s \Lambda / 2S$$

Optimal "risk management" delta < Δ_{BS}

What Is the Meaning of the Implied Tree?

- Imagine the world is described by a stochastic volatility model, but we hedge with the implied tree model
- Then the smile slope

$$\frac{d\bar{s}}{dx} = \frac{r\,s}{4\bar{s}}$$

- When we move the spot, keeping the implied tree fixed,

$$\frac{d\bar{s}^2(x,T)}{dx} = \frac{1}{T}\int_0^T dt\, E_{BB}\left[\left.\frac{\partial \bar{s}^2(x,t)}{\partial x}\right|_{x=x_{BB}(t)}\right]$$

Thus

$$\Delta_{IT} = \Delta_{BS} + r\,s\,\Lambda / 2S = y$$

Implied tree delta mimicks the risk management delta

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Summary and Overview

- Stochastic volatility and market jumps produce a skewed surface of implied volatilities
- The effect of volatility jumps on the skew is highly significant
- Perturbative expansions are a useful tool for understanding the smile
- The optimal delta depends on the dynamics of volatility