MODELING THE STATE–PRICE DEFLATOR AND THE TERM STRUCTURE OF INTEREST RATES

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Abstract. Generalizing the results of Kazemi (1992), we decompose the state–price deflator into the product of an exponential martingale (the permanent component) and the inverse of the price of the very-long discount (VLD) bond (a trend-stationary process). We further decompose the permanent component into a *term-structure martingale* correlated with bond returns and a *neutrino factor* uncorrelated with bond returns. We analyze the implications of the constancy of one or both of these permanent components. When the term-structure martingale is constant, the risk premia for all bonds are determined by the covariance with the return on the VLD bond. When the neutrino factor is constant, risk premia for all assets are determined by the price of term structure risk. When both permanent components are constant, the state–price deflator is trend-stationary and all assets are priced by the VLD bond. Since exchange rates can be modeled as the ratio of two state–price deflators, expected exchange-rate depreciation depends on the ratio of the two neutrino factors. When neutrino factors are constant, an important source of variation is missing. We apply our analysis to uncover the implicit restrictions in Constantinides (1992) and Rogers (1997).

1. Introduction

The state–price deflator (*a.k.a.* the pricing kernel) plays a central role in any general-equilibrium or arbitrage-free model of asset prices. The existence of a state–price deflator guarantees the absence of arbitrage opportunities, and (subject to technical conditions) the converse is also true. The central feature of the state–price deflator is that deflated trading gains are martingales. In the consumption-based capital asset pricing model (C–CAPM), for example, the state–price deflator is the subjectively discounted value of the marginal utility of consumption. In this case, the martingale property is embodied in the optimality condition that the expected marginal utility from liquidating a portfolio of assets in the future equals the marginal utility of its purchase price today.

In the context of a nominal version of the C–CAPM, Kazemi (1992) demonstrates that if the marginal utility of consumption has a limiting stationary distribution

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then the very-long discount (VLD) bond plays the same role as that of the market portfolio in the CAPM, namely risk-premia are determined by the covariance with the return on the VLD bond. Kazemi's point generalizes beyond his equilibrium model to *any* state–price deflator—real or nominal, general-equilibrium based or absence-of-arbitrage based: If the state–price deflator is trend-stationary, the value of the VLD bond is its inverse, with the consequence that all risk premia are determined by covariance with the return on the VLD bond.[1] This is a substantive restriction that has significant implications, some of which may or may not be desirable.[2]

The state–price deflator can be decomposed (subject to technical conditions) into the product of two factors: a "permanent" process and a trend-stationary process.[3] The permanent process is a martingale, which we compute as the *asymptotic revision process* for the state–price deflator: The permanent component changes if and only if the conditional expectation of the state–price deflator (suitable normalized) in the distant future changes. The a trend-stationary process is the inverse of the VLD bond price. When the state–price deflator is trend-stationary, the permanent component is identically one.

In general, changes to the permanent component will be correlated with bond returns, including the return on the VLD bond. This correlation can be used to decompose the permanent component itself into the product of two (exponential) martingales that are independent of each other (one or both of which may be identically one). Here is the recipe for computing the decomposition: Given the interest rate and the price of risk, which can be computed from the dynamics of the state–price deflator, one can compute bond returns. The difference between the price of term-structure risk and the volatility of the return on the VLD bond identifies the two components, one of which is correlated with bond returns (the *term-structure martingale*) and the other of which is not (the *neutrino factor*). The neutrino factor is so-named because it passes through the bond market without a trace (much as neutrinos pass through the ordinary matter undetected).[4]

If it turns out that the term-structure martingale is constant (in which case the neutrino factor comprises the entire permanent component of the state–price deflator), then the price of term-structure risk is identical to the volatility of the

[1]The returns process for the VLD bond is simply the limit of the returns process for zero-coupon bonds with finite horizons. Since the Treasury can issue zero-coupon bonds with any initial maturity, the return on the VLD bond can be approximated arbitrarily well.

[2]If the restriction were true, then of course it would be desirable to impose it. We suspect that it is far from true and that its imposition simply reduces the flexibility of models and may lead one to inappropriately abandon a model. One of the implications is spelled out in Appendix A where we show how the VLD bond can be used to extract market expectations when the state–price deflator is trend-stationary.

[3]This decomposition is similar in spirit to the decomposition of Beveridge and Nelson (1981). We discuss the relation between the two in Appendix C.

[4]We are able to identify the neutrino factor because we assume we know the state–price deflator. If instead we wished to construct a state–price deflator from bond prices, we would have no information about any neutrino factors. We would need other security prices to compute a neutrino factor, and even then there would always be the possibility that were other priced shocks beyond those captured by the set of assets chosen.

return on the VLD bond and consequently risk-premia for all bonds are determined by the covariance with the return on the VLD bond. In other words, the VLD bond prices the bond market. This by itself is a significant restriction that may or may not be desirable, even if the VLD bond does not price all assets (as it would if the state–price deflator were trend-stationary). For example, term premia in the bond market are constrained in such a way that they can be identified from knowledge of the formula for bond prices and the volatility of the state variables.[5]

The classic example of a model of the state–price deflator with a nonconstant neutrino factor and a constant term-structure martingale is that of Constantinides (1992), whose main purpose was to construct a model of the term structure with positive interest rates. More recently, Rogers (1997) has proposed a strategy for modeling both the term structure (also with positive interest rates) and foreign exchange rates that imposes trend-stationarity on the state–price deflator.[6] Not only does this strategy have the side-effects already described, but it also constrains the behavior of exchange rates in an important way. Exchange rates can be modeled as ratios of two state–price deflators. As Rogers points out, if one follows his strategy, there can be no shocks to the exchange rate that are independent of the two term structures. There is, however, evidence that expected exchange-rate depreciation has a component that is independent of both term structures. As we discuss, the only way for this to come about is if there is a *neutrino-factor component* to expected exchange-rate depreciation. A neutrino-factor component will be present if and only if the ratio of the two neutrino factors (one from each state–price deflator) is not identically one, and this requires additional shocks.

One can imagine two modeling strategies for introducing Markovian state variables. The first strategy is to directly Markovianize the state–price deflator itself. This is essentially what Constantinides and Rogers have done. The second strategy is to Markovianize the dynamics of the state–price deflator—the interest rate and the price of risk. Clearly a model derived via one strategy must be derivable at least in principle via the other strategy: One can always recover the dynamics from the solution using Ito's lemma; conversely, a model of the dynamics is valid only when the stochastic integral associated with those dynamics exists. Nevertheless, as we have indicated, the typical strategy of modeling the state–price deflator directly has lead to what could be called an "over-Markovianization."

The source of the restrictiveness of the strategy of modeling the state–price deflator can be understood by considering the relation between bond prices and the state–price deflator. In an economy where the price of a zero-coupon bond can be

[5]We note in passing that our analysis has implications for the macro literature that has examined (in the context of the C–CAPM) how the slope of the term structure depends on whether or not output, and hence the state-price deflator, is trend-stationary. In particular, the analysis has implicitly identified the permanent component of the state–price deflator with the term-structure martingale. (See, for example, Labadie (1994).) By contrast, if a neutrino factor were introduced into otherwise trend-stationary economy, the result would be a "trend-stationary" term-structure in an economy that is not trend-stationary.

[6]Jin and Glasserman (1997) show how positive interest rate models can be built from a family of positive martingales. In Appendix B we show that the permanent component of the state-price deflator is the limit of that family.

expressed as a function of a set of Markovian state variables, the state–price deflator cannot in general be expressed as a function of the same set state variables. Consequently, modeling strategies that impose a Markovian structure on the state–price deflator impose restrictions on the price of risk that can be undesirable.

To explore these restrictions, we model the state–price deflator as the ratio of an exponential martingale to the value of a Markovian bubble asset. We show that the state variables that drive the value of the bubble also drive the term structure of interest rates, and we show how the bubble is related to the VLD bond. We present a compendium of results regarding the VLD bond, extending the results of Kazemi (1992) that relate the volatility of the VLD bond to the price of risk.

Related work. Kazemi and Georgiev (1999) have independently covered some of the same ground that we do in this paper. In particular, they adopt the exponential-affine framework as delineated by Duffie and Kan (1996), where the market price of risk is unspecified and the dynamics of the state variables are specified only under the standard equivalent martingale measure ($\mathcal{Q}^\lambda$ in our notation; see Section 4). Within this framework they provide two examples: a single factor model and a multifactor model that includes a neutrino factor. In both examples the term-structure martingale is constant. They confirm that the price of (term-structure) risk equals the volatility of the return on the VLD bond in both examples. They emphasize the advantage of the constancy of the term-structure martingale: No additional parameters are required to complete the model, because the price of risk can be computed from parameters identified in bond prices.

Outline of the rest of the paper. In Section 2 we provide an introductory example. In Section 3 we provide an explicit statement of the information structure that we adopt in this paper. In Section 4 we define the asymptotic revision process and trend-stationarity, and we present a number of propositions in a non-Markovian setting. In Section 5 we apply these results in a Markovian setting and discuss Constantinides (1992). In Section 6 we treat the exponential-affine class of models and provide additional examples that illustrate a number of results from Section 5. In Section 7 we examine exchange rates, with particular focus on how the neutrino-factor component affects the dynamics of expected exchange rate depreciation, and we discuss Rogers (1997).

The paper has a number of appendices. In Appendix A we show how the VLD bond can be used to extract market expectations. In Appendix B we discuss the relation between the asymptotic revision process and the family of positive martingales that are central to positive interest-rate models. In Appendix C we discuss the relation between our decomposition and the Beveridge–Nelson decomposition. In Appendix D we relate a generalized VLD bond to the payoff bubbles of Gilles and LeRoy (1997).

2. An introductory example

Before proceeding to the general analysis, we illustrate several of the main points with a simple example.

Preliminaries. First we establish some notation and state quite briefly a few asset-pricing facts we use in the example.[7]

A state–price deflator is a positive process that guarantees the absence of arbitrage opportunities. Let n denote a state–price deflator, normalized so that $n(0) = 1$. Let $p(t,T)$ denote the value at time t of a default-free, zero-coupon bond that pays one unit when it matures at time T. The fact that the bonds are default-free implies $p(T,T) = 1$. Forward rates can be computed from bond prices:

$$f(t,T) = -\partial \log(p(t,T))/\partial T.$$

The absence-of-arbitrage condition provided by the state–price deflator is that deflated asset prices are martingales. For bond prices this means $n(t)\, p(t,T) = E_t[n(\tau)\, p(\tau,T)]$ for $t \le \tau \le T$, where $E_t[\,\cdot\,]$ is the expectation operator conditional on information at time t. For $\tau = T$, the absence-of-arbitrage condition has a convenient form:

$$p(t,T) = E_t[n(T)/n(t)]. \tag{2.1}$$

Let us assume that all uncertainty is driven by Brownian motion and that the state–price deflator and bond price are Ito processes. In this case, the dynamics of n are given by

$$\frac{dn(t)}{n(t)} = -r(t)\, dt - \lambda(t)^\top dW(t), \tag{2.2}$$

where r is the risk-free interest rate, λ is the price of risk vector, and W is a vector of independent Brownian motions.[8] The dynamics of bond prices can be written as

$$\frac{dp(t,T)}{p(t,T)} = \mu_p(t,T)\, dt + \sigma_p(t,T)^\top dW(t), \tag{2.3}$$

where μ_p is the expect return and σ_p is the volatility of the return. The absence-of-arbitrage condition can be expressed in terms of these dynamics. In particular, the fact that the deflated bond price, $n(t)\, p(t,T)$, is a martingale means that it has zero drift, which implies

$$\mu_p(t,T) = r(t) + \lambda(t)^\top \sigma_p(t,T). \tag{2.4}$$

Equation (2.4) says that the expected return equals the risk-free rate plus a risk premium that depends on the covariance with the state–price deflator.

The example. In this example there is a single state variable with Ornstein–Uhlenbeck dynamics:

$$dX(t) = \kappa\,(\theta - X(t))\, dt + \sigma\, dW_X(t),$$

where κ, θ, and σ are parameters and W_X is a Brownian motion. With these dynamics, the distribution of $X(T)$ conditional on $X(t)$ is Gaussian, with mean

[7] See Section 3 for more detail.

[8] The notation $x^\top y$ denotes the inner product of two vectors x and y.

$X(t,T) := E_t[X(T)]$ and variance $\Sigma(t,T) := E_t[(X(T) - E_t[X(T)])^2]$, where

$$X(t,T) = \theta \left(1 - e^{-\kappa\,(T-t)}\right) + X(t)\, e^{-\kappa\,(T-t)} \tag{2.5a}$$

$$\Sigma(t,T) = \frac{\sigma^2}{2\,\kappa} \left(1 - e^{-2\,\kappa\,(T-t)}\right). \tag{2.5b}$$

Assuming $\kappa > 0$, the limiting distribution as $T \to \infty$ is independent of any information at time t.

Model 1. In the first modeling strategy, we model the interest rate and the price of risk as functions of the state variable:

$$r(t) = a + b\,X(t) \qquad \text{and} \qquad \lambda(t) = q, \tag{Model 1}$$

where a, b, and q are parameters. Together r, λ, and the dynamics of X comprise an exponential-affine model of the term structure.[9] The price of a zero-coupon bond is given by $p(t,T) = P(X(t), T-t)$, where $P(x,\tau) = \exp(-A(\tau) - B(\tau)\,x)$. Given this setup, absence-of-arbitrage condition (2.4) becomes a partial differential equation (see (5.16) below), the solution to which is

$$B(\tau) = \frac{b}{\kappa}\left(1 - e^{-\kappa\,\tau}\right)$$

$$A'(\tau) = a + (\kappa\,\theta - q\,\sigma)\,B(\tau) - \frac{1}{2}\,\sigma^2\,B(\tau)^2.$$

Taking limits, we have

$$B(\infty) := \lim_{\tau\to\infty} B(\tau) = \frac{b}{\kappa}$$

$$A'(\infty) := \lim_{\tau\to\infty} A'(\tau) = a + b\,\theta - \frac{b\,q\,\sigma}{\kappa} - \frac{1}{2}\,\frac{b^2\,\sigma^2}{\kappa^2}.$$

The asymptotic forward rate is given by $A'(\infty)$. The volatility of bond returns is given by $-B(\tau)\,\sigma$, so the asymptotic bond-return volatility is

$$-B(\infty)\,\sigma = -b\,\sigma/\kappa. \tag{2.6}$$

(The asymptotic bond-return volatility is the volatility of the return for the VLD bond. We introduce the VLD bond formally in Section 4.)

Model 2. In the second modeling strategy, we model the state–price deflator $n_0(t)$ as a function of the state variable and time:

$$n_0(t) = \exp\left\{-\alpha\,t - \beta\left(X(t) - X(0)\right)\right\}, \tag{Model 2}$$

where α and β are parameters. (The factor $e^{\beta\,X(0)}$ ensures $n_0(0) = 1$.) We can compute bond prices using absence-of-arbitrage condition (2.1):

$$p_0(t,T) = E_t[n_0(T)/n_0(t)] = e^{-\alpha\,(T-t) + \beta\,X(t)}\,E_t\left[e^{-\beta\,X(T)}\right].$$

[9]We treat exponential-affine models in general in Section 6, where we also provide a number of other exponential-affine examples. In the current example, if we set $a = 0$ and $b = 1$ we have the term structure model of Vasicek (1977).

Since $-\beta\, X(T)$ is conditionally Gaussian, $e^{-\beta\, X(T)}$ is conditionally lognormally distributed, with

$$E_t\left[e^{-\beta\, X(T)}\right] = \exp\left(-\beta\, X(t,T) + \frac{1}{2}\,\beta^2\,\Sigma(t,T)\right). \tag{2.7}$$

Therefore, bond prices are given by $p_0(t,T) = P_0(X(t), T-t)$, where $P_0(x,\tau) = \exp(-A_0(\tau) - B_0(\tau)\,x)$ and where

$$A_0(\tau) = \alpha\,\tau + \beta\,\theta\left(1 - e^{-\kappa\,\tau}\right) - \frac{\beta^2\,\sigma^2}{4\,\kappa}\,\left(1 - e^{-2\,\kappa\,\tau}\right) \tag{2.8a}$$

$$B_0(\tau) = -\beta\left(1 - e^{-\kappa\,\tau}\right). \tag{2.8b}$$

We see that Model 2 produces a term-structure model of the same form as Model 1. From (2.8), we can compute the asymptotic forward rate $A_0'(\infty) = \alpha$ and the asymptotic bond return volatility $-B_0(\infty)\,\sigma = \beta\,\sigma$.

By applying Ito's lemma to the state–price deflator n_0 and identifying the relative drift and diffusion as in (2.2), we can compute the interest rate and price of risk for Model 2: $r_0(t) = a_0 + b_0\,X(t)$ and $\lambda_0(t) = q_0$, where

$$a_0 = \alpha + \beta\,\kappa\,\theta - \frac{1}{2}\,\beta^2\,\sigma^2, \qquad b_0 = -\beta\,\kappa, \qquad \text{and} \qquad q_0 = \beta\,\sigma.$$

Although the Model 2 is quite similar to Model 1, there is an important distinction between them. In Model 2, the price of risk λ_0 is necessarily identified with the asymptotic volatility of bond returns $-B_0(\infty)\,\sigma$, which is not true in Model 1. In other words, Model 2 is a restricted version of Model 1.

Model 3. We can modify the second modeling strategy to produce a model that is equivalent to Model 1. In order to have an equivalent model, we must incorporate an exponential martingale that is driven by the same Brownian that drives the term structure. We call this martingale the *term-structure* martingale. Let z_1 denote the term-structure martingale, and let $z_1(0) = 1$, and let the dynamics of z_1 be given by $dz_1(t)/z_1(t) = -g\,dW_X(t)$, where g is a parameter. Now model the state–price deflator as follows:

$$n_1(t) = z_1(t)\,n_0(t), \tag{Model 3}$$

where $n_0(t)$ is given in Model 2. The conditional expectation $E_t[z_1(t)\,n_0(t)]$ is not as simple to compute, since z_1 and n_0 are correlated. In any event, we can apply Ito's lemma to n_1 to compute the interest rate and price of risk for Model 3: $r_1(t) = a_1 + b_1\,X(t)$ and $\lambda_1(t) = q_1$, where

$$a_1 = a_0 - g\,\beta\,\sigma, \qquad b_1 = b_0, \qquad \text{and} \qquad q_1 = q_0 + g.$$

These equations provide a unique solution for α, β, and g in terms of a_1, b_1, and q_1, which we can identify with the parameters a, b, and q of Model 1. In particular, we have

$$\alpha = A_1'(\infty), \qquad \beta = -B_1(\infty), \qquad \text{and} \qquad g = q - \beta\,\sigma. \tag{2.9}$$

Equations (2.9) highlight the features of the Model 3 that are and are not changed relative to Model 2 by the addition of the term-structure martingale. Neither the asymptotic forward rate nor the asymptotic bond-return volatility are changed, but the price of risk is changed, driving a wedge between the price of risk and the asymptotic bond-return volatility.

Model 4. There is another point we can illustrate with this example. As it stands, the state–price deflator in Model 3 has the property that only term-structure-related risk is priced. In other words, the risk premium on equities (for example) can be computed from the price of risk that is identified solely from bond prices. If the interest rate were constant, then there would be no risk premium for *any* asset.

In order to introduce additional components to the price of risk vector without affecting the term structure, we can augment the model with what we call a *neutrino factor*, an exponential martingale that is independent of the term structure. Let z_2 denote this martingale (with $z_2(0) = 1$), and define a new state–price deflator:

$$n_2(t) = z_2(t)\, n_1(t), \qquad \text{(Model 4)}$$

where $n_1(t)$ is given in Model 3. The key feature of Model 4 in relation to Model 3 is

$$E_t[n_2(T)/n_2(t)] = E_t[n_1(T)/n_1(t)], \qquad (2.10)$$

so that bond prices are the same in the two models. As an example, let the dynamics of z_2 be given by $dz_2(t)/z_2(t) = -h(X(t), Y(t))\, dW_Y(t)$, where W_Y is a Brownian motion that is independent of W_X, $h(x, y)$ is a deterministic function, and Y is a state variable that may be driven by both W_X and W_Y. Applying Ito's lemma to n_2, we can see that $r_2(t) = r_1(t)$ and the price of risk now has an additional component:

$$\lambda_2(t) = \begin{pmatrix} \beta\,\sigma + g \\ h(X(t), Y(t)) \end{pmatrix}.$$

The first component is the *price of term-structure risk*, since the second component has no impact on bond prices as is clear from (2.10). In particular, a neutrino factor does not affect the relation between the price of term-structure risk and the asymptotic volatility of bond returns.

3. Description of the setting

Uncertainty. We refer the reader to Duffie (1996) or Karatzas and Shreve (1988), for all the missing details and definitions.

We fix a vector $W = (W_1, \ldots, W_d)$ of orthogonal standard Brownian motions defined on a complete probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ (the vector W is often split into vectors W_X and W_Y, with dimensions d_X and d_Y satisfying $d_X + d_Y = d$). All the components of W are initialized to 0, that is, $\mathcal{P}(W(0) = 0) = 1$. The filtration $\{\mathcal{F}_t \mid t \geq 0\}$ that describes the evolution of uncertainty is that generated by $W(t)$, augmented so that $\mathcal{F}_t$ includes all the $\mathcal{P}$-null sets of $\mathcal{F}$. This filtration satisfies the so-called *usual conditions*.

We define a stochastic process Z as a jointly measurable function on $[0, \infty) \times \Omega$ into some Euclidean space, and we write $Z(t)$ for the random variable $Z(t, \cdot)$ defined on Ω. We restrict attention to the stochastic processes Z that are adapted to the filtration $\{\mathcal{F}_t\}$ and have continuous paths (that is, $Z(t)$ is $\mathcal{F}_t$-measurable and, for $\mathcal{P}$-almost all ω, $t \to Z(t,\omega)$ is continuous).

We write $dZ(t) = \mu_z(t)\,dt + \sigma_z(s)^\top dW(t)$ to mean that $Z(t)$ is a process with representation as a (scalar) stochastic integral of the form

$$Z(t) = Z(0) + \int_0^t \mu_z(s)\,dt + \int_0^t \sigma_z(s)^\top dW(s).$$

The *drift* of Z, $\mu_z(t)$, and its *diffusion*, $\sigma_z(t)$, may be stochastic processes themselves, in which case we always assume that they satisfy the integrability conditions: $\mathcal{P}$-a.s. and for all $t \geq 0$, (a) $\int_0^t |\mu_z(s)|ds < \infty$ and (b) $\int_0^t \|\sigma_z(s)\|^2 ds < \infty$. This implies that all stochastic processes that we consider are square-integrable ($E_t[Z(T)^2] < \infty$ for all $t \leq T$, a.s., so that all conditional first and second moments exist).

We write E_t for the expectation conditional on the σ-field $\mathcal{F}_t$, that is, given a random variable X defined on Ω, $E_t[x] := E[X \mid \mathcal{F}_t]$; conditional variances and covariances are denoted in a similarly fashion.

The state-price deflator. The state-price deflator plays a central role in the analysis. It is a strictly positive stochastic process n normalized so that $n(0) = 1$ and such that the deflated gains process associated with any admissible trading strategy is a martingale—the so-called *no-arbitrage condition.* For ease of exposition, we restrict attention to assets that are claims to a single lumpy payoff. If $S(t)$, $0 \leq t \leq T$, is the value process of such an asset with payoff at time T and n is the state-price deflator, then $n\,S$ is a martingale; that is,

$$n(t)\,s(t) = E_t[n(T)\,s(T)] \tag{3.1}$$

for all $0 \leq t \leq \tau \leq T$, a.s.

We assume that all default-free zero-coupon bonds are traded. In other words, for all $T > 0$ the claim to one unit of numeraire payable at T is traded, and $p(t,T)$, $0 \leq t \leq T$ denotes its price process. Because $p(T,T) = 1$ and $n(t)\,p(t,T)$ is a martingale, we have that $p(t,T) = E_t[n(T)]/n(t)$ for all $t < T$; moreover, because $n(t)$ is strictly positive, $p(t,T) > 0$, so that $\log(p(t,T))$ is well defined.

For any $t \geq 0$, we define the instantaneous forward rate curve $\{f(t,T) \mid T \geq t\}$ by $f(t,T) := -\partial \log(p(t,T))/\partial T$, when such a derivative exist, and we define the short rate as $r(t) := \lim_{T\to t} f(t,T)$ when such a limit exists. We restrict attention to state-price deflators that are regular enough that the forward rate curve and the short rate exist a.s., for all t.

A state price deflator n can be represented as a stochastic integral, which in differential form can be expressed as

$$dn(t) = \mu_n(t)\,dt + \sigma_n(t)^\top dW(t). \tag{3.2}$$

If s is the value of a strictly positive asset, then its process can be written

$$ds(t)/s(t) = \mu_s(t)\,dt + \sigma_s(t)^\top dW(t). \tag{3.3}$$

We can express absence-of-arbitrage condition (3.1) in terms of the parameters in (3.2) and (3.3) by applying Ito's lemma to the deflated asset price and setting the drift to zero:

$$\mu_s(t) = -\left(\frac{\mu_n(t)}{n(t)}\right) - \left(\frac{\sigma_n(t)}{n(t)}\right)^\top \sigma_s(t). \tag{3.4}$$

Equation (3.4) characterizes the relative drift of this asset's value (the expected return of this asset) in terms of the relative drift of the state–price deflator and the covariance between relative changes in the state–price deflator and relative changes in this asset's value.

We define the value of the money market account (MMA) as the process $\beta(t) = \exp\left(\int_0^t r(s)ds\right)$, which obeys the locally risk-free dynamics

$$d\beta(t)/\beta(t) = r(t)\, dt.$$

The MMA can be interpreted as a *bubble*, the value of a claim to a payoff at infinity, as shown in the Appendix D. Applying (3.4) to the value of the MMA, we see that $r(t) = -\mu_n(t)/n(t)$. We can use this relation along with the definition $\lambda(t) := -\sigma_n(t)/n(t)$ to rewrite (3.4) as

$$\mu_s(t) = r(t) + \lambda(t)^\top \sigma_s(t). \tag{3.5}$$

The term $\lambda(t)^\top \sigma_s(t)$ is called the risk-premium and $\lambda(t)$ is called the price of risk since it determines component-by-component the risk premium in terms of the amount of risk as measured by $\sigma_s(t)$. In the case of bond prices, where the dynamics of $p(t,T)$ are given by (2.3), the no-arbitrage condition boils down to (2.4).

An equivalent martingale measure. Given a process ϑ, define

$$\zeta^\vartheta(t) = \exp\left(\int_{s=0}^t -\frac{1}{2}\,\|\vartheta(s)\|^2\, ds - \int_{s=0}^t \vartheta(s)^\top dW(s)\right). \tag{3.6}$$

Under suitable integrability conditions, ζ^ϑ is a martingale under the physical measure $\mathcal{P}$.[10] The dynamics of ζ^ϑ as given by $d\zeta^\vartheta(t)/\zeta^\vartheta(t) = -\vartheta(t)^\top dW(t)$. When n is a state–price deflator, we assume ζ^λ is a $\mathcal{P}$-martingale.

It is convenient to express expectations and dynamics under an equivalent measure, where an exponential martingale ζ^ϑ is the change-of-measure process. We refer to this equivalent measure as $\mathcal{Q}^\vartheta$. The Brownians for $\mathcal{Q}^\vartheta$ are related to Brownians for the original measure (the physical measure $\mathcal{P}$) via

$$dW^\vartheta(t) = dW(t) + \vartheta(t)\, dt, \tag{3.7}$$

where W^ϑ are standard independent Brownian motions under $\mathcal{Q}^\vartheta$. Conditional expectations under $\mathcal{Q}^\vartheta$ can be computed under $\mathcal{P}$ via

$$E_t^\vartheta[A_T] = \zeta^\vartheta(t)^{-1}\, E_t\left[\zeta^\vartheta(T)\, A_T\right], \tag{3.8}$$

[10]The Novikov condition is sufficient:

$$E_0\left[\exp\left(\frac{1}{2}\int_{s=0}^T \|\vartheta(s)\|^2\, ds\right)\right] < \infty \qquad \text{for all } 0 \le T < \infty.$$

See Karatzas and Shreve (1988, Section 3.5.D).

where A_T is any time-T measurable random variable and $E_t^{\vartheta}[\,\cdot\,]$ is the conditional expectation operator under $\mathcal{Q}^{\vartheta}$. Reversing direction, conditional expectations under $\mathcal{P}$ can be computed under $\mathcal{Q}^{\vartheta}$ via

$$E_t[A_T] = \zeta^{\vartheta}(t)\, E_t^{\vartheta}\left[\zeta^{\vartheta}(T)^{-1}\, A_T\right], \tag{3.9}$$

where $\zeta^{\vartheta}(t)^{-1}$ is a martingale under $\mathcal{Q}^{\vartheta}$.

4. The VLD bond in a Non-Markovian setting

Kazemi (1992) conducts his analysis in a specific equilibrium model in a Markovian setting in nominal terms. As we will show, the central idea extends beyond this setting to a non-Markovian no-arbitrage setting, where the state–price deflator may be measured in any units.

The very-long discount bond. The simplest positive asset is a default-free zero-coupon bond that makes a single unit payment when it matures. Recall that $p(t,T)$ is the value at time t of a zero-coupon bond that pays one unit when it matures at time T. Now let $v(t,T)$ denote the value of a zero-coupon bond that pays $1/p(0,T)$ units at maturity, so that $v(t,T) = p(t,T)/p(0,T)$. We define the value of the VLD bond as follows: $v(t) := \lim_{T\to\infty}\, v(t,T)$ if the limit exists, in which case we say the VLD bond exists.

If the VLD bond exists, then

$$\lim_{T\to\infty} \left.\frac{\partial v(t,s)}{\partial s}\right|_{s=T} = \lim_{T\to\infty}\, v(t,T)\,\big(f(0,T) - f(t,T)\big) = 0, \tag{4.1}$$

where $f(t,T)$ is the forward rate. Equation (4.1) indicates that, if the VLD bond exists, any changes in the long (*i.e.*, asymptotic) forward rate must be offset by the value of the VLD bond going to zero. Dybvig, Ingersoll, and Ross (1996) have shown that the long forward rate cannot fall absent arbitrage opportunities. Since the existence of a state–price deflator implies the absence of arbitrage opportunities, the long forward rate can never fall in our setting. However, the absence of arbitrage does not rule out the possibility that the long forward rate may rise. If it were to rise, then the value of the VLD bond must go to zero.[11]

The asymptotic revision process and trend-stationarity. Here we present some useful definitions and propositions.

[11] Dybvig, Ingersoll, and Ross (1996) provide two examples where the the long rate rises: A discrete-time example where the value of the VLD bond goes to zero each time the long rate rises (simultaneously, a different asset takes on the value of a VLD bond defined in terms of the limit of normalized bond prices) and a continuous-time example where the long rate rises from some finite value at time zero to infinity thereafter so that the VLD bond ceases to exist beyond time zero. They also refer to another continuous-time example, found in Ingersoll, Jr., Skelton, and Weil (1978), where the long forward rate exists for all $t \geq 0$. In this model, upward shifts in the yield curve are driven by Poison events.

Definition 1. We say that the forward rate is *asymptotically-deterministic* if, almost surely (hereafter a.s.),

$$\lim_{T\to\infty} f(0,T) - f(t,T) = 0 \quad \text{for all } t \geq 0. \tag{4.2}$$

A constant long forward rate is sufficient to guarantee (4.2), but not necessary. We could have $f(t,T) = \sin(T)$, for example, in which case $v(t) = e^{1-\cos(t)} > 0$.

Proposition 1. If the value of the VLD bond exists and is strictly positive a.s., the forward rate is asymptotically deterministic.

Proof. If value of the VLD bond is strictly positive, then (4.1) implies the forward rate is asymptotically-deterministic. □

Definition 2. Let n and $\hat{n}$ be two state–price deflators with forward rates f and $\hat{f}$ respectively. We say f and $\hat{f}$ are *asymptotically the same* if a.s.

$$\lim_{T\to\infty} f(t,T) - \hat{f}(t,T) = 0 \quad \text{for all } t \geq 0.$$

Definition 3. Let n be a strictly positive Ito process normalized so that $n(0) = 1$. If a.s.

$$z(t) = \lim_{T\to\infty} \frac{E_t[n(T)]}{E_0[n(T)]}$$

exists for all $t \geq 0$, then we say that z is the *asymptotic revision process* for n (under $\mathcal{P}$), which we write as $z = \varphi(n)$. Otherwise, we say that the asymptotic revision process for n does not exist. Similarly, if a.s.

$$z(t) = \lim_{T\to\infty} \frac{E_t^{\vartheta}[n(T)]}{E_0^{\vartheta}[n(T)]}$$

exists for all $t \geq 0$, then we say that z is the asymptotic revision process for n under $\mathcal{Q}^{\vartheta}$, which we write as $z = \varphi^{\vartheta}(n)$.

Proposition 2. For a positive Ito process n, suppose $z = \varphi(n)$ exists a.s. and is integrable. Then z is a martingale.

Proof. Define $x(t) := n(t)/E_0[n(t)]$, so that $z(t) = \lim_{T\to\infty} E_t[x(T)]$. Suppose the Proposition is false; then there exist $t \geq 0$, $T > 0$, $\epsilon > 0$, and a set of paths of positive probability such that for all these paths $\|z(t) - E_t[z(T)]\| > \epsilon$. Because $z(t)$ is a limit, there exists $S > t$ such that $\forall s > S$, $\|z(t) - E_t[x(s)]\| < \epsilon/2$, so that $\forall s > S$, $\|E_t[x(s)] - E_t[z(T)]\| > \epsilon/2$. Because of the integrability of $z(T)$, we have (a consequence of Fatou's lemma) that $E_t[z(T)] = \lim_{\tau\to\infty} E_t[x(\tau)]$, so that there is an $S' > \max\{S, T\}$ such that $\|E_t[z(T)] - E_t[x(S')]\| < \epsilon/4$. Therefore, $\forall s > S$, $\|E_t[x(s)] - E_t[x(S')]\| > \epsilon/4$. Taking $s = S'$ yields a contradiction. □

Definition 4. We say that n is *trend-stationary* (under $\mathcal{P}$) if a.s.

$$\lim_{T\to\infty} \frac{E_t[n(T)]}{E_0[n(T)]} = 1$$

for all $t \geq 0$, in which case we write $\varphi(n) = 1$. In other words, a trend-stationary process has an asymptotic revision process that is identically one, which is consistent with the absence of revisions to the long-horizon forecast. Otherwise we say that n is not trend-stationary. Similarly, we say that n is trend-stationary under $\mathcal{Q}^\vartheta$ if $\varphi^\vartheta(n) = 1$.

See Appendix C for a discussion of how our definition of trend-stationarity is related to the notion of trend-stationarity in an ARIMA setting.

Proposition 3. Let v denote the VLD bond price process and let n denote the state–price deflator. Then v exists if and only if $\varphi(n)$ exists, in which case

$$v = \varphi(n)/n.$$

Proof.

$$v(t) = \lim_{T\to\infty} v(t,T) = \lim_{T\to\infty} \frac{p(t,T)}{p(0,T)} = \frac{1}{n(t)} \lim_{T\to\infty} \frac{E_t[n(T)]}{E_0[n(T)]}.$$

□

Proposition 3 implies

$$v = 1/n \iff \varphi(n) = 1. \tag{4.3}$$

In other words, the VLD bond is the inverse of the state–price deflator if and only if the state–price deflator is trend-stationary. In this case, v is strictly positive (since n is never infinite), and the forward rate is asymptotically-deterministic (by Proposition 1).

We illustrate Proposition 3 with the example from Section 2. We start by computing the asymptotic revision process for the state–price deflator from Model 2:

$$\begin{aligned}
\lim_{T\to\infty} \frac{E_t[n_0(T)]}{E_0[n_0(T)]} &= \lim_{T\to\infty} \frac{E_t\left[e^{-\beta X(T)}\right]}{E_0\left[e^{-\beta X(T)}\right]} \\
&= \lim_{T\to\infty} \exp\left\{-\beta\big(X(t,T) - X(0,T)\big) + \frac{1}{2}\beta^2\left(\Sigma(t,T) - \Sigma(0,T)\right)\right\} \\
&= 1,
\end{aligned}$$

where $E_t\left[e^{-\beta X(T)}\right]$ is given in (2.7). Since $\varphi(n_0) = 1$, we have $v(t) = 1/n_0(t) = \exp\left\{\alpha t + \beta\big(X(t) - X(0)\big)\right\}$.

Next consider the following state–price deflator: $n = z_2\, n_0$, which combines the previous state–price deflator with a neutrino factor. In this case we have

$$\varphi(n) = \varphi(z_2\, n_0) = z_2\, \varphi(n_0) = z_2,$$

where the second equality uses the factor that z_2 is a martingale and is uncorrelated with n_0. Consequently $v = z_2/n = 1/n_0$ as before. Suppose now we compute the asymptotic revision process for $n_1 = z_1\, n_0$. This case is less straightforward, given the correlation between z_1 and n_0. We analyze this case following Proposition 4.

Proposition 4. Let z be an exponential martingale and let ω be a trend-stationary process. Then $\varphi(z\,\omega) = z$.

Proof. We can write

$$\lim_{T\to\infty} \frac{E_t[z(T)\,\omega(T)]}{E_0[z(t)\,\omega(T)]} = z(t)\left(\lim_{T\to\infty} \frac{\frac{E_t[\omega(T)]}{E_0[\omega(T)]} + \frac{1}{z(t)}\operatorname{Cov}_t[z(T), y(T)]}{1 + \operatorname{Cov}_0[z(T), y(T)]}\right),$$

where $y(t) := \omega(t)/E_0[\omega(t)]$; we want to show that $\lim_{T\to\infty} \operatorname{Cov}_t[z(T), y(T)] = 0$ for all t and all paths in a set of full measure.

Define z^n as the martingale z stopped at the random time $\tau^n := \inf\{\tau \mid z(\tau) \geq n\}$. Note that z^n is a bounded martingale, and therefore it converges a.s. and in L_1 to a random variable $z^n(\infty)$ such that $z^n(t) = E_t[z^n(\infty)]$. As a result, $\lim_{T\to\infty} \operatorname{Cov}_t[z^n(T), y(T)] = \lim_{T\to\infty} \operatorname{Cov}_t[z^n(\infty), y(T)]$; and the last expression is zero because otherwise $\lim_{T\to\infty} E[y(T) \mid z^n(\infty)]$ would be a non-constant function of $z^n(\infty)$, meaning that on some paths (those leading to particular values of $z^n(\infty)$) the asymptotic expectation of $y(T)$ is revised, so that y is not trend-stationary. Finally, note that

$$\lim_{T\to\infty} \operatorname{Cov}_t[z(T), y(T)] = \lim_{n\to\infty} \lim_{T\to\infty} \operatorname{Cov}_t[z^n(\infty), y(T)] = 0.$$

□

Let us return to consideration of $n_1 = z_1\, n_0$. Since $\zeta^\vartheta = z_1$ is an exponential martingale under the physical measure and we have already confirmed $\varphi(n_0) = 1$, it follows from Proposition 4 that $z_1 = \varphi(n_1)$ and $v = 1/n_0$.

The following proposition is an interesting corollary of Proposition 4.

Proposition 5. A process is trend-stationary under one measure if and only if it is trend-stationary under an equivalent measure.

Proof. Suppose a positive process u is trend-stationary under $\mathcal{P}$. Let $\mathcal{Q}^\vartheta$ be some equivalent measure. Then, using (3.8) and Proposition 4,

$$\lim_{T\to\infty} \frac{E_t^\vartheta[u(T)]}{E_0^\vartheta[u(T)]} = \zeta^\vartheta(t)^{-1} \lim_{T\to\infty} \frac{E_t[\zeta^\vartheta(T)\, u(T)]}{E_0[\zeta^\vartheta(t)\, u(T)]} = 1,$$

where ζ^ϑ is the change-of-measure process, an exponential martingale. Since $\mathcal{P}$ and $\mathcal{Q}^\vartheta$ are equivalent, the argument works the other way as well. □

We use Proposition 5 in the proof of the following proposition, which shows that the asymptotic forward rate is determined entirely by the trend-stationary component of the state–price deflator.

Proposition 6. If $n = \zeta^\vartheta\, \omega$ is a state–price deflator, where ζ^ϑ is an exponential martingale and u is a trend-stationary process, then the asymptotic forward rate is determined by ω.

Proof. Compute bond prices from ω: $p^\omega(t, T) = E_t[\omega(T)]/\omega(t)$. Note that $p^\omega(0, t) = E_0[\omega(T)]$ and define $m(t) := \omega(t)/E_0[\omega(t)]$, so that we can write $\omega(t) = m(t)\, p^\omega(0, t)$. Note that (by construction) $E_0[m(T)] = 1$ for all $T \geq 0$ and (by the trend-stationarity of u) $\lim_{T\to\infty} E_t[m(T)] = 1$ for all $t \geq 0$. We can compute forward rates from ω:

$$f^\omega(t,T) = \frac{-\partial \log(p^\omega(t,T))}{\partial T} = \frac{-\partial E_t[\omega(T)]/\partial T}{E_t[\omega(T)]} = f^\omega(0,T) - \frac{\partial E_t[m(T)]/\partial T}{E_t[m(T)]}.$$

Compute bond prices and forward rates from $n = \zeta^\vartheta\, \omega$:

$$p^n(t,T) = \frac{E_t[n(T)]}{n(t)} = \frac{E_t[\zeta^\vartheta(T)\, m(T)]\, p^\omega(0,T)}{\zeta^\vartheta(t)\, m(t)\, p^\omega(0,t)} = \frac{E_t^\vartheta[m(T)]\, p^\omega(0,T)}{m(t)\, p^\omega(0,t)}$$

and

$$f^n(t,T) = \frac{-\partial \log(p^n(t,T))}{\partial T} = f^\omega(0,T) - \frac{\partial E_t^\vartheta[m(T)]/\partial T}{E_t^\vartheta[m(T)]}.$$

Proposition 5 implies $\lim_{T\to\infty}\ E_t^\vartheta[m(T)] = 1$ for all $t \geq 0$, which in turn implies

$$\lim_{T\to\infty}\ f^n(t,T) - f^\omega(t,T) = 0,$$

which was to be shown. $\square$

Let ζ^η be an exponential $\mathcal{P}$-martingale and define yet another new state–price deflator:

$$n_2(t) := \zeta^\eta(t)\, n_1(t) = \zeta^\eta(t)\, \zeta^\vartheta(t)\, n(t) = \zeta^{\eta+\vartheta}(t)\, y(t)\, n(t),$$

where $\zeta^{\eta+\vartheta}$ is an exponential martingale and

$$y(t) = \exp\left(\int_{u=0}^t \vartheta(u)^\top \eta(u)\, du\right).$$

If $\vartheta(u)^\top\eta(u) = 0$, then $y(t) = 1$ and n_2 is formally the same as n_1. Consequently, f_2 is asymptotically the same as f. This would be the case if, for example, ζ^ϑ were the term-structure martingale and ζ^η were the neutrino factor. On the other hand, if $\vartheta(t)^\top\eta(t) \neq 0$, then $y(t) \neq 1$. In this case, y is not trend-stationary in general and the value of the VLD bond is not given by $y(t)\, n(t)$. We do not persue the analysis of this case further.

Proposition 7. Let $z = \varphi(n)$ and $v = \varphi(n)/n$. The following are equivalent:

$$(a)\ \ \mathrm{Cov}_t\,[1/v(T), z(T)] = 0, \qquad 0 \leq t \leq T < \infty$$
$$(b)\ \ p(t,T) = E_t\,[v(t)/v(T)]\,, \qquad 0 \leq t \leq T < \infty.$$

Proof. Compute bond prices using $n = z/v$:

$$p(t,T) = E_t\left[\frac{n(T)}{n(t)}\right] = E_t\left[\frac{z(T)}{z(t)}\ \frac{v(t)}{v(T)}\right] = E_t\left[\frac{v(t)}{v(T)}\right] + \frac{v(t)}{z(t)}\ \mathrm{Cov}_t\left[\frac{1}{v(T)}, z(T)\right].$$

$\square$

Proposition 7 says that if the conditional covariance is identically zero, then bond prices can be computed from the dynamics of the VLD bond. Any variation in z is due to variation in the neutrino factor; the term-structure martingale is constant. In this case we say that the VLD bond prices all bonds. Referring to the example in Section 2, given $n = z_2\, n_0$, the VLD bond prices all bonds.

Dynamics. Let n be a state–price deflator and let $\varphi(n)$ exist. Let $\zeta^\vartheta = \varphi(n)$ and $v = \varphi(n)/n$. We can express the dynamics of the exponential martingale ζ^ϑ as

$$d\zeta^\vartheta(t)/\zeta^\vartheta(t) = -\vartheta(t)^\top dW(t).$$

The dynamics of v can be written as

$$dv(t)/v(t) = \mu_v(t)\,dt + \sigma_v(t)^\top dW(t),$$

where, applying Ito's lemma to $v = \zeta^\vartheta/n$,[12]

$$\mu_v(t) = r(t) + \lambda(t)^\top \sigma_v(t) \qquad \text{and} \qquad \sigma_v(t) = \lambda(t) - \vartheta(t).$$

If n is trend-stationary, $z = 1$, $\vartheta(t) = 0$, and $\lambda(t) = \sigma_v(t)$. In other words, if n is trend-stationary, then the relative volatility of the VLD bond is the price of risk. In this case we say that the VLD bond prices all assets. Note that we can write

$$\lambda(t)^\top \sigma_v(t) = \|\sigma_v(t)\|^2 + \vartheta(t)^\top \sigma_v(t),$$

so that

$$\lambda(t)^\top \sigma_v(t) = \|\sigma_v(t)\|^2 \qquad \Longleftrightarrow \qquad \vartheta(t)^\top \sigma_v(t) = 0.$$

Note, however, that $\vartheta(t)^\top \sigma_v(t) \equiv 0$ does not imply $\mathrm{Cov}_t\left[1/v(T), z(T)\right] \equiv 0$, because v has a non-zero drift in general. (This point is illustrated in one of the examples in Section 6.)

5. The state–price deflator in a Markovian setting

In this section, we present a Markovian framework that is sufficiently general to encompass all of our examples as well as many models in the literature, including Constantinides (1992), which we discuss at the end of this section. The Markovian framework allows us to flesh-out the implications of the previous section and to put our examples into a more general context. In addition, the Markovian setting provides a convenient framework in which (*i*) to partition the shocks into those that drive the term structure (and hence the VLD bond) and those that do not, (*ii*) to define the price of term-structure risk, and (*iii*) to isolate the neutrino factor.

In a Markovian setting, asset prices can be expressed as functions of state variables. Bond prices, in particular, depend only on a subset of state variables. We show how to construct the VLD-bond price function from the limits of the partial derivatives of the zero-coupon bond price function.

[12]Also note that we can since $v(t,T) = p(t,T)/p(0,T)$,

$$\mu_v(t) = \lim_{T\to\infty} \mu_p(t,T) \qquad \text{and} \qquad \sigma_v(t) = \lim_{T\to\infty} \sigma_p(t,T).$$

State variables. Assume there is a vector of n_X state variables X and n_Y state variables Y. The dynamics of the state variables under the physical measure are given by

$$dX(t) = \mu_X(X(t))\,dt + \sigma_X(X(t))\,dW_X(t) \tag{5.1a}$$

$$dY(t) = \mu_Y(X(t), Y(t))\,dt + \sigma_Y(X(t), Y(t))\,dW(t), \tag{5.1b}$$

where

$$W = \begin{pmatrix} W_X \\ W_Y \end{pmatrix}$$

is a vector of $d_X + d_Y$ independent Brownian motions. Note that X and Y are jointly Markovian, and that X is Markovian by itself; in particular, X does not depend on W_Y.

The state–price deflator. Let the state–price deflator be given by $n(t) = \zeta^{\vartheta}(t)/u(t)$, where $\zeta^{\vartheta}(t)$ is given by (3.6), and where

$$u(t) = U(X(t), t) \qquad \text{and} \qquad \vartheta(t) = \begin{pmatrix} g(X(t)) \\ h(X(t), Y(t)) \end{pmatrix}. \tag{5.2}$$

We can factor the martingale $\zeta^{\vartheta} = z\,\nu$ into two independent components, the term-structure martingale z and the neutrino factor ν, where

$$\frac{dz(t)}{z(t)} = -g(X(t))^{\top} dW_X(t) \quad \text{and} \quad \frac{d\nu(t)}{\nu(t)} = -h(X(t), Y(t))^{\top} dW_Y(t).$$

The state-price deflator cannot in general be expressed as a function of the state variables unless both components are constant; *i.e.*, $g = h \equiv 0$.

Let $U_t(x, t)$, $U_x(x, t)$, and $U_{xx}(x, t)$ denote the obvious partial derivatives. The dynamics of u are given by

$$du(t)/u(t) = \mu_u(t)\,dt + \sigma_u(t)^{\top} dW(t),$$

where, by Ito's lemma,

$$\mu_u = U_t/U + \mu_X^{\top}\,U_x/U + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X\,\sigma_X^{\top}\,U_{xx}/U\right] \tag{5.3a}$$

$$\sigma_u = \sigma_X^{\top}\,U_x/U. \tag{5.3b}$$

The interest rate and the price of risk. Applying Ito's lemma to $n(t) = \zeta^{\vartheta}(t)/u(t)$, we get the Markovian dynamics of the state–price deflator:

$$r(t) = R(X(t)) \qquad \text{and} \qquad \lambda(t) = \Lambda(X(t), Y(t)) = \begin{pmatrix} \Lambda_X(X(t)) \\ \Lambda_Y(X(t), Y(t)) \end{pmatrix},$$

where

$$R = \frac{U_t}{U} + \left\{\mu_X - \sigma_X\,\left(g + \sigma_X^{\top}\,\frac{U_x}{U}\right)\right\}^{\top} \frac{U_x}{U} + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X\,\sigma_X^{\top}\,U_{xx}/U\right] \tag{5.4a}$$

and

$$\Lambda = \begin{pmatrix} \Lambda_X \\ \Lambda_Y \end{pmatrix} = \begin{pmatrix} g + \sigma_X^{\top}\,U_x/U \\ h \end{pmatrix}. \tag{5.4b}$$

Note that R and Λ_X depend only on X. We refer to Λ_X as the *price of term-structure risk.*

Asset pricing. It is convenient to express expectations and dynamics under the equivalent measure $\mathcal{Q}^\vartheta$, where ζ^ϑ is the change-of-measure process. The dynamics of the state variables X and Y under $\mathcal{Q}^\vartheta$ are given by

$$dX(t) = \mu_X^\vartheta(X(t))\,dt + \sigma_X(X(t))\,dW_X^\vartheta(t) \tag{5.5a}$$

$$dY(t) = \mu_Y^\vartheta(X(t), Y(t))\,dt + \sigma_Y(X(t), Y(t))\,dW^\vartheta(t), \tag{5.5b}$$

where

$$\mu_X^\vartheta(x) = \mu_X(x) - \sigma_X(x)\,g(x) \tag{5.6a}$$

$$\mu_Y^\vartheta(x, y) = \mu_Y(x, y) - \sigma_Y(x, y)\begin{pmatrix} g(x) \\ h(x, y) \end{pmatrix}. \tag{5.6b}$$

Thus X and (X, Y) remain Markovian under $\mathcal{Q}^\vartheta$.

We can use (3.9) to compute asset prices:

$$s(t, T) = E_t\left[\frac{n(T)}{n(t)}\,\omega(T)\right] = E_t^\vartheta\left[\frac{n(T)\,\zeta^\vartheta(t)}{n(t)\,\zeta^\vartheta(T)}\,\omega(T)\right] = u(t)\,E_t^\vartheta\left[\frac{\omega(T)}{u(T)}\right],$$

where $\omega(t) = \Omega(X(t), Y(t), t)$. Note that $E_t^\vartheta[\omega(T)/u(T)]$ depends only on $X(t)$, $Y(t)$, and the dynamics of (X, Y) under $\mathcal{Q}^\vartheta$; in particular, it does not depend on ζ^ϑ. Moreover, for any $\omega(t) = \Omega(X(t), t)$ that does not depend on Y, $E_t^\vartheta[\omega(T)/u(T)]$ depends only on $X(t)$ and its dynamics under $\mathcal{Q}^\vartheta$; it does not depend on Y or ζ^ϑ.

For example, consider the value of a zero-coupon bond $p(t, T)$ that pays one unit of the numeraire at time $T \ge t$ without fail (*i.e.*, for which $\omega(T) \equiv 1$):

$$\begin{aligned} p(t, T) &= u(t)\,E_t^\vartheta\left[u(T)^{-1}\right] \\ &= U(X(t), t)\,E_t^\vartheta\left[U(X(T), T)^{-1}\right] \\ &= P(X(t), t, T), \end{aligned} \tag{5.7}$$

where $P(x, t, T)$ is a deterministic function of x, t, and T only. In other words, conditional on the value of $X(t)$ and its dynamics under $\mathcal{Q}^\vartheta$, $Y(t)$ and its dynamics are irrelevant. The expression for bond prices in (5.7) shows that the exponential martingale plays no role in bond pricing once the measure has been changed. Note that the price of a zero-coupon bond is a function of X even if the state–price deflator cannot be expressed as a function of (X, Y).

The value of the VLD bond. Given $n = \zeta^\vartheta/u$, Proposition 4 shows that if ζ^ϑ is an exponential martingale and $\varphi(1/u) = 1$, then u is the value of the VLD bond. Given our Markovian setup, it is sufficient that X have a limiting stationary distribution under $\mathcal{P}$. With this assumption, we can compute $U(x, t)$ from bond prices.

At this point, it is convenient to consider a bond price function of the form $p(t, T) = P(X(t), T - t)$, where $P(x, \tau)$ is a function of t and T only through the remaining time to maturity τ. In addition we assume $P_\tau(x, \tau) = -\partial P(x, T - t)/\partial t$, which rules out the possibility that any of the state variables behaves like time. Given this bond price function, we have

$$U(x,t) = \lim_{T\to\infty} P(x, T-t)/P(x_0, T). \tag{5.8}$$

Taking the log of both sides of (5.8), we have

$$\begin{aligned}
\log\big(U(x,t)\big) &= \lim_{T\to\infty} \log\big(P(x,T-t)\big) - \log\big(P(x_0,T)\big) \\
&= \lim_{T\to\infty} \int_{(\xi=x_0,\zeta=0)}^{(x,t)} d\log(P(\xi,T-\zeta)) \\
&= \lim_{T\to\infty} \int_{(\xi=x_0,\zeta=0)}^{(x,t)} -\frac{P_\tau(\xi,T-\zeta)}{P(\xi,T-\zeta)}\,d\zeta + \left(\frac{P_x(\xi,T-\zeta)}{P(\xi,T-\zeta)}\right)^{\top} d\xi \\
&= \int_{(\xi=x_0,\zeta=0)}^{(x,t)} \left(\lim_{T\to\infty} -\frac{P_\tau(\xi,T-\zeta)}{P(\xi,T-\zeta)}\right) d\zeta + \left(\lim_{T\to\infty} \frac{P_x(\xi,T-\zeta)}{P(\xi,T-\zeta)}\right)^{\top} d\xi \\
&= \int_{(\xi=x_0,\zeta=0)}^{(x,t)} \alpha\, d\zeta + \beta(\xi)^{\top} d\xi \\
&= \alpha\, t + \int_{\xi=x_0}^{x} \beta(\xi)^{\top} d\xi,
\end{aligned} \tag{5.9}$$

where

$$\alpha := \lim_{\tau\to\infty} -P_\tau(x,\tau)/P(x,\tau) \qquad \text{and} \qquad \beta(x) := \lim_{\tau\to\infty} P_x(x,\tau)/P(x,\tau). \tag{5.10}$$

Note that α is the long forward rate.[13] The second line of (5.9) re-expresses the first line as a path-independent line integral of the total differential $d\log P$. The third line re-expresses the second line in terms of the differentials of the arguments. The fifth line is justified by the existence of the VLD bond under the conditions stated, which in turn implies the existence of the limits. The last line follows from the path-independence of the integral. To conclude,

$$U(x,t) = \exp(\alpha\, t + f(x)),$$

where $f(x) := \int_{\xi=x_0}^{x} \beta(\xi)^{\top} d\xi$. For example, if $\beta(x) = \beta$, then $f(x) = \beta^{\top}(x - x_0)$.

Specialize the bubble. Changing perspective, let us now assume

$$U(x,t) = \exp\{\alpha\, t + f(x) - f(x_0)\}, \tag{5.11}$$

where α is constant and $f(x)$ is an arbitrary twice-differentiable function of the vector x. This form of U will produce a constant long forward rate equal to α and bond prices that depend only on maturity. In this case, we have

$$U_t/U = \alpha, \quad U_x/U = f_x, \quad \text{and} \quad U_{xx}/U = f_x\, f_x^{\top} + f_{xx}.$$

[13] Since bond prices do not separately depend on T in this formulation, the long forward rate must be constant if the forward rate is asymptotically deterministic.

We can rewrite R and Λ_X in (5.4) in terms of f:

$$R = \alpha + \left\{\mu_X - \sigma_X\,\Lambda_X\right\}^\top f_x + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X\,\sigma_X^\top\left(f_x\,f_x^\top + f_{xx}\right)\right] \tag{5.12a}$$

$$\Lambda_X = g + \sigma_X^\top\,f_x. \tag{5.12b}$$

In addition, we can specialize (5.7):

$$P(x,\tau) = e^{-\alpha\,\tau + f(x)}\,E^\vartheta\left[e^{-f(X(t+\tau))} \mid X(t) = x\right]. \tag{5.13}$$

Note that $g(x) \equiv 0$ implies $\mu_X(x) \equiv \mu_X^\vartheta(x)$, so that the dynamics of X are the same under $\mathcal{Q}^\vartheta$ as under $\mathcal{P}$. In this case we can compute bond prices under the physical measure via

$$P(x,\tau) = e^{-\alpha\,\tau + f(x)}\,E\left[e^{-f(X(t+\tau))} \mid X(t) = x\right]. \tag{5.14}$$

As a result, the physical drift μ_X is separately identified in bond prices. In general, the price of term-structure risk is not identified from the bond price function $P(x,\tau)$. However, if X has a limiting stationary distribution and $g(x) \equiv 0$, then the price of term-structure risk equals the volatility of the VLD bond, $\Lambda_X = \sigma_X\,f_x$, which is identifiable from bond prices. Clearly, (5.14) embodies a substantial restriction.

PDE bond pricing. Given the dynamics of the state variables (*i.e.*, the functions μ_X, and σ_X), we can use Ito's lemma to write $\mu_p(t,T) = \mu_P(X(t), T-t)$ and $\sigma_p(t,T) = \sigma_P(X(t), T-t)$ where the functions $\mu_P(x,\tau)$ and $\sigma_P(x,\tau)$ are given by

$$\mu_P = -\frac{P_\tau}{P} + \mu_X^\top\frac{P_x}{P} + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X\,\sigma_X^\top\,\frac{P_{xx}}{P}\right] \qquad \text{and} \qquad \sigma_P = \sigma_X^\top\,\frac{P_x}{P}. \tag{5.15}$$

With these functions, absence-of-arbitrage condition (2.4) can be written as the partial differential equation

$$R = -\frac{P_\tau}{P} + \left(\mu_X - \sigma_X\,\Lambda_X\right)^\top\frac{P_x}{P} + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X\,\sigma_X^\top\,\frac{P_{xx}}{P}\right], \tag{5.16}$$

subject to $P(x,0) = 1$.

The instantaneous term premium, $\mu_p(t,T) - r(t)$, is given by

$$\mu_P - R = \Lambda_X^\top\left(\sigma_X^\top\,\frac{P_x}{P}\right) = \left(g + \sigma_X^\top\,\frac{U_x}{U}\right)^\top\left(\sigma_X^\top\,\frac{P_x}{P}\right). \tag{5.17}$$

In (5.17), both P_x/P and U_x/U are completely determined by the bond price function P when u is the value of the VLD. If, in addition, $g \equiv 0$, term premia can be computed directly from P and σ_X. No direct knowledge of μ_X is required.

Constantinides. Constantinides (1992) produces a model of the term structure by modeling the state–price deflator directly. Constantinides introduces non-trend-stationarity into the state–price deflator with a neutrino factor. (Indeed, we consider this to be the classic example.) By contrast, the term-structure martingale is constant in his model.

Constantinides model of state price deflator can be expressed as follows. Let[14]

$$u(t) = \exp(\alpha\, t - \|X(t) - c\|^2),$$

where c is a constant vector and X is a vector of independent, mean reverting Gaussian variables:

$$dX(t) = -K\, X(t)\, dt + S\, dW_X(t).$$

K and S are diagonal matrices of positive constants. The dynamics of the exponential martingale can be specified in terms of

$$g(x) = 0 \quad \text{and} \quad h(\,\cdot\,) = \sigma_0,$$

where σ_0 is a scalar constant. Thus $d\zeta^\vartheta(t)/\zeta^\vartheta(t) = -\sigma_0\, dW_Y(t)$, where W_Y is a scalar Brownian motion independent of W_X. Using (5.4) we can compute the interest rate and price of risk:

$$\begin{aligned} R(x) &= \alpha - \operatorname{tr}[S^2] + 2\,(x-c)^\top \big\{(K - S^2)\,(x-c) + K\, c\big\} \\ \Lambda(x) &= \begin{pmatrix} 2\, S\,(c-x) \\ \sigma_0 \end{pmatrix}. \end{aligned}$$

Constantinides computes bond prices using (5.14).

Since the state variables are stationary, u is the value of the VLD bond. Moreover, since $g \equiv 0$, the volatility of the VLD bond is the price of term-structure risk. The volatility of the VLD bond is

$$\sigma_u(x) = \begin{pmatrix} 2\, S\,(c-x) \\ 0 \end{pmatrix}.$$

This restriction can be relaxed, extending Constantinides' model, by allowing a nonzero $g(x)$ that is linear in x and interpreting the specified dynamics to be those under $\mathcal{Q}^\vartheta$. To proceed along these lines, let

$$\mu_X^\vartheta(x) = -K\, x.$$

Choose a suitable $g(x)$ and define

$$\mu_X(x) = \mu_X^\vartheta(x) + \sigma_X(x)\, g(x) \qquad \text{and} \qquad \Lambda_X(x) = \sigma_X(x)^\top \frac{U_x}{U} + g(x).$$

For example, let

$$g(x) = S^{-1}\,\big\{K\, x + B\,(\theta - x)\big\},$$

so that

$$\mu_X(x) = B\,(\theta - x) \quad \text{and} \quad \Lambda_X(x) = 2\, S\,(c-x) + S^{-1}\,\big\{K\, x + B\,(\theta - x)\big\}.$$

Constantinides' expressions for bond prices, bond yields, and the interest rate are all unchanged, but his expressions for term premia must be modified since they were computed with $g \equiv 0$ (see (5.17)). We have introduced parameters (in B and θ) that are not identified using purely cross-sectional information.

[14]To avoid clutter, we have not imposed $u(0) = 1$.

6. Exponential-affine models of the term structure

Exponential-affine models of the term structure are sufficiently important and useful to merit explicit treatment.

Fisher and Gilles (1996) define a term-structure model to be "exponential affine" whenever (a) yields of zero-coupon bonds and (b) the first and second conditional moments of the distribution of future state variables are both affine in the current state variables. This definition is more restrictive than the original definition of Duffie and Kan (1996), which covers only condition (a); as a result, the circumstances under which our conditions hold are a little bit more restrictive than those set out in Duffie and Kan. The term "exponential affine" comes from the fact that for condition (a) to hold, the bond price function $P(x, \tau)$ must have the form

$$P(X(t), \tau) = \exp\Big(- A(\tau) - \sum_{i=1}^{n_X} B_i(\tau)\, X_i(t)\Big), \tag{6.1}$$

where $A(\tau)$ and $B_i(\tau)$ are fixed functions of τ, the time to maturity. As explained in Fisher and Gilles (1996), a model has an exponential-affine structure if and only if

$$\mathcal{M}(x) \text{ and } \mathcal{S}(x)^\top \mathcal{S}(x) \text{ are affine in } x, \tag{6.2}$$

where

$$\mathcal{M}(x) := \begin{pmatrix} \mu_X(x) \\ R(x) \end{pmatrix} \quad \text{and} \quad \mathcal{S}(x) := \begin{pmatrix} \sigma_X(x)^\top & \Lambda_X(x) \end{pmatrix}.$$

Conditions (6.2) are more restrictive than the conditions that Duffie and Kan assume since they do not model the price of risk. They assume only that

$$R(x), \qquad \mu_X(x) - \sigma_X(x)\,\Lambda_X(x), \qquad \text{and} \qquad \sigma_X(x)\,\sigma_X(x)^\top$$

are affine in x.

It will be convenient to refer to explicitly affine representations for the relevant variables. Starting with $\mathcal{M}(x)$, let

$$\mu_X(x) = M_0 + \sum_{i=1}^{n_X} M_i\, x_i \quad \text{and} \quad R(x) = R_0 + \sum_{i=1}^{n_X} R_i\, x_i\,.$$

Next, turning to $\mathcal{S}(x)$, adapting a proof in Duffie and Kan to the present context shows that $\mathcal{S}(x)^\top \mathcal{S}(x)$ is affine in x if and only if $\mathcal{S}(x)$ can be written as

$$\mathcal{S}(x) = \begin{pmatrix} \sigma_X(x)^\top & \Lambda_X(x) \end{pmatrix} = D(x)\, \begin{pmatrix} \Omega & \gamma \end{pmatrix}$$

where Ω is a $n_X \times d_X$ matrix of constants, γ is a $n_X \times 1$ vector of constants, and $D(x)$ is a diagonal $n_X \times n_X$ matrix whose *i-th* diagonal entry, $u_i(x)$ satisfies

$$u_i(x) = \sqrt{\alpha_{i0} + \sum_{j=1}^{n_X} \alpha_{ij}\, x_j};$$

Now, we can write $\sigma_X(x)\,\sigma_X(x)^\top = \Omega^\top D^2(x)\,\Omega$, $\Lambda(x)^\top\Lambda(x) = \gamma^\top D^2(x)\,\gamma$ and $\sigma_X(x)\,\Lambda(x) = \Omega^\top D^2(x)\,\gamma$, which are all affine since $D^2(x)$ has the representation

$$D^2(x) = D_0^2 + \sum_{i=1}^{n_X} D_i^2\, x_i,$$

where each D_i^2 is an $n_X \times n_X$ diagonal matrix with *j-th* diagonal entry $D_{ij} = \alpha_{ij}$. To show that bond prices satisfy (6.1), assume that they do and show that for specific functions $A(\tau)$ and $B_i(\tau)$, the PDE (5.16) is satisfied.

Now let us further specialize (5.12), letting $f(x) = \beta^\top x$, so that

$$R = \alpha + \left\{\mu_X - \sigma_X\,\Lambda_X\right\}^\top \beta + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X\,\sigma_X^\top\,\beta\,\beta^\top\right] \tag{6.3a}$$

$$\Lambda_X = g + \sigma_X^\top\,\beta. \tag{6.3b}$$

We can use (6.3) to reproduce any exponential-affine model of the term structure. Given μ_X, σ_X, R, and Λ_X, first choose α and β to match coefficients in (6.3a), then use (6.3b) to determine g. For Gaussian models (*i.e.*, models for which the volatility matrix σ_X is constant), there is a unique solution. For non-Gaussian models, there may be more than one solution. However, if the state variables have a limiting stationary distribution, then there is a unique solution where α and β are the limits of the log bond-price derivatives.

Going the other way, we can choose U and g to deliver an exponential-affine term structure: Given $\mu_X(x)$ and $\sigma_X(x)^\top = D(x)\,\mathit{\Omega}$, choose

$$\begin{aligned} U(x,t) &= \exp(\alpha\,t + \beta^\top x) \\ g(x) &= D(x)\,(\gamma - \mathit{\Omega}\,\beta). \end{aligned}$$

Then $R(x)$ is affine in x and $\Lambda_X(x) = D(x)\,\gamma$. In this case, the restriction $g(x) \equiv 0$ is equivalent to $\gamma = \mathit{\Omega}\,\beta$.

We now provide a number of examples from the exponential-affine class that illustrate many of the points from the preceding sections.

One-factor CIR. In this example, we model the interest rate and the price of risk directly and derive the solution to the state–price deflator.[15] To keep things simple, we take the interest rate to be the single state variable, $r(t) = X(t)$, and model it as a square-root process along the lines of CIR:

$$dX(t) = \kappa\,(\theta - X(t))\,dt + s\,\sqrt{X(t)}\,dW_X(t), \tag{6.4}$$

so that $\mu_X(x) = \kappa\,(\theta - x)$ and $\sigma_X(x) = s\,\sqrt{x}$. Let the price of risk be given as follows,[16]

$$\Lambda(x) = \begin{pmatrix} (q/s)\,\sqrt{x} \\ h(x,y) \end{pmatrix},$$

for some function h and some state variables Y.

[15] We will relate the model in this section to the C–CAPM framework in the following section.

[16] Our q is CIR's parameter λ.

We can solve (6.3) for $\alpha = -\kappa\,\theta\,\beta$, $g(x) = \sqrt{x}\,(q/s - s\,\beta)$, and

$$\beta = \frac{\kappa + q}{s^2} \pm \frac{\gamma}{s^2}, \qquad \text{where } \gamma = \sqrt{(\kappa+q)^2 + 2\,s^2}.$$

Thus there are two representations for the state–price deflator that are consistent with our restrictions on functional forms. The drift of x under $\mathcal{Q}^{\vartheta}$ is given by

$$\mu_X^{\vartheta}(x) = \kappa\,\theta \pm \gamma\,x.$$

Thus X has a limiting stationary distribution with the second solution for β, $\beta = (\kappa + q - \gamma)/s^2$, but not with the first solution. Therefore, it is the second solution that delivers the decomposition into the asymptotic revision process and the inverse of the VLD bond price.

It is well-known that the bond price function is given by $P(x,\tau) = \exp(-A(\tau) - B(\tau)\,x)$, where

$$\begin{aligned} B(\tau) &= \frac{2\,(e^{\gamma\,\tau} - 1)}{(\gamma + \kappa + q)\,(e^{\gamma\,\tau} - 1) + 2\,\gamma} \\ A'(\tau) &= \kappa\,\theta\,B(\tau). \end{aligned}$$

Note that $P_x(x,\tau)/P(x,\tau) = -B(\tau)$, $-P_\tau(x,\tau)/P(x,\tau) = A'(\tau) + B'(\tau)\,x$, and

$$\begin{aligned} \lim_{\tau\to\infty} -B(\tau) &= \frac{-2}{\kappa + q + \gamma} = \frac{\kappa+q}{s^2} - \frac{\gamma}{s^2} \\ \lim_{\tau\to\infty} A'(\tau) + B'(\tau)\,x &= -\kappa\,\theta\left(\lim_{\tau\to\infty} -B(\tau)\right), \end{aligned}$$

which agrees with the stationary solution computed above.

The parameters κ and q do not appear in the bond price function separately, but only as the sum $\kappa + q$. As a result, these parameters are not separately identified using the bond price function $P(x,\tau)$ alone, although the sum is. In general, one needs time-series data to identify the price-of-risk parameter q via the drift under the physical measure. However, if we restrict the model to have $g \equiv 0$, then $q = s^2\,\beta$ is identified without resorting to time-series data. This identification of q flows from the equivalence of the price of term-structure risk with the volatility of the VLD bond.

An equilibrium interpretation. Consider the C-CAPM with power utility. In this case we have $n(t) = e^{-\delta\,t}\,c(t)^{-1/\eta}$ where δ is the rate of time preference and η is the elasticity of intertemporal substitution. Given the dynamics of consumption,

$$d\log(c(t)) = \widetilde{\mu}_c(t)\,dt + \sigma_c(t)^\top dW(t),$$

or equivalently

$$c(t) = \exp\left(\int_{s=0}^{t} \widetilde{\mu}_c(s)\,ds + \int_{s=0}^{t} \sigma_c(s)^\top dW(s)\right),$$

the interest rate and price of risk are given by

$$r(t) = \delta + \frac{1}{\eta}\,\widetilde{\mu}_c(t) - \frac{1}{2}\,\|\lambda(t)\|^2 \qquad \text{and} \qquad \lambda(t) = \frac{1}{\eta}\,\sigma_c(t)$$

As long as the *dynamics* of consumption (*i.e.*, $\widetilde{\mu}_c(t)$ and $\sigma_c(t)$) are deterministic functions of Markovian state variables, bond prices will also be deterministic functions of those state variables—even though consumption and the state–price deflator are not. Our approach provides an explicit decomposition of the state–price deflator into (i) trend-stationary and (ii) martingale components.[17]

As an illustration, consider a case where X and Y each contain a single state variable, and there are two Brownians. In particular, let

$$\widetilde{\mu}_c(t) = -\eta\,\delta + \eta\,\left(1 + \frac{1}{2}\,(q/s)^2\right) X(t) + \frac{1}{2}\,\eta\,\zeta^2\,Y(t)$$

$$\sigma_c(t) = \begin{pmatrix} \eta\,(q/s)\,\sqrt{X(t)} \\ \eta\,\zeta\,\sqrt{Y(t)} \end{pmatrix}.$$

Let the dynamics of X be given by (6.4) and the dynamics of Y be given by

$$dY(t) = \kappa_Y\,(\theta_Y - Y(t))\,dt + s_Y\,\sqrt{Y(t)}\,dW_Y(t).$$

In this case,

$$r(t) = X(t) \qquad \text{and} \qquad \lambda(t) = \begin{pmatrix} (q/s)\,\sqrt{X(t)} \\ \zeta\,\sqrt{Y(t)} \end{pmatrix}.$$

This interest rate and price of risk deliver CIR bond prices. As shown above, the VLD bond is the Markovian bubble asset, which can be computed from the appropriate limit. In this example, $g(x) = \sqrt{x}\,(\gamma - \kappa)/s$ and $h(x, y) = \zeta\,\sqrt{y}$. Thus if $\kappa = \gamma$, $g(x) \equiv 0$ and the VLD bond will price all bonds, even though it does not price all assets as long as $\zeta \neq 0$.[18]

A Markovian MMA-bubble with stochastic interest rates. Under what conditions can the money-market account be a Markovian bubble asset? Value of the MMA is $B(t) = \exp\left(\int_{s=0}^{t} r(s)\,ds\right)$, so its dynamics are given by $dB(t)/B(t) = r(t)\,dt$. Therefore, if the value of the MMA is proportional to $U(X(t), t)$, we must have (i) $\mu_u(t) = r(t)$ and (ii) $\sigma_u(t) = 0$. Condition (ii) implies $\sigma_X\,U_x/U \equiv 0$, which in turn implies condition (i). (Compare (5.3) with (5.4a)). Assuming σ_X is invertible, condition (ii) implies $U_x \equiv 0$. In other words, if σ_X is invertible the MMA can only be the Markovian bubble when U is a function of time only, in which case the interest rate is deterministic. However, if σ_X has less than full rank, the MMA can be a Markovian bubble even with stochastic rates. We can summarize this discussion by saying that if the bubble depends on any stochastic state variables, they must be locally riskless in order for the bubble to be the MMA. Finally note that condition (ii) implies the price of term-structure risk is given by $\Lambda_X(x) = g(x)$.

We illustrate the foregoing analysis with an example in which the VLD bond exists and the MMA is not a Markovian bubble in general. However, the example

[17] We could of course separately decompose consumption into its trend-stationary and martingale components: $\varphi(c)/c$ and $\varphi(c)$.

[18] Campbell, Kazemi, and Nanisetty (1999) examine risk-premia in the bond market to see whether the VLD bond prices bonds by itself. The example shows that even for the C–CAPM, the finding that the VLD bond prices all bonds is not sufficient to conclude that the VLD bond prices all assets.

can be specialized in two ways (one way making the interest rate deterministic and the other way leaving it stochastic), both of which identify the VLD bond with the MMA. Let the dynamics of the state variables be given by

$$dX_1(t) = \kappa_1 (\theta - X_1(t))\, dt + s_1\, dW_{X_1}(t) \tag{6.5a}$$

$$dX_2(t) = \kappa_2 (X_1(t) - \delta\, X_2(t))\, dt + s_2\, dW_{X_2}(t). \tag{6.5b}$$

Let $u(t) = \exp(\alpha\, t + \beta\, X_2(t))\, e^{-\beta\, X_2(0)}$,

$$g(x) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad \text{and} \qquad h(x, y) = 0.$$

For future reference, note that $n(t) = z(t)/u(t)$, where

$$\frac{dz(t)}{z(t)} = -\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}^{\top} \begin{pmatrix} dW_{X_1}(t) \\ dW_{X_2}(t) \end{pmatrix}.$$

Assuming $\kappa_1 > 0$ and $\kappa_2\, \delta > 0$, there is a limiting stationary distribution for $X = (X_1, X_2)$ under both $\mathcal{P}$ and $\mathcal{Q}^{\vartheta}$. Consequently, u is the value of the VLD bond. We can compute the interest rate and the price of term-structure risk,

$$R(x) = \left(\alpha - q_2\, \beta\, s_2 - \frac{1}{2} (\beta\, s_2)^2\right) + (\beta\, \kappa_2)\, x_1 - (\delta\, \beta\, \kappa_2)\, x_2 \tag{6.6a}$$

$$\Lambda_X(x) = \begin{pmatrix} q_1 \\ q_2 + \beta\, s_2 \end{pmatrix}, \tag{6.6b}$$

which together with (6.5) comprise an exponential-affine model of the term structure.The volatility of the bubble asset is

$$\sigma_X\, \frac{U_x}{U} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \beta\, s_2 \end{pmatrix}.$$

By making this volatility zero, we identify the VLD bond with the MMA. One way we can do this is by setting $\beta = 0$, in which case the interest rate becomes deterministic: $r(t) = \alpha$. The other way we can do this is by setting $s_2 = 0$, which makes X_2 locally risk-free and reduces the rank of σ_X. In this case the interest rate is stochastic: $r(t) = \alpha + \beta\, \kappa_2\, (X_1(t) - \delta\, X_2(t))$, the relative drift of $u(t)$.

This example provides an illustration of the fact that $\vartheta^{\top} \sigma_v \equiv 0$ does not imply $\Lambda_X = \sigma_v$. Let $q_2 = 0$. In this case, $\vartheta^{\top} \sigma_u = 0$ and $\Lambda_X^{\top} \sigma_u = \|\sigma_u\|^2$ even though $\sigma_u \neq \Lambda_X$.

A change of state variables. We can examine a different issue by specializing the model in the previous section in another direction. Given the setup in the preceding section, bond prices are given by

$$p(t, T) = \exp\{-A(T - t) - B(T - t)\, X_1(t) - C(T - t)\, X_2(t)\},$$

where

$$B(\tau) = \frac{\beta\, \kappa_2\, \left(e^{-\kappa_2\, \delta\, \tau} - e^{-\kappa_1\, \tau}\right)}{\kappa_1 - \kappa_2\, \delta} \quad \text{and} \quad C(\tau) = \beta\, \left(e^{-\kappa_2\, \delta\, \tau} - 1\right).$$

Note that for $\delta \neq 0$, $B(\infty) = 0$ and $C(\infty) = -\beta$.

Now let $\delta = 0$. X_2 no longer has a limiting stationary distribution. In this case $C(\tau) \equiv 0$, and $B(\infty) = \kappa_2\,\beta/\kappa_1$. The VLD bond can be expressed as a function of X_1 alone, which is Markovian and has a limiting stationary distribution:

$$V(x,t) = \exp\left\{\hat\alpha\,t + \hat\beta\big(x_1 - X_1(0)\big)\right\},$$

where

$$\hat\alpha = \alpha + \beta\left(\kappa_2\,\theta - \frac{\kappa_1\,q_2\,s_2 + \kappa_2\,q_1\,s_1}{\kappa_1}\right) - \beta^2\left(\frac{\kappa_1^2\,s_2^2 + \kappa_2^2\,s_1^2}{2\,\kappa_1^2}\right) \tag{6.7a}$$

$$\hat\beta = -\frac{\kappa_2\,\beta}{\kappa_1}. \tag{6.7b}$$

Since X_2 does not enter either $R(x)$ of $\Lambda_X(x)$, it is irrelevant to the term structure and can be removed from the X vector. In this case, given

$$\hat R(x_1) = \left(\alpha - q_2\,\beta\,s_2 - \frac{1}{2}\,(\beta\,s_2)^2\right) + (\beta\,\kappa_2)\,x_1,$$

$\hat\Lambda_X(x_1) = q_1$, $\hat\mu_X(x_1) = \kappa_1\,(\theta - x_1)$, and $\hat\sigma_X(x_1) = s_1$, the unique solution to a version of (6.3) with hats is given by (6.7) and

$$\hat g(x) = q_1 + \frac{\beta\,\kappa_2\,s_1}{\kappa_1}.$$

Given $\hat u(t) = \exp(\hat\alpha\,t + \hat\beta\,X_1(t))$ and $\hat g(x)$, we can find an $\hat h(\,\cdot\,)$ such that the dynamics of $\hat n(t) = \hat z(t)/\hat u(t)$ are identical to those of $n(t)$. The volatility of $n(t)$ involves the second component of $\Lambda_X(\,\cdot\,)$ in (6.6b). In order to keep the price of risk unchanged, define $\hat\Lambda_Y(\,\cdot\,) = h(\,\cdot\,) = q_2 + \beta\,s_2$, so that

$$\frac{d\hat z(t)}{\hat z(t)} = -\begin{pmatrix} q_1 + \frac{\beta\,\kappa_2\,s_1}{\kappa_1} \\ q_2 + \beta\,s_2 \end{pmatrix}^{\top} \begin{pmatrix} dW_{X_1}(t) \\ dW_{Y_1}(t) \end{pmatrix},$$

where we have relabeled W_{X_2} as W_{Y_1}. Since $n(t)$ and $\hat n(t)$ have the same dynamics, they price all assets identically, including $u(t)$ in particular.

No VLD bond. We present an example of an affine economy in which there is no VLD bond. The example was introduced by Fisher and Gilles (1998). There are two state variables whose dynamics are given by

$$\begin{aligned} dX_1(t) &= (\theta - X_2(t))\,dt - \zeta\,dW_{X_1}(t) \\ dX_2(t) &= X_1(t)\,dt + \zeta\,dW_{X_2}(t), \end{aligned}$$

where θ and ζ are constants. The conditional expectation of the state variables is given by $E_t[X(T)] = \mathcal{E}(X(t), T-t)$, where

$$\mathcal{E}(x,\tau) = \begin{pmatrix} x_1\,\cos(\tau) + (x_2 - \theta)\,\sin(\tau) \\ \theta + x_1\,\sin(\tau) + (x_2 - \theta)\,\cos(\tau) \end{pmatrix}.$$

The point $x_0 = (0,\theta)$ is a stationary point in state space: $\mathcal{E}(x_0,\tau) = x_0$. However, there is no tendency for the state variables to move toward this stationary point. Consequently, there is no limiting stationary distribution.

Let the interest rate and price of term-structure risk be given by

$$R(x) = x_2 \quad \text{and} \quad \Lambda_X(x) = \begin{pmatrix} \zeta \\ 0 \end{pmatrix},$$

which produces an exponential-affine model of the term structure. Bond prices are given by

$$P(x, \tau) = \exp(-A_0(\tau) - A_1(\tau)\, x_1 - A_2(\tau)\, x_2),$$

where

$$A_0(\tau) = \theta\,(\tau - \sin(\tau)), \quad A_1(\tau) = 1 - \cos(\tau), \quad \text{and} \quad A_2(\tau) = \sin(\tau).$$

The forward rate function is

$$F(x, \tau) = \theta + x_1\, \sin(\tau) + (x_2 - \theta)\, \cos(\tau),$$

which is identical to the conditional expectation of the interest rate, X_2. In other words, the "strong form" of the expectations hypothesis holds in this model.

Let us examine the state–price deflator. The solution to (6.3) is given by

$$\alpha = \theta + \frac{1}{2}\,\zeta^2, \quad \beta = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad g(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so that

$$n(t) = \exp\left\{-\left(\theta + \zeta^2/2\right) t + \left(X_1(t) - X_1(0)\right)\right\},$$

assuming $h(x, y) \equiv 0$. Perhaps surprisingly, X_2 does not appear in the state–price deflator. Since n has no asymptotic revision process, the VLD bond does not exist, and the long forward rate is not asymptotically-deterministic. Finally note that even though $g = 0$, so that the price of risk is the volatility of the bubble asset ($u = 1/n$), the price-of-risk parameter ζ is not identified by bond prices since the bubble asset is not a bond.

7. Exchange rates

Absence-of-arbitrage conditions can be expressed in more than one numeraire. There will be a state–price deflator for each numeraire. Let n_1 be the state–price deflator for the original numeraire, and let s_1 be the value of an asset denominated in that numeraire. Let $\psi(t)$ be the exchange rate between the two numeraires, such that $s_2(t) = \psi(t)\, s_1(t)$.[19] Then

$$n_1(t)\, s_1(t) = E_t[n_1(\tau)\, s_1(\tau)] \quad \text{and} \quad n_2(t)\, s_2(t) = E_t[n_2(\tau)\, s_2(\tau)]$$

are equivalent if

$$n_2(t) = \frac{n_1(t)}{\psi(t)}. \tag{7.1}$$

[19]The exchange rate can refer to either foreign exchange or, if one numeraire is nominal and the other is real, to the price level.

By the parallel structure of the absence-of-arbitrage conditions in the two numeraires, we have

$$\frac{dn_i(t)}{n_i(t)} = -r_i(t)\,dt - \lambda_i(t)^\top dW(t) \quad \text{for } i = 1, 2.$$

Solving (7.1) for ψ and applying Ito's lemma produces

$$d\log(\psi(t)) = \mu_\psi(t)\,dt + \sigma_\psi(t)^\top dW(t),$$

where

$$\begin{aligned}\mu_\psi(t) &= r_2(t) - r_1(t) + \frac{1}{2}\left(\|\lambda_2(t)\|^2 - \|\lambda_1(t)\|^2\right)\\ \sigma_\psi(t) &= \lambda_2(t) - \lambda_1(t).\end{aligned}$$

At this point we adopt the Markovian setting of Section 5. We fix the dynamics of the state variables in (5.1), and let

$$n_i(t) = \frac{\zeta^{\vartheta_i}(t)}{U_i(X(t),t)}, \tag{7.2}$$

where

$$U_i(x,t) = \exp(\alpha_i\,t + f_i(x)) \quad \text{and} \quad \vartheta_i(t) = \begin{pmatrix} g_i(X(t)) \\ h_i(X(t),Y(t)) \end{pmatrix}.$$

With this structure, we can write the Markovian dynamics of the exchange rate,

$$\psi(t) = \frac{n_1(t)}{n_2(t)} = \frac{\zeta^{\vartheta_1}(t)}{\zeta^{\vartheta_2}(t)}\,\frac{U_2(X(t),t)}{U_1(X(t),t)},$$

as follows:

$$\begin{aligned}\mu_\Psi(x,y) = R_2(x) - R_1(x) + \frac{1}{2}\left(\|\Lambda_{X_2}(x)\|^2 - \|\Lambda_{X_1}(x)\|^2\right) + \\ \frac{1}{2}\left(\|h_2(x,y)\|^2 - \|h_1(x,y)\|^2\right),\end{aligned} \tag{7.3}$$

where the functions R_i and Λ_{Xi} are the appropriately subscripted versions of (5.4), and

$$\sigma_\Psi(x,y) = \Lambda_2(x,y) - \Lambda_1(x,y) = \begin{pmatrix} g_2(x) - g_1(x) + \sigma_X(x)\,(f_{1x}(x) - f_{2x}(x)) \\ h_2(x,y) - h_1(x,y) \end{pmatrix}. \tag{7.4}$$

We refer to the last term on the right-hand side of (7.3) as the *neutrino factor component* of expected exchange-rate depreciation.

First, note that

$$\zeta^{\vartheta_1}(t) = \zeta^{\vartheta_2}(t) \iff \psi(t) = \frac{u_2(t)}{u_1(t)} \iff \mathcal{Q}^{\vartheta_1} = \mathcal{Q}^{\vartheta_2}. \tag{7.5}$$

Thus when $\zeta^{\vartheta_1} = \zeta^{\vartheta_2}$, the exchange rate is the ratio of the values of the two bubble assets and there a single equivalent martingale measure for both numeraires.

Moreover, the neutrino factor component is identically zero and the volatility of the exchange rate is reduced to

$$\sigma_\Psi(x,y) = \begin{pmatrix} \sigma_X(x)\big(f_{1x}(x) - f_{2x}(x)\big) \\ 0 \end{pmatrix}. \tag{7.6}$$

Second, note that one of the central stylized facts of foreign exchange rates is that the instantaneous expected rate of depreciation of the exchange rate, μ_ψ, has a component that is orthogonal to both term structures.[20] In order for this feature to obtain in (7.3), the neutrino factor component must depend on Y in a non-trivial way (and Y must depend on W_Y in a non-trivial way).[21]

This requires that $\vartheta_1 \not\equiv \vartheta_2$, in which case there are two different equivalent martingale measures, $\mathcal{Q}^{\vartheta_1}$ and $\mathcal{Q}^{\vartheta_2}$. Nevertheless, we could still have $g_1 = g_2$, in which case X would have identical dynamics under each of the two martingale measures. In this case, the only difference in the Markovian bond prices across the two numeraires would come form $U_1(x,t) \neq U_2(x,t)$. Nevertheless, converting the numeraire-1 value of a bond to its numeraire-2 value (for example) would still require the exchange rate which, if it is to capture the afore-mentioned stylized fact, involves Y. Therefore, given $g_1 = g_2$, additionally imposing $h_1 = h_2$ would impose empirically incorrect cross-currency bond pricing restrictions. We provide an example of an exponential-affine model of the exchange rate below.

Exponential-affine model the exchange rate. This example is in the spirit of Saá-Requejo (1994). We model two yield curves that share one factor: $r_1(t) = X_1(t) + X_3(t)$ and $r_2(t) = X_2(t) + X_3(t)$, where X_1, X_2, and X_3 are independent square-root processes:

$$dX_i(t) = \kappa_i\,(\theta_i - X_i(t))\,dt + s_i\,\sqrt{X_i(t)}\,dW_{X_i}(t).$$

Let Y be an independent scalar square-root process

$$dY(t) = \kappa_Y\,(\theta_Y - Y(t))\,dt + s_Y\,\sqrt{Y(t)}\,dW_Y(t).$$

Let

$$\lambda_1(t) = \begin{pmatrix} \alpha_1\sqrt{X_1(t)} \\ \alpha_2\sqrt{X_2(t)} \\ \alpha_3\sqrt{X_3(t)} \\ \alpha_4\sqrt{Y(t)} \end{pmatrix} \qquad \text{and} \qquad \lambda_2(t) = \begin{pmatrix} \beta_1\sqrt{X_1(t)} \\ \beta_2\sqrt{X_2(t)} \\ \beta_3\sqrt{X_3(t)} \\ \beta_4\sqrt{Y(t)} \end{pmatrix}.$$

Evidently, $h_1(x,y) = \alpha_4\sqrt{y}$ and $h_2(x,y) = \beta_4\sqrt{y}$. The dynamics of expected exchange-rate depreciation are given by

$$\begin{aligned} d\mu_\psi(t) &= dr_2(t) - dr_1(t) + \frac{1}{2}\left\{d\left(\|\lambda_2(t)\|^2\right) - d\left(\|\lambda_2(t)\|^2\right)\right\} \\ &= \sum_{i=1}^{3} a_i\,dX_i(t) + \left(\frac{\beta_4^2 - \alpha_4^2}{2}\right) dY(t), \end{aligned} \tag{7.7}$$

[20]See Saá-Requejo (1994) and the citations therein.

[21]This point can be found in Saá-Requejo (1994).

where the coefficients a_i can be easily computed. We see that as long as $\alpha_4^2 \neq \beta_4^2$, expected inflation is driven by Y which is uncorrelated with both yield curves.

Rogers. Rogers (1997) models the state–price deflator as $n(t) = 1/u(t)$, where $u(t) = \exp(\alpha\, t + f(X(t)))$.[22] For most of his examples, he specifies the dynamics of the state variables as

$$\mu_X(x) = -B\, x \quad \text{and} \quad \sigma_X(x) = I, \tag{7.8}$$

where B is a general $d \times d$ matrix and I is the $d \times d$ identity matrix. Given this setup, we have $g(x) \equiv 0$, $h(x, y) \equiv 0$, and $u = v$, the value of the VLD bond. Rogers shows that with (7.8), the expectation in (5.14) is straightforward to compute for a variety of functional forms f.

Rogers uses the foregoing setup to model foreign exchange rates, which are given by

$$\psi(t) = \frac{n_1(t)}{n_2(t)} = \frac{u_2(t)}{u_1(t)}, \tag{7.9}$$

where $u_i(t) = \exp(\alpha_i\, t + f_i(X(t)))$. As a result, there is no neutrino-factor component to exchange rates. As discussed above, these are strong restrictions. In particular, it is not possible for expected exchange-rate depreciation to move in ways that are orthogonal to both term structures. Moreover, the volatility of the log of the exchange rate is constrained to be as given in (7.6). Rogers, however, points out a "practical advantage" to his approach: " ... once the term structure has been modeled in the two countries, the exchange rate between them is determined; *no further Brownian motions are needed!*"[23] The upshot is that all foreign exchange-rate risk can be hedged in the bond markets.

In our analysis of Constantinides (1992) in Section 5, we showed that by assuming the drift of the state variables was specified under an equivalent martingale measure $\mathcal{Q}^\vartheta$ rather than under the physical measure $\mathcal{P}$ as in the original model, one can relax the restriction that imposes the constancy of the term-structure martingale and yet keep the bond price function unchanged. We can apply the same approach here. Modeling under $\mathcal{Q}^\vartheta$ is not restrictive, since one is free to introduce $g(x)$, $h(x, y)$, and the dynamics of Y to obtain a more general model of the state–price deflator.[24] What are the consequences of this reinterpretation for exchange rates? Given (7.9), we must have $\zeta^{\vartheta_1}(t) \equiv \zeta^{\vartheta_2}(t)$ and consequently

$$g_1(x) = g_2(x) \quad \text{and} \quad h_1(x, y) = h_2(x, y). \tag{7.10}$$

Therefore, even with the benefit of the reinterpretation, the neutrino-factor component must be absent from exchange-rate dynamics.

[22] Rogers' function $f(x)$ is equivalent to $\exp(-f(x))$ in our notation.

[23] Emphasis in the original.

[24] Of course if we wish to maintain the same functional form for the drift of X under both measures, we must restrict the form of g. Given (7.8) for example, as long as $g(x)$ is linear in x, the form of μ_X will be the same as μ_X^ϑ.

Appendix A. Using the VLD to reveal market expectations

Throughout this appendix, we assume the VLD bond exists. Let v denote the value of the VLD bond, n denote the state–price deflator, and $z = n/v$ be the asymptotic revision process for the state–price deflator. Let $\delta(T)$ be some time-T measurable random variable for $T \geq t$. First we show how to extract the conditional expectation of $\delta(T)$. Let $s(t)$ be the value at time t of an asset that has a single payoff $v(T)\delta(T)$ at time T. In this case, the martingale property of deflated asset prices (see (3.1)) can be written as

$$n(t)\, s(t) = E_t[n(T)\, v(T)\, \delta(T)]. \tag{A.1}$$

Dividing both sides of (A.1) by $z(t)$ produces

$$\frac{s(t)}{v(t)} = E_t\left[\frac{z(T)}{z(t)}\,\delta(T)\right] = E_t\left[\delta(T)\right] + \mathrm{Cov}_t\left[\frac{z(T)}{z(t)},\, \delta(T)\right].$$

Thus, if

$$\mathrm{Cov}_t\left[\frac{z(T)}{z(t)},\, \delta(T)\right] = 0, \tag{A.2}$$

for all $t \leq T$, then

$$\frac{s(t)}{v(t)} = E_t\left[\delta(T)\right],$$

in which case asset values can be constructed using the VLD to reveal the market's expectation about $\delta(T)$. We consider two cases. First, if $g \equiv 0$, then (A.2) holds for any $\delta(T)$ that is measurable with respect to the filtration generated by W_X. Second, if in addition $h \equiv 0$, then (A.2) holds for any time-T measurable $\delta(T)$.

Thus far, we have shown how to extract the conditional expectation of $\delta(T)$ using the VLD bond. Now we show how to extract the entire conditional distribution of $\delta(T)$. Let $s(t,T,K)$ be the value of the asset with payoff $v(T)\,(\delta(T) - K)^+$, where $(x)^+ := \max(x, 0)$. In this case we have

$$\frac{s(t,T,K)}{v(t)} = E_t\left[(\delta(T) - K)^+\right] + \mathrm{Cov}_t\left[\frac{z(T)}{z(t)},\, (\delta(T) - K)^+\right].$$

If the covariance term is zero (for one reason or another), then we have

$$\frac{s(t,T,K)}{v(t)} = \int_K^\infty (x - K)\, \phi(x; t, T)\, dx,$$

where $\phi(x; t, T)$ is the probability density function for $\delta(T)$ conditional on the information at time t. Differentiating twice with respect to the strike price produces

$$\frac{\partial^2}{\partial x^2}\, \frac{s(t,T,x)}{v(t)} = \phi(x; t, T).$$

Proxy VLD bond. Since it is infeasible to issue a zero-coupon bond with an infinite maturity, we investigate the effects of using a finite-maturity bond as a proxy for the VLD bond. A natural proxy for $v(t)$ is $v(t,\tau) = p(t,\tau)/p(0,\tau)$ for some large τ. Next we show how to use the proxy VLD bond to extract an approximation to the market expectation, where the approximation error can be made arbitrarily small.

Let $s(t)$ be the value of an asset that makes a single payment of $v(T,\tau)\,\delta(T)$, where $\tau \gg T$. Assume, as above, $\mathrm{Cov}_t[z(T)/z(t), \delta(T)] = 0$. Then

$$\begin{aligned}\frac{s(t)}{v(t,\tau)} &= E_t\left[\frac{v(T,\tau)/v(T)}{v(t,\tau)/v(t)}\,\frac{z(T)}{z(t)}\,\delta(T)\right]\\ &= E_t[\delta(T)] + \mathrm{Cov}_t\left[\frac{v(T,\tau)/v(T)}{v(t,\tau)/v(t)}, \frac{z(T)}{z(t)}\delta(T)\right].\end{aligned} \tag{A.3}$$

In the second equality we have used the fact that $v(t,\tau)/v(t)$, the deflated value of an asset, is a martingale. The covariance term in (A.3) is the approximation error, which can be made arbitrarily small by choosing τ sufficiently large. (A similar argument shows that we can approximate arbitrarily-well the entire conditional distribution of $\delta(T)$ using assets with payoffs equal to $v(t,\tau)\,(\delta(T)-K)^+$.)

For example, the Treasury could issue zero-coupon bonds with an original maturities of 1000 years (a very long Treasury security). The Treasury would fix the issue-date value at unity, and the bidders would bid on the face value at maturity with the lowest bids being accepted. The Federal Reserve could require the Primary Dealers to make markets in a sequence of assets with payoffs $r(T)\,v(T,\tau)$ where $r(T)$ is the interest rate at time T in the future. (The Fed could ensure liquidity by actively trading with the Primary Dealers if necessary.) Then (assuming the appropriate covariance were zero) the values of these assets, deflated by the current value of the very long Treasury security would reveal the market's expectation for the path of the interest rate. After 10 or 20 years, the Treasury could issue new very long Treasury securities and retire the outstanding ones.

Appendix B. Relationship to positive interest rate models

Given (3.2), we can write the conditional expectation of the state–price deflator as follows:

$$E_t[n(T)] = n(t) + \int_{u=t}^{T} E_t[\mu_n(u)]\,du. \tag{B.1}$$

Given (B.1), we can express $z = \varphi(n)$ as

$$z(t) = \lim_{T\to\infty} \frac{n(t) + \int_{u=t}^{T} E_t[\mu_n(u)]\,du}{1 + \int_{u=0}^{T} E_0[\mu_n(u)]\,du}. \tag{B.2}$$

For strictly positive interest rates, $\lim_{T\to\infty} E_t[n(T)] = 0$, so both numerator and denominator in (B.2) go to zero. Applying L'Hôpital's rule, we have

$$z(t) = \lim_{T\to\infty} M(t,T) \tag{B.3}$$

where

$$M(t,T) := \frac{E_t[\mu_n(T)]}{E_0[\mu_n(T)]}.$$

$M(t,T)$ is the family of positive martingales that is central to the positive interest rate models of Jin and Glasserman (1997). In particular, they express the state-price deflator as

$$n(t) = \int_{s=t}^{\infty} h(s)\, M(t,s)\, ds,$$

where $h(s) := -E_0[\mu_n(s)]$.[25]

Appendix C. Beveridge–Nelson decomposition

When there are no state-dependent volatilities, there is a close correspondence between our decomposition via the asymptotic revision process and the decomposition of Beveridge and Nelson (1981) for discrete-time ARIMA models. We illustrate this with an example.

First, we introduce two definitions.

Definition 5. Let n be a positive process with $n(0) = 1$. If

$$\xi(t) = \lim_{T\to\infty} \; E_t[\log(n(T))] - E_0[\log(n(T))]$$

exists a.s. for all $t \geq 0$, then we say that ξ is the *permanent component* of n, which we write as $\xi = \chi(\log(n))$.

Definition 6. We say that n is *log-trend-stationary* if $\chi(\log(n)) = 0$.

Let us take $n_2 = z_2\, z_1\, n_0$ from the example in Section 2 as the state–price deflator, but replace the function $h(x,y)$ with the parameter h. Then we can write

$$\log(n_2(t)) = \log(n_0(t)) - \frac{1}{2}\left(g^2 + h^2\right) t - g\, W_X(t) - h\, W_Y(t). \tag{C.1}$$

One can verify that the permanent component of $\log(n_2)$ is given by $\xi(t) = -g\, W_X(t) - h\, W_Y(t)$, and that $\log(n_0(t)) - \frac{1}{2}\left(g^2 + h^2\right) t$ is trend-stationary.

Let us assume that we sample this process discretely, at unit intervals. We introduce the standard lag operator L, where $L\, x(t) = x(t-1)$ for any process x. By applying the differencing operator $1 - L$ to $X(t+1)$ we can decompose

$$\begin{aligned}(1-L)\, X(t+1) &= (X(t,t+1) - X(t)) + \int_{s=t}^{t+1} dX(s,t+1) \\ &= \left(1 - e^{-\kappa}\right)(\theta - X(t)) + \epsilon(t+1)\end{aligned} \tag{C.2}$$

into expected and unexpected components, where

$$\epsilon(t+1) = \int_{s=t}^{t+1} dX(s,t+1) = \int_{s=t}^{t+1} \sigma\, e^{-\kappa\,(t+1-s)}\, dW_X(s)$$

[25] Equation (B.3) implies $\varphi(n) = \varphi(-\mu_n) = \varphi(n\, r)$, where r is the interest rate. See Appendix D for a discussion of generalized VLD bonds.

is conditionally Gaussian with mean zero and variance $\Sigma(t, t+1)$.[26] Thus discretely-sampled X follows an AR(1), which we can write as

$$(1 - \rho\, L)\, X(t+1) = m + \epsilon(t+1), \tag{C.3}$$

where $m = (1 - e^{-\kappa})\,\theta$ and $\rho = e^{-\kappa}$. Consequently, if $g = h = 0$, $\log(n_2)$ is an AR(1) plus deterministic drift.

Differencing (C.1) produces

$$\begin{aligned}(1 - L)\,\log(n_2(t+1)) = \\ &- \hat{\alpha} - \beta\,(1 - L)\, X(t+1) - g\,\varepsilon_X(t+1) - h\,\varepsilon_Y(t+1),\end{aligned} \tag{C.4}$$

where $\hat{\alpha} = \alpha + \frac{1}{2}\left(g^2 + h^2\right)$ and $\varepsilon_i(t+1) = (1 - L)\, W_i(t+1)$. The $\varepsilon_i(t+1)$ have conditional mean zero and variance one. Also note, the covariance between $\varepsilon_X(t+1)$ and $\epsilon(t+1)$ is given by $(\sigma/\kappa)\,(1 - e^{-\kappa})$. The variance of the first difference of the permanent component is $g^2 + h^2$. Substituting (C.3) into (C.4) produces

$$\begin{aligned}(1 - L)\,\log(n_2(t+1)) = \\ &- \hat{\alpha} - \beta \left(\frac{1 - L}{1 - \rho\, L}\right) \epsilon(t+1) - g\,\varepsilon_X(t+1) - h\,\varepsilon_Y(t+1),\end{aligned} \tag{C.5}$$

where we can expand

$$\left(\frac{1 - L}{1 - \rho\, L}\right) = 1 + (\rho - 1)\left(L + \rho\, L^2 + \rho^2\, L^3 + \cdots\right).$$

We can compute the autocovariance function from (C.5):

$$\begin{aligned}\gamma(0) &= C + g^2 + h^2 \\ \gamma(1) &= \frac{(\rho - 1)\, C}{2} \\ \gamma(j) &= \rho\,\gamma(j-1) \qquad \text{for } j = 2, 3, \ldots,\end{aligned} \tag{C.6}$$

where

$$C = \frac{\beta^2\,\sigma^2}{(1 + \rho)\,\kappa}\left(1 - e^{-2\,\kappa}\right) + \frac{2\,g\,\beta\,\sigma}{\kappa}\left(1 - e^{-\kappa}\right).$$

This autocovariance function is characteristic of an ARIMA(1,0,1) with AR coefficient ρ. Therefore, we can parameterize the process for $(1 - L)\,\log(n_2(t+1))$ as follows:

$$(1 - \rho\, L)\,(1 - L)\,\log(n_2(t+1)) = (1 - \rho)\,\hat{\alpha} + (1 + \Theta\, L)\, e(t+1) \tag{C.7}$$

is

$$\begin{aligned}\gamma(0) &= \frac{1 + \Theta^2 + 2\,\rho\,\Theta}{1 - \rho^2}\,\sigma_e^2 \\ \gamma(1) &= \frac{(1 + \rho\,\Theta)\,(\rho + \Theta)}{1 - \rho^2}\,\sigma_e^2 \\ \gamma(j) &= \rho\,\gamma(j-1) \qquad \text{for } j = 2, 3, \ldots,\end{aligned} \tag{C.8}$$

[26]The expressions for $X(t, T)$ and $\Sigma(t, T)$ are given in (2.5).

where σ_e^2 is the conditional variance of $e(t+1)$.[27] Matching expressions for $\gamma(0)$ and $\gamma(1)$ in (C.6) and (C.8), we can solve for Θ and σ_e^2.

We now turn to the Beveridge–Nelson decomposition. We can normalize (C.7) into an infinite-order MA repersentation for the growth rates of n_2:

$$(1 - L)\, \log(n_2(t+1)) = \hat{\alpha} + c(L)\, e(t+1),$$

where $c(L) := (1 + \Theta\, L)/(1 - \rho\, L)$. Beveridge and Nelson decompose this operator into permanent and transitory components: $c(L) = c(1) + c^*(L)$, where $c^*(L) = c(L) - c(1)$ is the transitory component.[28] The one-step ahead conditional variance of the permanent component is given by

$$c(1)\, \sigma_e^2 = \frac{\sigma_e^2\, (1+\Theta)^2}{(1-\rho)^2} = \gamma(0) - \frac{2\,\gamma(1)}{\rho - 1} = g^2 + h^2,$$

which (as noted) is the variance of the permanent component. (The second equality comes from solving (C.8) for Θ and σ_e^2, and the third equality comes from using (C.6) to eliminate $\gamma(0)$ and $\gamma(1)$.)

Appendix D. The generalized VLD bond and payout bubbles

In this appendix we define a generalization of the VLD bond and we show that these assets are payout bubbles in the sense of Gilles and LeRoy (1997).

The generalized VLD bond. Define

$$s(t,T) = E_t\left[\frac{n(T)}{n(t)}\, \omega(T)\right],$$

where $\omega(t)$ is some strictly positive process. In other words, $s(t,T)$ is the value at time t of an asset that makes a single payment of $\omega(T)$ at time T. Now define

$$\tilde{s}(t,T) = \frac{s(t,T)}{s(0,T)} = \frac{E_t\left[\left(\frac{n(T)}{n(t)}\right)\omega(T)\right]}{E_0\left[n(T)\,\omega(T)\right]} = E_t\left[\left(\frac{n(T)}{n(t)}\right)\tilde{\omega}(T)\right],$$

where

$$\tilde{\omega}(T) = \frac{\omega(T)}{E_0\left[n(T)\,\omega(T)\right]}.$$

In other words, $\tilde{s}(t,T)$ is the value at time t of an asset that makes a single payment of $\tilde{\omega}(T)$ at time T. Note that $s(0,T) \equiv 1$. Now define

$$u(t) = \lim_{T\to\infty} \tilde{s}(t,T),$$

if the limit exists for all $t \geq 0$, which we can write as $u = \varphi(n\,\omega)/n$. We refer to u as the value process of a generalized VLD (GVLD) bond and to ω as its payoff process. The VLD bond is a special case, where $\omega = 1$.

[27] See Harvey (1981, chapter 2), for example.

[28] $c(1) = c(L=1) = \sum_{i=0}^{\infty} c_i$.

Proposition 8. Let ω be the payoff process for the GVLD bond u and let $v = \varphi(n)/n$ be the value of the VLD bond. Then

$$u = v \iff \varphi(\omega/v) = 1.$$

Proof. Let $\zeta^\vartheta = \varphi(n)$. Then

$$u = \varphi(n\,\omega)/n = \varphi(\zeta^\vartheta\,\omega/v)/n = \zeta^\vartheta\,\varphi^\vartheta(\omega/v)/n = v\,\varphi^\vartheta(\omega/v).$$

But by Proposition 5, $\varphi^\vartheta(\omega/v) = 1 \iff \varphi(\omega/v) = 1$. □

Relation to payoff bubbles. In this section, we sketch how the MMA and the VLD can be given a mathematical representation, following the argument in Gilles and LeRoy (1997). For ease of exposition, we assume that traded securities are claims to lumpy dividends payable at any dates that are in a fixed countable set, say $\mathcal{T} = \{0, 1, 2.\dots\}$, endowed with the counting measure on $2^{\mathcal{T}}$ (the set of all subsets of $\mathcal{T}$ (the analysis can be carried out on the half-line $[0, \infty)$ with Lebesgue measure, but the technical difficulties would obfuscate the main idea).

With fixed state-price deflator $n(t)$, consider the space of cashflows

$$c = \{c(0), c(1), c(2), \dots\}$$

such that $c(t)$ is $\mathcal{F}_t$-measurable and $\|c\|_1 := \sum_{t\in\mathcal{T}} E_0[n(t)\,|c(t)|] < \infty$. This space can be thought of as an L_1 space with norm $\|\cdot\|_1$ over the measurable space $\{\Omega \times \mathcal{T}, \mathcal{F} \times 2^{\mathcal{T}}\}$; its norm dual is the L_∞ space consisting of the processes m such that $m(t)/n(t)$ is essentially bounded on $\mathcal{T}$, and the norm of m is $\|m\|_\infty :=$ ess $\sup_{t\in\mathcal{T}}\{|m(t)|/n(t)\}$. The norm dual of this L_∞ space is a space, denoted **ba**, of finitely-additive measures ν on $\Omega \times \mathcal{T}$, such that $\|\nu\| := \sup\{\int_A n(T)\,d|\nu| \mid A \in \mathcal{F} \times 2^{\mathcal{T}}\} < \infty$.

The space L_1 of cashflows is embedded in **ba** in such a way that the unit ball of L_1 is dense in the unit ball of **ba**, when L_1 is endowed with the weak* topology. Consider a sequence $(x_1, x_2, \dots)$ of cashflows in L_1, with x_i consisting of a single non-negative cashflow $c(i)$ payable at time i satisfying $E_0[n(i)\,c(i)] > 0$, so that $\|x_i\| = E_0[n(i)\,c(i)] > 0$. Defining $y_i := x_i/\|x_i\|$, the sequence $y := (y_1, y_2, \dots)$ is contained in the unit sphere in L_1, and each y_i has unit value. Therefore y has a limit point b in **ba**, a *bubble*, or payoff at infinity with value $\int n(\,\cdot\,)\,db = 1$.

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