Modelling Volatility and Volatility Derivatives

Andrew Matytsin

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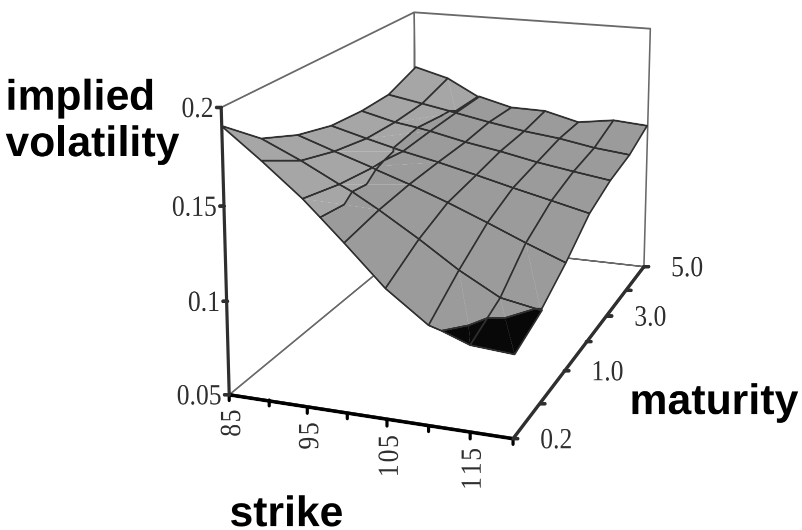
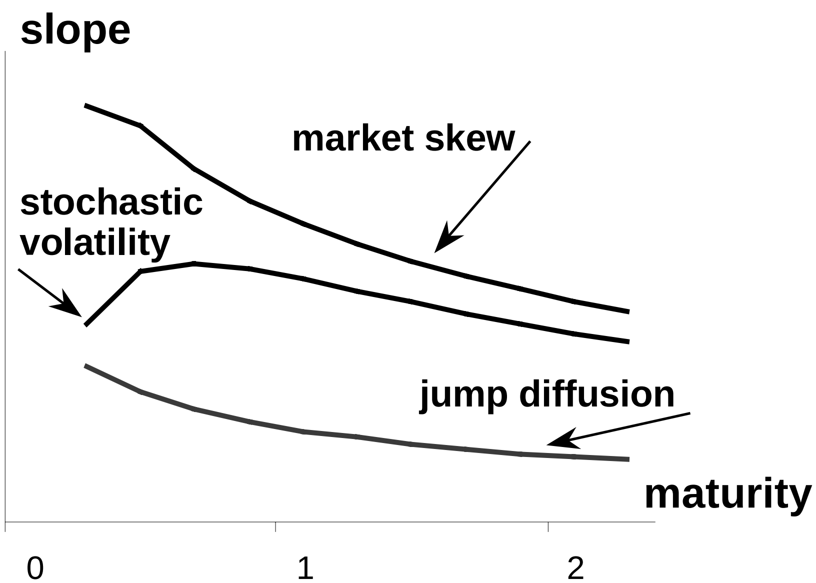
What Determines the Smile Term Structure?

S&P500 volatility surface on January 11, 1996



skew slope vs. maturity



Market crashes drive the short-term smile
Uncertainty in volatility drives the long-term smile

How to Combine Stochastic Volatility and Jump Diffusion ?

- **between jumps** $\begin{cases} dS/S = \mu dt + \sqrt{v}dz_1 \\ dv = \kappa(\theta - v)dt + \sigma\sqrt{v}dz_2 \end{cases}$ $\quad Corr(dz_1, dz_2) = \rho$

- **market crashes form a Poisson process with rate** λ

$$\begin{cases} \log S \rightarrow \log S + \gamma_s + \delta_s \varepsilon \\ v \rightarrow v + \gamma_v \end{cases} \qquad \varepsilon \propto N(0,1)$$

- **the option price obeys the equation**

$$\frac{\partial f}{\partial t} + \mu^* S \frac{\partial f}{\partial S} + \kappa(\theta - v)\frac{\partial f}{\partial v} + \frac{1}{2}v\left\{S^2 \frac{\partial^2 f}{\partial S^2} + \sigma^2 \frac{\partial^2 f}{\partial v^2} + 2\rho\sigma S \frac{\partial^2 f}{\partial S \partial v}\right\}$$

$$+ \lambda E^*\left[f(Se^{\gamma_s + \delta_s \varepsilon}, v + \gamma_v) - f(S, v)\right] = rf$$

European option prices can be computed analytically

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What are the European Option Prices?

- **Call prices equal** $C = S\,P_1 - K\,e^{-rT}\,P_0$
- **The Fourier Transforms of** P_1 **and** P_0 **have the affine form**

$$\hat{P}_n = e^{C(T-t,\varphi)+D(T-t,\varphi)v}$$

- $C(\tau,\varphi)$ **and** $D(\tau,\varphi)$ **obey the first order equations in** τ

$$\begin{cases} C(\tau,\varphi) = C_H(\tau,\varphi) + \lambda\tau\left[e^{i\varphi\gamma_s - \varphi^2\delta_s^2/2} I(\tau) - 1\right] \\ D(\tau,\varphi) = D_H(\tau,\varphi) \end{cases}$$

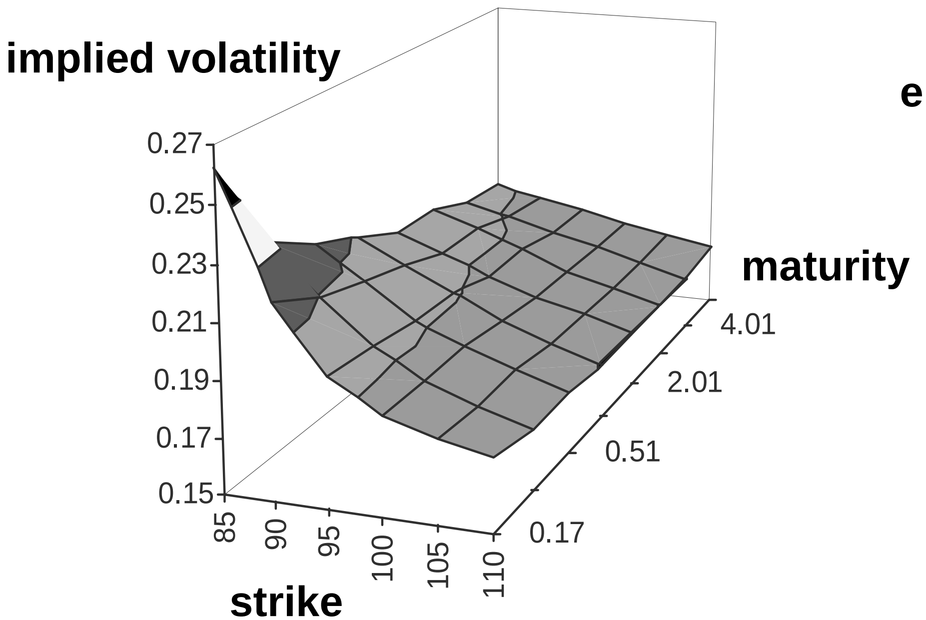
$$p_\pm = \frac{\gamma_v}{\sigma^2}(b - \rho\sigma\varphi i \pm d)$$

$$I(\tau) = \frac{1}{\tau}\int_0^\tau e^{\gamma_v D(t,\varphi)} dt = -\frac{2\gamma_v}{p_+ p_-}\int_0^{-\gamma_v D(\tau,\varphi)} \frac{e^{-z}dz}{(1+z/p_+)(1+z/p_-)}$$

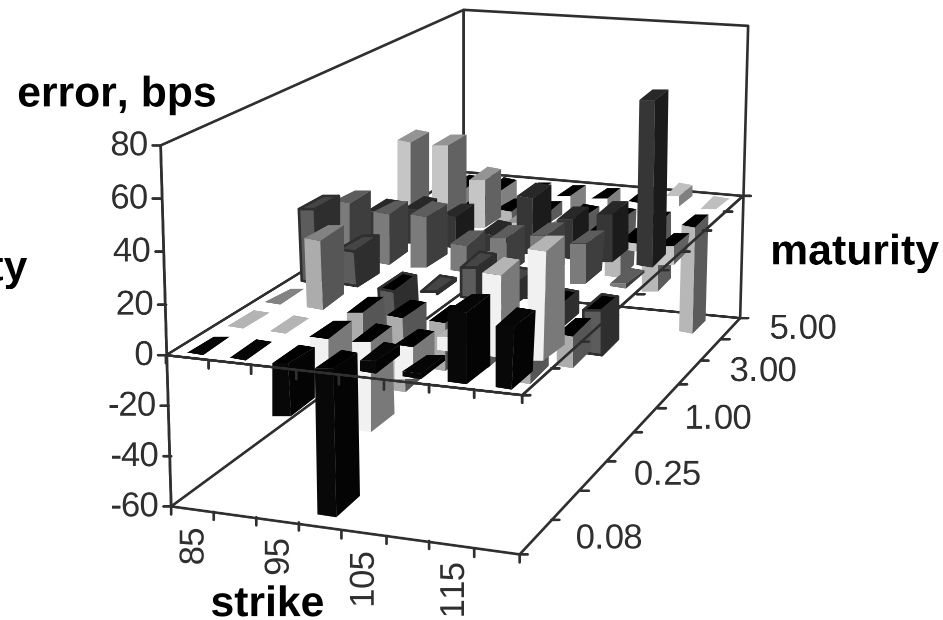
Computed efficiently using the Fast Fourier Transform
There is no need to sum over jumps

Does the Model Fit the Smile?

S&P500 volatility surface on June 11, 1997



Calibration errors



The whole volatility surface is described by one set of constant parameters

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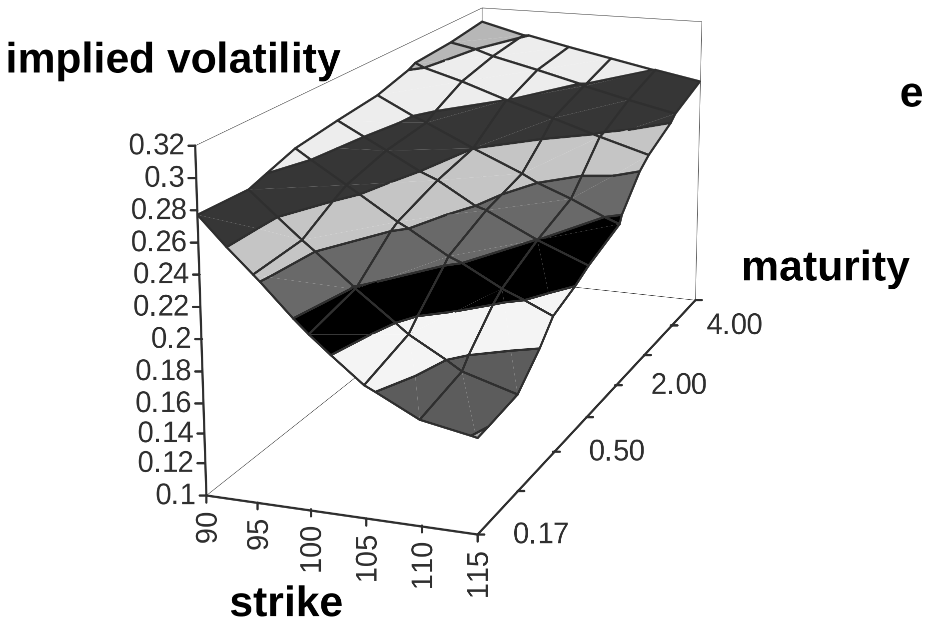
Does the Model Fit the Smile?

S&P500 volatility surface in August, 1999



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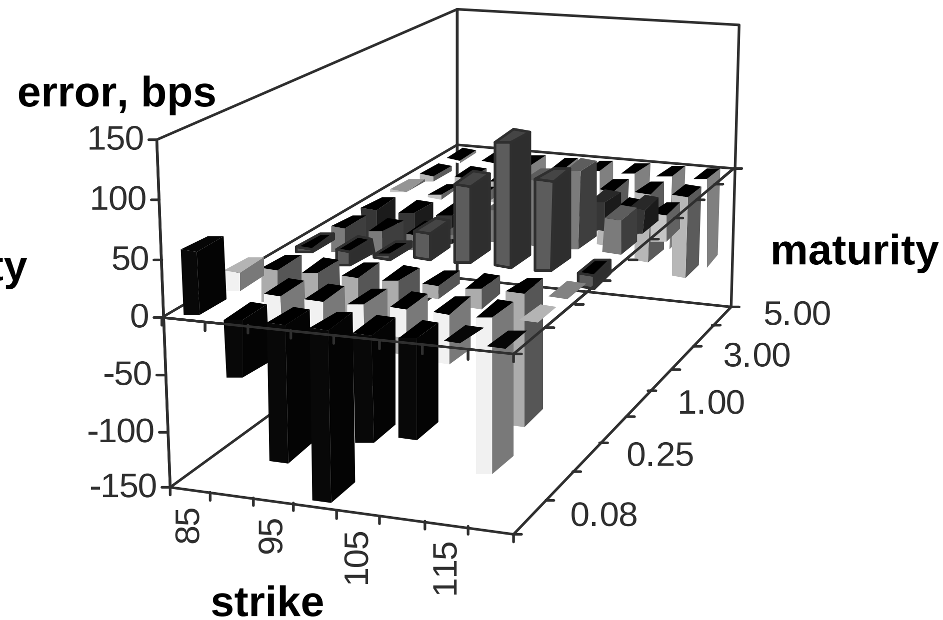
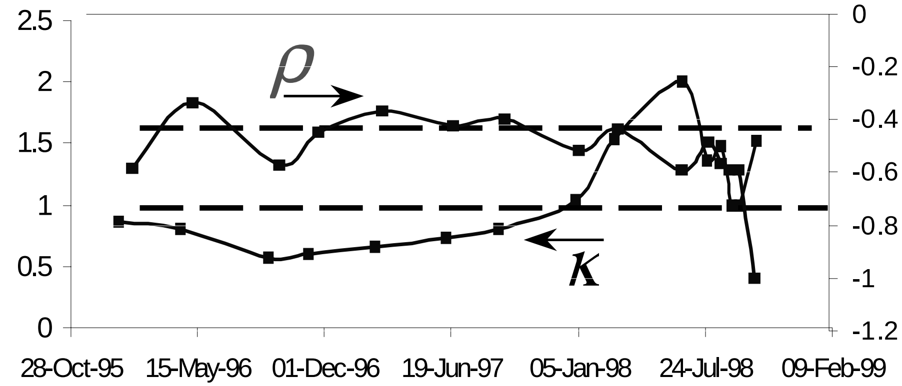
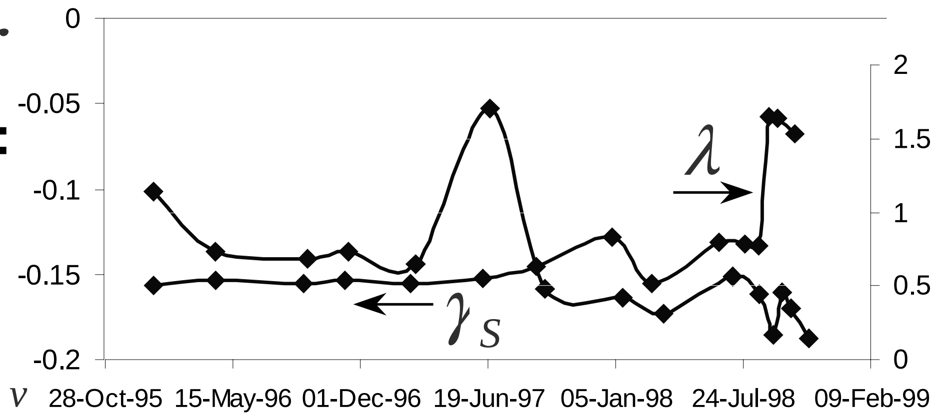


The whole volatility surface is described by one set of constant parameters

Are Smile Parameters Stable Over Time?

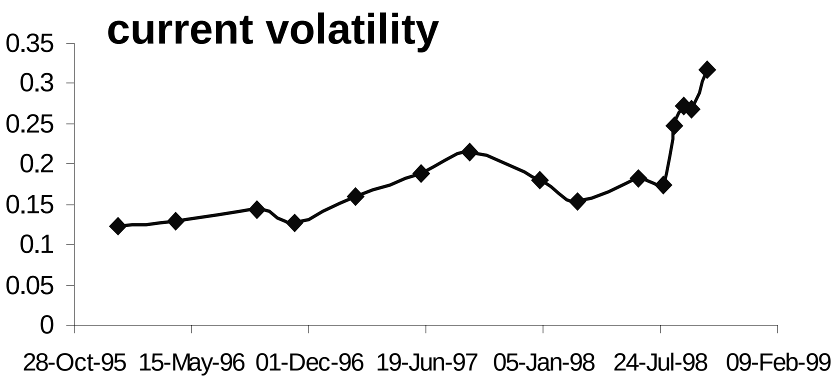
- **Volatility parameters:**
  - current volatility $\sqrt{v}$
  - correlation ρ
  - vol of vol σ
  - long run volatility $\sqrt{\theta}$
  - mean reversion rate κ
- **Market crash parameters:**
  - crash rate λ
  - crash magnitude γ_S
  - vol jump magnitude γ_v

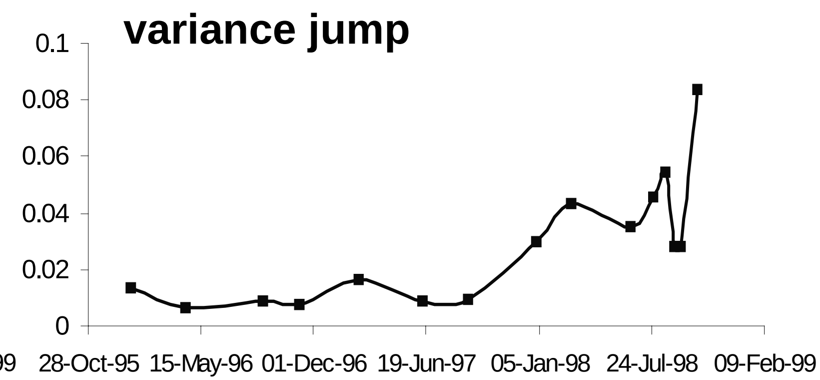


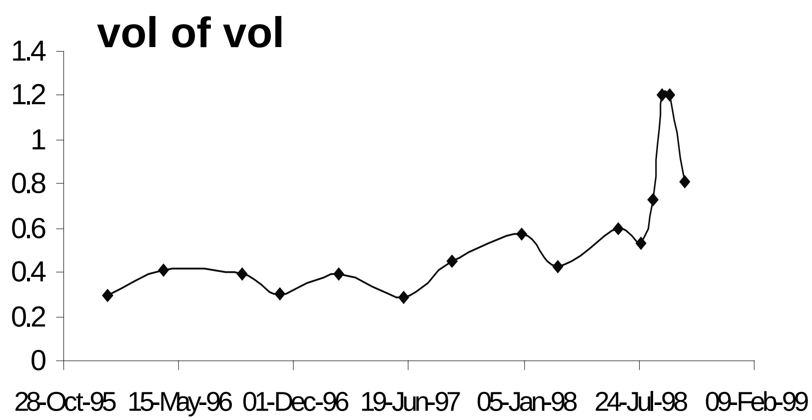


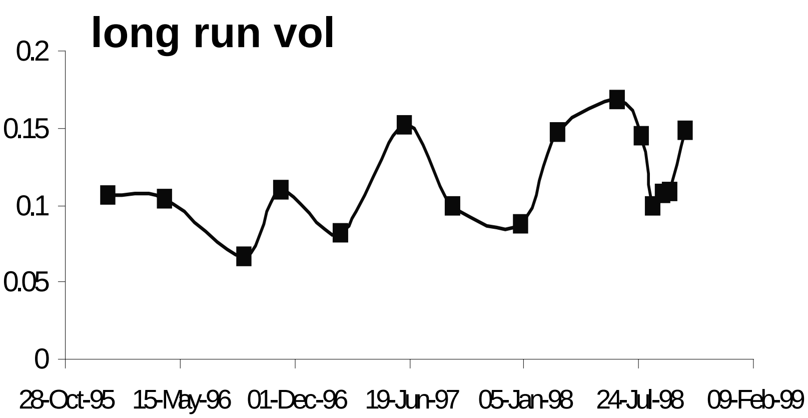
Mean reversion, correlation and crash size are constant

Patterns in Stochastic Volatility Parameters





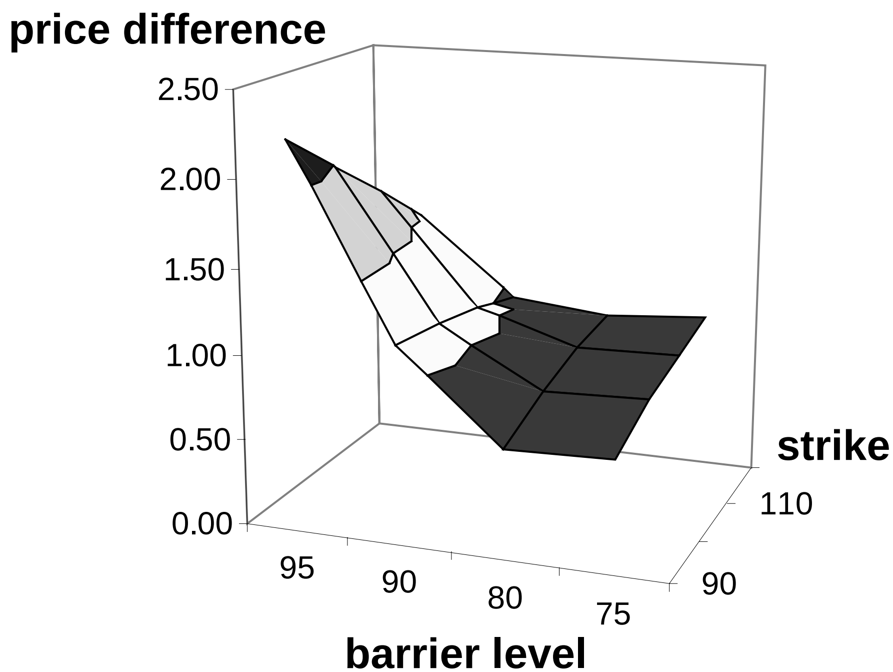




Long run diffusion volatility is relatively stable

Are Exotics Prices Different?

Down-and-out call
maturity 3 years



Deltas in the two models

Stochastic Volatility / Jump Diffusion

	95	90	80	75
90	1.46	1.13	0.98	0.86
100	1.29	1.01	0.87	0.78
110	1.12	0.89	0.77	0.70
120	0.95	0.77	0.68	0.63

Implied Tree / Dupire

	95	90	80	75
90	1.11	1.24	1.02	0.94
100	0.97	1.08	0.89	0.83
110	0.84	0.92	0.76	0.72
120	0.71	0.77	0.64	0.61

Deterministic volatility models may misprice barrier options

What are the Risk Premia?

- **The risk-neutral drift of variance and the jump rate contain risk premia:** $\sigma_m^* = \sqrt{\theta^*} > \sigma_m = \sqrt{\theta}$ $\lambda^* > \lambda$
- **In the power utility model with** $U(C) \propto C^\alpha$

$$dC_t / C_t = \mu_c dt + \sigma_c \sqrt{v_t} dz_c + (e^{\gamma_c} - 1) dq$$

  - **crash rate** $\lambda^* = \lambda e^{(\alpha-1)\gamma_c}$
  - **mean reversion** $\kappa^* = \kappa - (\alpha - 1)\sigma_c \sigma \rho_{cv}$
  - **total return** $R = r - (\alpha - 1) v_t \sigma_c \sigma_x \rho_{cx} + (e^{\gamma_x} - 1)(\lambda - \lambda^*)$
- **Rough estimates for S&P500:** $\lambda \propto 0.1, \quad \lambda^* \propto 0.5$

$$\gamma_x \propto -0.1, \quad \alpha - 1 \propto -16, \quad \gamma_c \propto -0.1 \quad \Rightarrow \quad \kappa^* - \kappa \propto -1.5, \quad \lambda^* \propto 5\lambda$$

Risk premia are roughly consistent with historic observations

Derivatives on Realized Volatility

- **Volatility swap:**

$$payout\ at\ maturity = notional * \left[\sigma_{historic} - \sigma_{agreed\ \ vol}\right]$$

- **Variance swap:**

$$payout\ at\ maturity = notional * \left[\sigma^2_{historic} - \sigma^2_{agreed\ \ var}\right]$$

- $\sigma_{historic}$ **is the standard deviation of realized returns**

$$\sigma_{historic} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(r_i - \bar{r}\right)^2}, \quad r_i = \ln\left(\frac{S_{i+1}}{S_i}\right)$$

- **To hedge a variance swap, buy 2 log contracts** $f(S_T) = \log(S_T / F)$

$$\Delta t\left[\frac{1}{2}\sigma^2_{realized}S^2\Gamma_f - \frac{1}{2}\sigma^2_{implied}S^2\Gamma_f\right] = \frac{1}{2}\Delta t\left(\sigma^2_{implied} - \sigma^2_{realized}\right)$$

Without jumps, the log hedge is model-independent

How to Find Expected Volatility?

- **Evaluate the characteristic functional** $f(v,t \mid z) = E_t^*\left[e^{-zV_t} \mid F_t\right]$

$$V_t = \int_t^T v(\tau)d\tau + \sum_{n=1}^{N_j} (\gamma_s + \delta_s \varepsilon)^2$$

- **Expected variance:** $E\{V_t\} = -(\partial f / \partial z)\,|_{z=0}$

- **Expected volatility:** $E\left\{\sqrt{V_t}\right\} = \dfrac{1}{2\sqrt{\pi}} \displaystyle\int_0^\infty \frac{1 - E(e^{-zV_t})}{z^{3/2}} dz$

- **Expected payout of a variance call option:**

$$E\left\{\max(V_t - K, 0)\right\} = \frac{1}{\pi} \mathrm{Re} \int_0^\infty \frac{1 - E\{e^{i\varphi(V_t - K)}\}}{\varphi^2} d\varphi + \frac{1}{2}\left[E\{V_t\} - K\right]$$

The characteristic functional is easily found in closed form

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What are the Resulting Prices?

- **Characteristic functional has the form** $f(v,t\,|\,z) = e^{C(t,z)+D(t,z)v}$
- **The final result is** $f(v,t\,|\,z) = \psi_{sv}(z)\;\psi_j(z)$

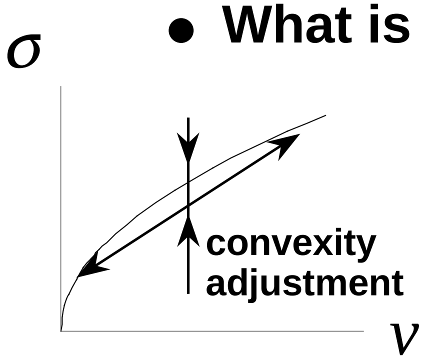
$$\psi_{sv}(z) = e^{\zeta\tau/2}\left[\cosh\left(\frac{\mu\tau}{2}\right) + \frac{1}{\mu}\sinh\left(\frac{\mu\tau}{2}\right)\right]^{-\zeta} \exp\left\{-\frac{2zv}{\kappa[1+\mu\coth(\mu\tau/2)]}\right\}$$

$$\begin{cases} \tau = \kappa T \\ \mu = \sqrt{1+2z\sigma^2/\kappa^2} \\ \zeta = 2\kappa\theta/\sigma \end{cases}$$

$$\psi_j(k) = \exp\left\{-\lambda T\left(1 - \frac{1}{\sqrt{1+2k\delta^2}}\, e^{-k\tilde{\gamma}^2/(1+2k\delta^2)}\left(\frac{1}{T}\int_0^T e^{D(t)\gamma_v}\,dt\right)\right)\right\}$$

$$D(t\,|\,z) = -\frac{2z}{\kappa}\,\frac{1}{1+\mu\coth(\mu\tau/2)}$$

- **What is the difference between expected vol and variance?**



	Swap				
	volatility	variance	implied vol	spread to implied	convexity adjustment
combined model	26.34%	31.20%	26.53%	-0.19%	4.87%
stochastic vol	28.64%	31.29%	26.53%	2.11%	2.66%
jump diffusion	27.55%	30.06%	26.53%	1.02%	2.51%

The convexity adjustment due to jumps may be substantial

How to Hedge Against Jumps?

- **Constant diffusion volatility ⇒ no risk between crashes**
- **An exact hedge against a crash is** $x = \gamma^2 / (e^{\gamma} - 1 - \gamma)$
  of the log security together with its delta $\Delta = -(1/S)\, e^{-r(T-t)}$
- **During a crash, gain** $\left[\log(S_{i+1} / S_i)\right]^2 = \gamma^2$ **on the variance contract which offsets exactly by the loss on the hedge:**

$$x\left\{\left[\log(S_t e^{\gamma}) - \log(S_t)\right] + \Delta\,(S_t e^{\gamma} - S_t)\right\} = -x(e^{\gamma} - 1 - \gamma)$$

- **For S&P500, with jump size** $\gamma \approx -0.15$
  the hedge ratio $x \approx 2(1 - \gamma / 3) \approx 2.10$

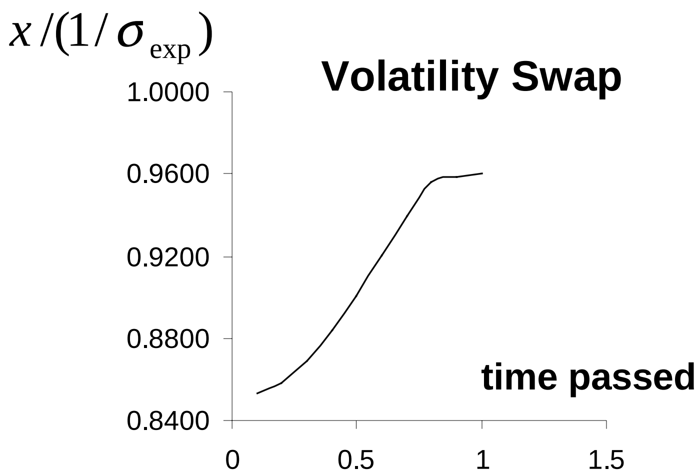
In the stochastic volatility & jump diffusion model the optimal hedge ratio is closer to 2.0

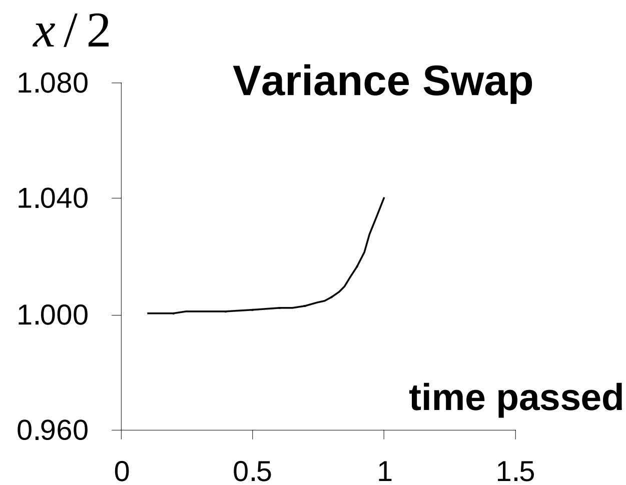
What is the Optimal Log Hedge?

- Hedge with x log contracts and y shares
- Find x and y to minimize the expected P/L variance

$$\lambda\{\Delta C + x\Delta L + y\Delta S\}^2 + v_t\{\sigma^2(\Lambda_C + x\Lambda_L)^2 + y^2 + 2\rho\sigma y(\Lambda_C + x\Lambda_L)\}$$

- The optimal amounts and exposures change over time





Summary and Overview

- Model matches the whole smile with one set of parameters
- It provides a realistic representation of risks
- The impact of jumps on the pricing and hedging of volatility derivatives is significant