RESEARCH STATEMENT

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My research interests center on interactions between topology, geometry, and functional analysis. The specific interactions that are most relevant to my work are organized by Higson and Roe’s analytic surgery exact sequence (see [6], [7], and [8]), an analytic version of the surgery exact sequence from algebraic topology. My thesis research develops a relative counterpart of this machinery and uses it to solve various problems in Riemannian geometry concerning positive scalar curvature obstructions.

My area of research is made possible by the Atiyah-Singer index theorem, an important result in differential topology which extracts topological information from an elliptic differential operator on a smooth compact manifold. Since its first proofs in the 1960’s, the deep connection between analysis and topology made by the index theorem has been recast using the theory of operator algebras, a subfield of functional analysis invented by von Neumann to give a mathematical foundation for quantum mechanics in the 1930’s. The link between the index theorem and operator algebras was forged in part by Kasparov’s far reaching functional-analytic synthesis of K-theory and K-homology, the topological tools used by Atiyah and Singer to prove their theorem (explicitly in the case of K-theory and implicitly in the case of K-homology). This helped pave the way for a flurry of activity aimed at illuminating further connections between topology, geometry and analysis, including Connes’ noncommutative geometry program.

The analytic surgery exact sequence is a tool which uses operator algebras to model the passage from local data to global invariants in a way which is particularly relevant to the index theorem. The local data depend only on the small-scale (topological) structure of a space, while the global invariants depend only on large-scale geometry. Information on each of these two scales is captured by a well-adapted algebra of Hilbert space operators, and a third algebra - an analogue of the mysterious structure set in surgery theory - mediates between them. My research focuses on the role of this third algebra in index theory and on some of the interesting geometric and topological invariants that it houses. In the remainder of this essay I will present the analytic surgery exact sequence in greater detail, outline some of my contributions, and discuss directions for future research.

1. THE ANALYTIC SURGERY EXACT SEQUENCE

Atiyah and Singer’s first published proof of the index theorem (see [3]) relied heavily on Atiyah and Hirzebruch’s topological K-theory, a generalized cohomology theory in algebraic topology. They showed that an elliptic operator $D$ on a smooth compact manifold $M$ gives rise to a class in the K-theory of the cotangent bundle of $X$, and constructed an analytic index map $K^0(T^*X) \to \mathbb{Z}$ which sends the K-theory class of $D$ to the integer $\dim \ker(D) - \dim \text{coker}(D)$ called the index of $D$. The main content of the index theorem lies in the fact that the analytic index map agrees with another map $K^0(T^*X) \to \mathbb{Z}$ which depends only on the topology of $X$.

Later Atiyah realized that $D$ gives rise to a class in the K-homology of $M$ - that is, the generalized homology theory naturally dual to topological K-theory - and that the index of $D$ depends only on its K-homology class (see [1]). Thus there is a Poincare duality pairing between K-theory and K-homology which yields analytic index map $K_0(M) \to \mathbb{Z}$, and this map is the first conceptual ancestor of the analytic surgery exact sequence. Motivated by Atiyah’s idea as well as some seemingly unrelated breakthroughs in the operator algebra community, Kasparov constructed an explicit analytic model of the K-homology of $M$ using the theory of C*-algebras.

A C*-algebra is an algebra of bounded operators on Hilbert space which satisfies some appropriate algebraic and analytic axioms. Given a proper metric space $X$ we can represent the algebra $C_0(X)$ of continuous functions on $X$ which vanish at infinity as a C*-algebra of bounded multiplication operators on $L^2(X)$.
(relative to a suitably chosen Borel measure) and extract subalgebras of $\mathcal{B}(L^2(X))$ which capture certain desirable features of $X$.

The analytic surgery exact sequence uses three different C*-algebras which capture three different aspects of the geometry of $X$:

- $C^*(X) \hookrightarrow$ coarse structure
- $D^*(X) \hookrightarrow$ uniform structure (large and small scales)
- $Q^*(X) \hookrightarrow$ topological structure

The coarse structure of $X$ ignores all local information; any bounded metric space is coarsely equivalent to a point, and $\mathbb{R}$ is coarsely equivalent to its integer lattice $\mathbb{Z}$. Similarly, standard examples of bounded metrics on $\mathbb{R}$ which induce the same topology as the standard one show that the topological structure of $X$ ignores its global geometry.

The C*-algebras above fit into a short exact sequence:

$$0 \to C^*(X) \to D^*(X) \to Q^*(X) \to 0 \tag{1.1}$$

Topological K-theory extends to a system of algebraic invariants associated to any C*-algebra, and it turns out that there is an isomorphism $K_p(Q^*(X)) \cong K_{p-1}(X)$ between the K-theory of the C*-algebra $Q^*(X)$ and the K-homology of $X$. By general principles in K-theory there is a long exact sequence of abelian groups:

$$\cdots \to K_{p+1}(C^*_r(G)) \to K_{p+1}(D^*_G(\tilde{X})) \to K_p(\tilde{X}) \to K_p(C^*_r(G)) \to \cdots \tag{1.2}$$

If $X$ is a simply connected smooth manifold then the long exact sequence (1.2) is precisely the analytic surgery exact sequence for $X$. If $X$ is any compact smooth manifold with fundamental group $G$, then the analytic surgery exact sequence for $X$ is obtained by passing to the K-theory of a $G$-equivariant counterpart of (1.1) applied to the universal cover $\tilde{X}$ of $X$. Since $\tilde{X}$ is equivalent to $G$ on large scales and to $X$ on small scales, we obtain:

$$\cdots \to K_{p+1}(C^*_r(G)) \to K_{p+1}(D^*_G(\tilde{X})) \to K_p(\tilde{X}) \to K_p(C^*_r(G)) \to \cdots \tag{1.3}$$

Here $C^*_r(G)$ is the reduced group C*-algebra of $G$, an object well-studied by operator algebraists and representation theorists.

Those familiar with surgery theory may guess the connection between the analytic surgery exact sequence and algebraic topology: elements of $K_p(X)$ correspond to normal invariants of $X$, $K_p(C^*_r(G))$ corresponds to the algebraic L-groups of $G$, and $K_{p+1}(D^*_G(\tilde{X}))$ corresponds to the structure set of $X$. Higson and Roe constructed an explicit map from the topological surgery exact sequence to (1.3) which makes these correspondences precise. Just as the structure set is often the most mysterious part of the topological surgery exact sequence, the structure algebra $D^*(X)$ and its equivariant counterpart is often the most complicated part of the analytic surgery exact sequence.

My work is more concerned with the connection between the analytic surgery exact sequence and the index theorem than its connection with surgery theory (though the latter is mediated by the former). If $X$ is a compact manifold then it turns out that $K_0(C^*(X)) \cong \mathbb{Z}$, and thus the boundary map in the long exact sequence 1.2 is a homomorphism $K_0(X) \to \mathbb{Z}$. This turns out to be precisely the analytic index map, and hence the homomorphism $K_p(X) \to K_p(C^*(X))$ and its equivariant counterpart can be considered to be a generalized index map. These maps carry a considerable amount of topological and geometric information about $X$.

2. Mayer-Vietoris Sequences

My contributions involved developing a relative version of the short exact sequence (1.1) and using it to solve problems in geometry. Given a proper metric space $X$ and a closed subspace $Y$, there are ideals $C^*(Y \subseteq X)$, $D^*(Y \subseteq X)$, and $Q^*(Y \subseteq X)$ which fit into a short exact sequence

$$0 \to C^*(Y \subseteq X) \to D^*(Y \subseteq X) \to Q^*(Y \subseteq X) \to 0$$

Intuitively these ideals consist of operators which are “supported near $Y$” in an appropriate sense, and indeed the K-theory for each of these ideals is isomorphic to the K-theory of the intrinsic C*-algebras associated
to $Y$. The ideals are useful because in the right circumstances they fit into a Mayer-Vietoris sequence in K-theory associated to a decomposition $A = J_1 + J_2$ of a C*-algebra as the sum of two ideals.

For example, given a decomposition $X = Y_1 \cup Y_2$ of $X$ as the union of two closed subspaces, these remarks yield a Mayer-Vietoris sequence in K-homology:

$$\ldots \rightarrow K_p(Y_1 \cap Y_2) \rightarrow K_p(Y_1) \oplus K_p(Y_2) \rightarrow K_p(X) \rightarrow K_{p-1}(Y_1 \cap Y_2) \rightarrow \ldots$$

If the decomposition satisfies a coarse excision condition identified by Higson, Roe, and Yu in [11] then there is a similar Mayer-Vietoris sequence associated to the coarse C*-algebras of these spaces. I proved that a decomposition of $X$ into subspaces which are closed and which satisfy the coarse excision condition yields a Mayer-Vietoris sequence for the K-theory of the structure algebras. I also constructed equivariant counterparts of these sequences.

- **Main Result 1**: The three Mayer-Vietoris sequences fit together with the analytic surgery exact sequence into the following main diagram:

$$\begin{array}{ccccccccc}
\downarrow & K_{p+1}(D^*(Y_1 \cap Y_2)) & \rightarrow & K_p(Y_1 \cap Y_2) & \rightarrow & K_p(C^*(Y_1 \cap Y_2)) & \rightarrow & \\
\downarrow & K_{p+1}(D^*(Y_1)) \oplus K_{p+1}(D^*(Y_2)) & \rightarrow & K_p(Y_1) \oplus K_{p+1}(Y_2) & \rightarrow & K_p(C^*(Y_1)) \oplus K_p(C^*(Y_2)) & \rightarrow & \\
\downarrow & K_{p+1}(D^*(X)) & \rightarrow & K_p(X) & \rightarrow & K_p(C^*(X)) & \rightarrow & \\
\downarrow & K_p(D^*(Y_1 \cap Y_2)) & \rightarrow & K_{p-1}(Y_1 \cap Y_2) & \rightarrow & K_{p-1}(C^*(Y_1 \cap Y_2)) & \rightarrow & \\
\end{array}$$

This diagram contains a wealth of interesting geometric information. I discovered that it is an ideal environment for conceptualizing Roe’s partitioned manifold index theorem (appearing in [12]), and I used it to prove some useful generalizations.

Roe’s theorem concerns a non-compact manifold $M$ partitioned by a compact oriented hypersurface $N$, meaning $M$ is the union of two submanifolds $M_1$ and $M_2$ with common boundary $N$. The index of an elliptic operator $D_M$ on $M$ need not be finite since $M$ is not compact, but Roe was able to associate a natural generalized index to $D_M$ and prove that it agrees with the ordinary index of the restriction of $D_M$ to the compact manifold $N$. Higson noticed in [4] that this result implies the cobordism invariance of the index: if $W$ is a smooth manifold equipped with an elliptic operator $D$ and the boundary of $W$ is the disjoint union of two compact submanifolds $M_1$ and $M_2$ then the index of the restriction of $D$ to $M_1$ agrees with the index of the restriction of $D$ to $M_2$. To prove this using the partitioned manifold theorem, simply attach cylindrical ends to $W$ and observe that the indices of $D|_{M_1}$ and $D|_{M_2}$ each agree with the generalized index of $D$.

I used the main diagram to give a geometric proof of this result which avoids some of the analytic complications in earlier arguments. Earlier proofs of the theorem did not generalize in an obvious way to equivariant indices, but my argument generalizes easily and yields a new proof of the cobordism invariance of equivariant indices (following Higson).

- **Main Result 2**: The (equivariant) partitioned index of $D_M$ agrees with the (equivariant) index of the restriction of $D_M$ to $N$.

My proof of Roe’s theorem proceeds as follows. Since $N$ is compact $M$ can be equipped with a metric in such a way that it becomes coarsely equivalent to $\mathbb{R}$, and thus there are index maps $K_p(M) \rightarrow K_p(C^*(\mathbb{R}))$
obtained from (1.2). Consider the decomposition \( M = M_1 \cup M_2 \), so that \( M_1 \cap M_2 = N \) as above. This decomposition turns out to be coarsely excisive, so we obtain from the following subdiagram of the main diagram:

\[
\begin{array}{ccc}
K_1(M) & \rightarrow & K_1(C^*(\mathbb{R})) \\
\downarrow & & \downarrow \\
K_0(N) & \rightarrow & K_0(C^*(\{0\}))
\end{array}
\]

Basic calculations show that there is an isomorphism \( K_1(C^*(\mathbb{R})) \cong \mathbb{Z} \) which sends the K-homology class of \( D_M \) to Roe’s generalized index and an isomorphism \( K_0(C^*(\{0\})) \cong \mathbb{Z} \) which sends the K-homology class of \( D_M|_N \) to its ordinary index. The partitioned manifold theorem thus follows from calculations which show that the Mayer-Vietoris boundary map \( K_1(M) \rightarrow K_0(N) \) sends the K-homology class of \( D_M \) to the K-homology class of its restriction to \( N \) and that the Mayer-Vietoris boundary map \( K_1(C^*(\mathbb{R})) \rightarrow K_0(C^*(\{0\})) \) corresponds to the identity map \( \mathbb{Z} \rightarrow \mathbb{Z} \).

One advantage of this approach is that it works even when the partitioning hypersurface is non-compact. This allowed me to prove a higher dimensional counterpart of the partitioned manifold index theorem. To be precise, I constructed a \( k \)-partitioned index map associated to a manifold \( M \) equipped with codimension \( k \) submanifold \( N \) which is the level set of a coarse submersion \( M \rightarrow \mathbb{R}^k \). Assuming that \( D_M \) is an elliptic operator on \( M \) which has a favorable local structure near \( N \), I proved:

- **Main Result 3**: The (equivariant) \( k \)-partitioned index of \( D_M \) agrees with the (equivariant) index of the restriction of \( D_M \) to \( N \).

I proved this by iterating the ordinary partitioned manifold index theorem \( k \) times.

### 3. Positive Scalar Curvature Invariants

One important and nontrivial application of index theory is to positive scalar curvature obstructions in Riemannian geometry. The scalar curvature of a Riemannian manifold \( X \) is a real-valued function on \( X \) which measures the difference between the volume of an infinitesimal metric ball in \( X \) and the volume of the Euclidean ball with the same radius. It is a very “floppy” invariant: in dimension at least three, any function on \( X \) which takes negative values somewhere is the scalar curvature function for some metric. However, Schoen and Yau proved using variational techniques that the \( n \)-torus admits no metric of positive scalar curvature for \( n \leq 7 \). Later Gromov and Lawson used index theory to prove that no torus has a metric of positive scalar curvature; their proof led to a general theory of positive scalar curvature obstructions which is compatible with the analytic surgery exact sequence.

The starting point for Gromov and Lawson’s program is the existence of a special kind of elliptic operator on a Riemannian spin manifold called the spinor Dirac operator. The spin condition can be regarded as a generalized orientability condition, and the details need not concern us here. Lichnerowicz showed that the spinor Dirac operator \( D \) on a Riemannian spin manifold \( X \) satisfies

\[
D^2 = \text{positive operator} + \frac{\kappa}{4}
\]

where \( \kappa \) is the scalar curvature function of \( X \). By spectral theory, this implies that \( D \) is an invertible operator and hence its index vanishes. By the Atiyah-Singer index theorem, the vanishing of the index of \( D \) corresponds to the vanishing of a certain topological invariant of \( X \), so to show that a spin manifold has no metric of positive scalar curvature one need only calculate that invariant and show that it is nonzero. In some cases, such as tori, one must consider the equivariant indeces discussed above, but the basic idea is just the same.

Another way to conceptualize this argument, as explained in [5], is to use the Lichnerowicz formula to construct an explicit lift of the K-homology class of \( D \) in \( K_n(X) \) to the analytic structure set \( K_{n+1}(D^*(X)) \). This lifting procedure actually occurs at the C*-algebra level: one identifies an element of \( Q^*(X) \) whose K-theory class corresponds to the K-homology class of \( D \), lifts this element to \( D^*(X) \), and uses the fact that \( D \) has a gap around 0 in its spectrum to show that the lifted element gives rise to a class in the K-theory of \( D^*(X) \). This class is called the **positive scalar curvature invariant** of the metric on \( X \), and the fact that it maps to the K-homology class of \( D \) automatically implies the vanishing of the index of \( D \). Thus \( D^*(X) \)
provides a context for manipulating and comparing positive scalar curvature invariants in a way that can easily be related to index theory.

The relative analytic surgery exact sequences yield some appealing proofs of a few folklore results about positive scalar curvature.

- **Main Result 3**: If $X$ is a noncompact manifold coarsely equivalent to $\mathbb{R}^n$ and $X$ has a metric whose scalar curvature is positive on a subspace $Y \subseteq X$ which is coarsely equivalent to a half-space then the index of the spinor Dirac operator vanishes.

This can be proven by arguing that the image of the K-homology class of $D$ under the index map $K_0(X) \to K_0(C^*(\mathbb{R}^n))$ lands in the group $K_0(\mathbb{C}^*\mathbb{R}(\mathbb{R}^n-1 \times \mathbb{R}^+ \subseteq \mathbb{R}^n))$; standard calculations in coarse geometry show that $K_0(\mathbb{C}^*\mathbb{R}(\mathbb{R}^n-1 \times \mathbb{R}^+ \subseteq \mathbb{R}^n)) \cong K_0(\mathbb{C}^*\mathbb{R}(\mathbb{R}^n-1 \times \mathbb{R}^+)) = 0$, and thus the index of $D$ must be 0. Clearly the possible generalizations of this result are limited only by one’s ability to compute the K-theory of coarse C*-algebras.

Another folklore result about positive scalar curvature concerns a geometric description of the positive scalar curvature invariant using an analogue of the partitioned manifold index theorem. Given a compact spin manifold $M$ of positive scalar curvature, this description uses the coarse structure of a manifold homeomorphic to $M \times \mathbb{R}$ which is equipped with a metric such that the diameter of $M \times \{t\}$ grows very rapidly as $t \to \pm \infty$.

- **Main Result 4 (in progress)**: The “coarse positive scalar curvature invariant” agrees with the “analytic positive scalar curvature invariant” arising from the Lichnerowicz formula.

This result is sometimes used implicitly in the literature to compare different metrics of positive scalar curvature, but to the best of my knowledge nobody has written up a correct proof. The result would follow from the existence of the commuting square:

$$
\begin{array}{ccc}
K_{p+1}(D^*(M \times \mathbb{R})) & \longrightarrow & K_{p+1}(M \times \mathbb{R}) \\
\downarrow & & \downarrow \\
K_p(D^*(M)) & \longrightarrow & K_p(M)
\end{array}
$$

where the vertical maps are Mayer-Vietoris boundary maps so long as the map $K_{p+1}(D^*(M \times \mathbb{R})) \to K_p(D^*(M))$ sends the positive scalar curvature invariant of $M \times \mathbb{R}$ to the positive scalar curvature invariant of $M$. This is analogous to the statement that the Mayer-Vietoris boundary map $K_{p+1}(M \times \mathbb{R}) \to K_p(M)$ sends the K-homology class of the Dirac operator on $M \times \mathbb{R}$ to the K-homology class of the Dirac operator on $M$, and this is a classical theorem in K-homology theory. It is proved using a powerful tool in K-homology called the Kasparov product together with the suspension map in K-homology; and I have made considerable progress in building comparable tools in the context of the structure algebra.

- **Main Result 5**: Let $M$ be a Riemannian spin$^c$ manifold and let $N$ be a Riemannian spin manifold with positive scalar curvature. There is a product

$$K_p(M) \times K_1(D^*(N)) \to K_1(D^*(M \times N))$$

and the product of the K-homology class of the spin$^c$ Dirac operator on $M$ with the positive scalar curvature invariant on $N$ is the positive scalar curvature invariant on $M \times N$.

I have also defined a suspension map for the K-theory of the structure algebra, and I believe I can prove Main Result 4 if I can show that the product above is compatible with this suspension map.

### 4. Future Directions

The analytic surgery exact sequence and its geometric implications provides fertile territory for future research, and I am interested in several specific programs which I believe are well-suited to my talents.

First, I would like to explore how the tools that I have developed - particularly the product between K-homology and K-theory of the structure algebra - interact with the surgery exact sequence in algebraic topology. The relationship between the structure algebra and the structure set in surgery theory is mediated by the signature operator, and it would be interesting to consider how secondary index invariants associated to the signature operator interact with the Mayer-Vietoris sequence and the product.
Second, a theorem due to Atiyah and Hirzebruch asserts that the index of the spinor Dirac operator vanishes on a spin manifold which admits a circle action (see [2]). I would like to explicitly construct (if possible) an invariant in the K-theory of the structure algebra of such a manifold which explains this result. Atiyah and Hirzebruch’s proof makes clever use of complex analysis, so this project might introduce some interesting tools into index theory and topology.

Third, the partitioned manifold index theorem hints at a possible connection with index theory for manifolds with boundary. In [10] Atiyah, Patodi, and Singer were able to generalize the classical index theorem to manifolds with boundary, and in doing so they introduced an extremely subtle and nontrivial construction known as the \textit{eta invariant}. Higson and Roe showed in [9] that a relative version of the eta invariant can be captured using the analytic surgery exact sequence, and it would be interesting to understand how their construction behaves when applied to a partitioned manifold.

Finally, I would like to understand how the tools of \textit{local index theory} fit with the analytic surgery exact sequence. After Atiyah and Singer published their “global” proof of the index theorem, Atiyah, Bott, and Patodi gave a “local” proof based on the analysis of the heat equation. Modern versions of the proof use a rescaling procedure which, informally, appears to parallel the passage from local geometry to global invariants in the analytic surgery exact sequence. I made some small progress in this direction using some ideas of Connes earlier in my graduate career, and I hope to one day revisit this program.

\textbf{References}