The Analytic Surgery Exact Sequence and Partitioned Manifolds

Paul Siegel

Penn State University

Felix Klein Seminar - Notre Dame - 5/5/11
The Analytic Surgery Exact Sequence and Partitioned Manifolds

Paul Siegel

1. The Atiyah-Singer Index Theorem

2. Operator Algebras and K-theory

3. Coarse Geometry

4. Mayer-Vietoris Sequences
Dirac Operators on Manifolds

Setup: let $D$ be a first order self adjoint differential operator acting on smooth sections of a Hermitian bundle $E$ over a manifold $M$. 
Dirac Operators on Manifolds

Setup: let $D$ be a first order self adjoint differential operator acting on smooth sections of a Hermitian bundle $E$ over a manifold $M$.

$E$ is a Dirac bundle if it is equipped with an appropriate action of $TM$ by endomorphisms. A Dirac bundle comes equipped with a Dirac operator, whose square is a Laplacian.
Dirac Operators on Manifolds

Setup: let $D$ be a first order self adjoint differential operator acting on smooth sections of a Hermitian bundle $E$ over a manifold $M$.

$E$ is a Dirac bundle if it is equipped with an appropriate action of $TM$ by endomorphisms. A Dirac bundle comes equipped with a Dirac operator, whose square is a Laplacian.

More generally we can consider any elliptic operator, i.e. an operator which is invertible up to a finite dimensional error.
Dirac Operators on Manifolds

Setup: let $D$ be a first order self adjoint differential operator acting on smooth sections of a Hermitian bundle $E$ over a manifold $M$.

$E$ is a Dirac bundle if it is equipped with an appropriate action of $TM$ by endomorphisms. A Dirac bundle comes equipped with a Dirac operator, whose square is a Laplacian.

More generally we can consider any elliptic operator, i.e. an operator which is invertible up to a finite dimensional error.

If $E = E^+ \oplus E^-$ is a $\mathbb{Z}/2\mathbb{Z}$ graded vector bundle, then $D$ is graded if it has the form:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$
Theorem (Gelfand)

*If D is a graded Dirac operator on a closed manifold M then $D^+$ and $D^- = (D^+)^*$ have finite dimensional kernel. The Fredholm index:*

$$\text{Index}(D) = \dim \ker(D^+) - \dim \ker(D^-)$$

*is a homotopy invariant of M.*
Index of Dirac Operators

Theorem (Gelfand)

If $D$ is a graded Dirac operator on a closed manifold $M$ then $D^+$ and $D^- = (D^+)^*$ have finite dimensional kernel. The Fredholm index:

$$ \text{Index}(D) = \dim \ker(D^+) - \dim \ker(D^-) $$

is a homotopy invariant of $M$.

Index Problem: Calculate the index of $D$ in terms of geometric data on $M$. 
Example - The De Rham Operator

Let $M$ be a closed, even dimensional Riemannian manifold, and consider the operator:

$$D^+ = d + d^* : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$$
Example - The De Rham Operator

Let $M$ be a closed, even dimensional Riemannian manifold, and consider the operator:

$$D^+ = d + d^* : \Omega^{\text{even}}(M) \rightarrow \Omega^{\text{odd}}(M)$$

The Fredholm index of $D$ is the Euler characteristic of $M$, and it can be calculated via the Chern-Gauss-Bonnet theorem:

$$\chi(M) = \int_M \text{Pf}(R)$$

where $\text{Pf}(R)$ is the Pfaffian of a Riemannian curvature form for $M$. 
Example - The Signature Operator

Let $M$ be a closed Riemannian manifold of dimension $4k$, and denote by $\Omega^{\text{even}'}$ and $\Omega^{\text{odd}'}$ the spaces of differential forms which commute and anti-commute, respectively, with the map $\Omega^p(M) \to \Omega^{n-p}(M)$ given by $-i^p(p-1)\star$, where $\star$ is the Hodge star operator. Consider the operator:

$$D = d + d^* : \Omega^{\text{even}'}(M) \to \Omega^{\text{odd}'}(M)$$
Example - The Signature Operator

Let $M$ be a closed Riemannian manifold of dimension $4k$, and denote by $\Omega^\text{even}'$ and $\Omega^\text{odd}'$ the spaces of differential forms which commute and anti-commute, respectively, with the map $\Omega^p(M) \to \Omega^{n-p}(M)$ given by $-i^p(p-1)\star$, where $\star$ is the Hodge star operator. Consider the operator:

$$D = d + d^* : \Omega^\text{even}'(M) \to \Omega^\text{odd}'(M)$$

The Fredholm index of $D$ is the signature of $M$, and it can be calculated via the Hirzebruch Signature theorem:

$$\text{Sign}(M) = \int_M L(TM)$$

where $L(TM)$ is the Hirzebruch $L$ class.
Example - The Dolbeault Operator

Let $M$ be a Kahler manifold, and consider the Dolbeault complex $\Omega^{(0,\ast)}(M)$ (the anti-holomorphic part of the De Rham complex). Consider the operator:

$$D = \sqrt{2}(\overline{\partial} + \overline{\partial}^*) : \Omega^{(0,\text{even})}(M) \to \Omega^{(0,\text{odd})}(M)$$
Example - The Dolbeault Operator

Let $M$ be a Kahler manifold, and consider the Dolbeault complex $\Omega^{(0,\ast)}(M)$ (the anti-holomorphic part of the De Rham complex). Consider the operator:

$$D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*) : \Omega^{(0,\text{even})}(M) \to \Omega^{(0,\text{odd})}(M)$$

The Fredholm index of $D$ is the Euler characteristic of $M$, and it can be calculated via the Hirzebruch-Riemann-Roch theorem:

$$\chi(M) = \int_M \text{Todd}(TM)$$
The Atiyah-Singer Index Theorem

The Atiyah-Singer index theorem generalizes each of the above examples to an arbitrary Dirac operator $D$ on a Dirac bundle $E$ over a closed manifold $M$. 
The Atiyah-Singer Index Theorem

The Atiyah-Singer index theorem generalizes each of the above examples to an arbitrary Dirac operator $D$ on a Dirac bundle $E$ over a closed manifold $M$.

Key idea: The index of $D$ depends only on the $K$-theory class of $E$. 
The Atiyah-Singer Index Theorem

The Atiyah-Singer index theorem generalizes each of the above examples to an arbitrary Dirac operator $D$ on a Dirac bundle $E$ over a closed manifold $M$.

Key idea: The index of $D$ depends only on the $K$-theory class of $E$. There is a map $K^*(M) \to \mathbb{Z}$ which sends $[E]$ to the index of $D$, and it agrees with another map $K^*(M) \to \mathbb{Z}$ built out of standard constructions in algebraic topology.
The Atiyah-Singer Index Theorem

The Atiyah-Singer index theorem generalizes each of the above examples to an arbitrary Dirac operator $D$ on a Dirac bundle $E$ over a closed manifold $M$.

Key idea: The index of $D$ depends only on the $K$-theory class of $E$. There is a map $K^*(M) \to \mathbb{Z}$ which sends $[E]$ to the index of $D$, and it agrees with another map $K^*(M) \to \mathbb{Z}$ built out of standard constructions in algebraic topology.

What about non-compact $M$?
A Non-Compact Index Problem

Let $M$ be an oriented manifold partitioned by a closed hypersurface $N$, meaning $M = M^+ \cup M^-$ where $\partial M^+ = \partial M^- = N$. 
A Non-Compact Index Problem

Let $M$ be an oriented manifold \textit{partitioned} by a closed hypersurface $N$, meaning $M = M^+ \cup M^-$ where $\partial M^+ = \partial M^- = N$. Example: $M = N \times \mathbb{R}$. 
A Non-Compact Index Problem

Let $M$ be an oriented manifold partitioned by a closed hypersurface $N$, meaning $M = M^+ \cup M^-$ where $\partial M^+ = \partial M^- = N$. Example: $M = N \times \mathbb{R}$.

Assume $M$ is equipped with a Dirac bundle $E_M$ and Dirac operator $D_M$ which restricts to $E_N, D_N$. 

Write $U^+=\phi^+ (D_M-i) (D_M+i) - 1$ where $\phi^+$ is a smooth function which agrees with the characteristic function of $M^+$ outside a compact set.

Fact: $U^+$ has an index which is independent of $\phi^+$.

Theorem (Roe) $\text{Index} (U^+) = \text{Index} (D_N)$.
A Non-Compact Index Problem

Let $M$ be an oriented manifold partitioned by a closed hypersurface $N$, meaning $M = M^+ \cup M^-$ where $\partial M^+ = \partial M^- = N$. Example: $M = N \times \mathbb{R}$.

Assume $M$ is equipped with a Dirac bundle $E_M$ and Dirac operator $D_M$ which restricts to $E_N$, $D_N$.

Write $U_+ = \phi_+(D_M - i)(D_M + i)^{-1}$ where $\phi_+$ is a smooth function which agrees with the characteristic function of $M^+$ outside a compact set.
A Non-Compact Index Problem

Let $M$ be an oriented manifold partitioned by a closed hypersurface $N$, meaning $M = M^+ \cup M^-$ where $\partial M^+ = \partial M^- = N$. Example: $M = N \times \mathbb{R}$.

Assume $M$ is equipped with a Dirac bundle $E_M$ and Dirac operator $D_M$ which restricts to $E_N$, $D_N$.

Write $U_+ = \phi_+(D_M - i)(D_M + i)^{-1}$ where $\phi_+$ is a smooth function which agrees with the characteristic function of $M^+$ outside a compact set. Fact: $U_+$ has an index which is independent of $\phi_+$. 
A Non-Compact Index Problem

Let $M$ be an oriented manifold partitioned by a closed hypersurface $N$, meaning $M = M^+ \cup M^-$ where $\partial M^+ = \partial M^- = N$. Example: $M = N \times \mathbb{R}$.

Assume $M$ is equipped with a Dirac bundle $E_M$ and Dirac operator $D_M$ which restricts to $E_N, D_N$.

Write $U_+ = \phi_+(D_M - i)(D_M + i)^{-1}$ where $\phi_+$ is a smooth function which agrees with the characteristic function of $M^+$ outside a compact set. Fact: $U_+$ has an index which is independent of $\phi_+$.

Theorem (Roe)

$\text{Index}(U_+) = \text{Index}(D_N)$
Let \((N_1, E_1, D_1)\) and \((N_2, E_2, D_2)\) be closed oriented manifolds equipped with Dirac bundles and Dirac operators, and let \((W, E_W, D_W)\) be a cobordism between them (with the right orientation conventions).
Let $(N_1, E_1, D_1)$ and $(N_2, E_2, D_2)$ be closed oriented manifolds equipped with Dirac bundles and Dirac operators, and let $(W, E_W, D_W)$ be a cobordism between them (with the right orientation conventions).

Attach cylindrical ends to $W$ (and extend the Dirac bundle and Dirac operator). The result is a manifold partitioned by both $N_1$ and $N_2$; by the partitioned manifold theorem, $\text{Index}(D_1) = \text{Index}(D_2)$.
A $C^*$-algebra is an object which blends algebra and Hilbert space theory. Basic examples:
A $C^*$-algebra is an object which blends algebra and Hilbert space theory. Basic examples:

- The bounded operators $B(H)$ and the ideal of compact operators $k(H)$.
A \textit{C*-algebra} is an object which blends algebra and Hilbert space theory. Basic examples:

- The bounded operators $B(H)$ and the ideal of compact operators $k(H)$
- Their quotient $Q(H) = B(H)/k(H)$
A $C^*$-algebra is an object which blends algebra and Hilbert space theory. Basic examples:

- The bounded operators $B(H)$ and the ideal of compact operators $k(H)$
- Their quotient $Q(H) = B(H)/k(H)$
- $C_0(X)$ where $X$ is a locally compact Hausdorff space
If $A$ is a C*-algebra, the K-theory group $K_0(A)$ is defined in the usual way...
If $A$ is a C*-algebra, the K-theory group $K_0(A)$ is defined in the usual way...

Higher K-theory: $K_p(A) = K_0(A \otimes C_0(\mathbb{R}^p))$
K-Theory for C*-Algebras

If $A$ is a C*-algebra, the K-theory group $K_0(A)$ is defined in the usual way...

Higher K-theory: $K_p(A) = K_0(A \otimes C_0(\mathbb{R}^p))$

A short exact sequence of C*-algebras

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$
K-Theory for C*-Algebras

If $A$ is a C*-algebra, the K-theory group $K_0(A)$ is defined in the usual way...

Higher K-theory: $K_p(A) = K_0(A \otimes C_0(\mathbb{R}^p))$

A short exact sequence of C*-algebras

$$0 \to I \to A \to B \to 0$$

gives rise to a long exact sequence in K-theory:

$$\to K_p(I) \to K_p(A) \to K_p(B) \to K_{p-1}(I) \to$$
K-Theory for C*-Algebras

If $A$ is a C*-algebra, the K-theory group $K_0(A)$ is defined in the usual way...

Higher K-theory: $K_p(A) = K_0(A \otimes C_0(\mathbb{R}^p))$

A short exact sequence of C*-algebras

$$0 \to I \to A \to B \to 0$$

gives rise to a long exact sequence in K-theory:

$$\to K_p(I) \to K_p(A) \to K_p(B) \to K_{p-1}(I) \to$$

Bott Periodicity: $K_p(A) \cong K_{p+2}(A)$, so the degree $p$ can be taken mod 2.
Fact: a bounded operator on $H$ has an index if and only if its image in $B(H)/k(H)$ is invertible.
K-Theory and the Fredholm Index

Fact: a bounded operator on $H$ has an index if and only if its image in $B(H)/k(H)$ is invertible.

Consider the short exact sequence of C* algebras:

$$0 \to k(H) \to B(H) \to Q(H) \to 0$$
Fact: a bounded operator on $H$ has an index if and only if its image in $B(H)/k(H)$ is invertible.

Consider the short exact sequence of $C^*$ algebras:

$$0 \to k(H) \to B(H) \to Q(H) \to 0$$

Calculation: $K_0(k(H)) \cong \mathbb{Z}$, and the boundary map $K_1(Q(H)) \to K_0(k(H))$ sends the K-theory class of a Fredholm operator to its Fredholm index.
Let $A$ be a C*-algebra represented on a Hilbert space $H$. A bounded operator $T$ on $H$ is \textit{pseudolocal} if it commutes with $A$ modulo compact operators.
Let $A$ be a C*-algebra represented on a Hilbert space $H$. A bounded operator $T$ on $H$ is *pseudolocal* if it commutes with $A$ modulo compact operators.

$T$ is *locally compact* if $aT$ and $Ta$ are compact for every $a \in A$. 
Let $A$ be a C*-algebra represented on a Hilbert space $H$. A bounded operator $T$ on $H$ is *pseudolocal* if it commutes with $A$ modulo compact operators. 

$T$ is *locally compact* if $aT$ and $Ta$ are compact for every $a \in A$. 

(Motivation: pseudodifferential operator theory)
Let $A$ be a C*-algebra represented on a Hilbert space $H$. A bounded operator $T$ on $H$ is pseudolocal if it commutes with $A$ modulo compact operators.

$T$ is locally compact if $aT$ and $Ta$ are compact for every $a \in A$. (Motivation: pseudodifferential operator theory)

**Definition**

- $D(A)$ (the dual algebra) is the C*-algebra of pseudolocal operators.
Let $A$ be a C*-algebra represented on a Hilbert space $H$. A bounded operator $T$ on $H$ is *pseudolocal* if it commutes with $A$ modulo compact operators.

$T$ is *locally compact* if $aT$ and $Ta$ are compact for every $a \in A$. (Motivation: pseudodifferential operator theory)

**Definition**

- $\mathcal{D}(A)$ (the *dual algebra*) is the C*-algebra of pseudolocal operators.
- $\mathcal{C}(A)$ is the C*-algebra of locally compact operators.
Let $A$ be a C*-algebra represented on a Hilbert space $H$. A bounded operator $T$ on $H$ is pseudolocal if it commutes with $A$ modulo compact operators.

$T$ is locally compact if $aT$ and $Ta$ are compact for every $a \in A$. (Motivation: pseudodifferential operator theory)

**Definition**

- $\mathcal{D}(A)$ (the dual algebra) is the C*-algebra of pseudolocal operators.
- $\mathcal{C}(A)$ is the C*-algebra of locally compact operators.
- The K-homology of $A$ is $K^p(A) = K_{1-p}(\mathcal{D}(A)/\mathcal{C}(A))$
K-Homology and Index Theory

Every Dirac operator $D$ on a manifold $M^n$ gives rise to a class $[D]$ in $K_n(M) \equiv K^n(C_0(M))$. 
K-Homology and Index Theory

Every Dirac operator $D$ on a manifold $M^n$ gives rise to a class $[D]$ in $K_n(M) \equiv K^n(C_0(M))$.

There is a well-defined map $K_n(M) \to \mathbb{Z}$ which sends $[D]$ to its index.
Every Dirac operator $D$ on a manifold $M^n$ gives rise to a class $[D]$ in $K_n(M) \equiv K^n(C_0(M))$.

There is a well-defined map $K_n(M) \to \mathbb{Z}$ which sends $[D]$ to its index.

We can construct this map, and generalize it, using large-scale geometry.
A map $f: X \to Y$ between proper metric spaces is *coarse* if it preserves the large-scale structure of $X$. 
The Coarse Category

A map $f : X \rightarrow Y$ between proper metric spaces is *coarse* if it preserves the large-scale structure of $X$.

Examples:

- The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$
- The map $\mathbb{R} \rightarrow \mathbb{Z}$ which sends $[n, n + 1)$ to $n$
The Coarse Category

A map \( f : X \to Y \) between proper metric spaces is *coarse* if it preserves the large-scale structure of \( X \).

Examples:

- The inclusion \( \mathbb{Z} \hookrightarrow \mathbb{R} \)
- The map \( \mathbb{R} \to \mathbb{Z} \) which sends \([n, n + 1)\) to \( n \)

Two coarse maps \( f, g : X \to Y \) are *close* if \( d_Y(f(x), g(x)) < C \) for some global constant \( C \).
The Coarse Category

A map \( f : X \to Y \) between proper metric spaces is **coarse** if it preserves the large-scale structure of \( X \).

Examples:

- The inclusion \( \mathbb{Z} \hookrightarrow \mathbb{R} \)
- The map \( \mathbb{R} \to \mathbb{Z} \) which sends \([n, n+1)\) to \( n \)

Two coarse maps \( f, g : X \to Y \) are **close** if \( d_Y(f(x), g(x)) < C \) for some global constant \( C \).

The **coarse category** is the category whose objects are proper metric spaces and whose morphisms are closeness classes of coarse maps.
The Coarse Category

A map \( f : X \to Y \) between proper metric spaces is \textit{coarse} if it preserves the large-scale structure of \( X \).

Examples:

- The inclusion \( \mathbb{Z} \hookrightarrow \mathbb{R} \)
- The map \( \mathbb{R} \to \mathbb{Z} \) which sends \( [n, n+1) \) to \( n \)

Two coarse maps \( f, g : X \to Y \) are \textit{close} if \( d_Y(f(x), g(x)) < C \) for some global constant \( C \).

The \textit{coarse category} is the category whose objects are proper metric spaces and whose morphisms are closeness classes of coarse maps.

The maps above are inverses as coarse morphisms.
Coarse Algebras

Let $X$ be a proper metric space and suppose $C_0(X)$ is represented on a Hilbert space $H$. 
Coarse Algebras

Let $X$ be a proper metric space and suppose $C_0(X)$ is represented on a Hilbert space $H$.

The *support* of an operator $T \in B(H)$, a subset of $X \times X$, is the (distributional) support of its Schwartz kernel.
Coarse Algebras

Let $X$ be a proper metric space and suppose $C_0(X)$ is represented on a Hilbert space $H$.

The *support* of an operator $T \in B(H)$, a subset of $X \times X$, is the (distributional) support of its Schwartz kernel.

$T$ is *controlled* if its support lies in a bounded neighborhood of the diagonal in $X \times X$. 
Coarse Algebras

Let $X$ be a proper metric space and suppose $C_0(X)$ is represented on a Hilbert space $H$.

The support of an operator $T \in B(H)$, a subset of $X \times X$, is the (distributional) support of its Schwartz kernel.

$T$ is controlled if its support lies in a bounded neighborhood of the diagonal in $X \times X$.

- The structure algebra $D^*(X)$ is the $C^*$-subalgebra of $B(H)$ generated by pseudolocal ($Tf - fT$ is compact) controlled operators.
Let $X$ be a proper metric space and suppose $C_0(X)$ is represented on a Hilbert space $H$.

The support of an operator $T \in B(H)$, a subset of $X \times X$, is the (distributional) support of its Schwartz kernel.

$T$ is controlled if its support lies in a bounded neighborhood of the diagonal in $X \times X$.

- The structure algebra $D^*(X)$ is the C*-subalgebra of $B(H)$ generated by pseudolocal ($Tf - fT$ is compact) controlled operators.
- The coarse algebra for $X$ is the C*-ideal in $D^*(X)$ generated by locally compact ($Tf$ and $fT$ are compact) controlled operators.
The structure algebra and the coarse algebra enjoy the following functoriality properties:
K-theory and Coarse Algebras

The structure algebra and the coarse algebra enjoy the following functoriality properties:

- $K_\ast(C^*(\cdot))$ is a functor from the coarse category to the category of abelian groups
The structure algebra and the coarse algebra enjoy the following functoriality properties:

- $K_\ast(C^*(\cdot))$ is a functor from the coarse category to the category of abelian groups
- $K_\ast(D^*(\cdot))$ is functorial for uniform maps, i.e. continuous coarse maps
K-theory and Coarse Algebras

The structure algebra and the coarse algebra enjoy the following functoriality properties:

- $K_*(C^*(\cdot))$ is a functor from the coarse category to the category of abelian groups
- $K_*(D^*(\cdot))$ is functorial for \textit{uniform maps}, i.e. continuous coarse maps

Theorem

$$K_p(D^*(X)/C^*(X)) \cong K_{1-p}(X)$$
The Index Map

Let $M$ be a manifold. The short exact sequence

$$0 \to C^*(M) \to D^*(M) \to D^*(M)/C^*(M) \to 0$$
Let $M$ be a manifold. The short exact sequence

$$0 \rightarrow C^*(M) \rightarrow D^*(M) \rightarrow D^*(M)/C^*(M) \rightarrow 0$$

gives rise to a long exact sequence in K-theory:

$$
\rightarrow K_p(C^*(M)) \rightarrow K_p(D^*(M)) \rightarrow K_{1-p}(M) \rightarrow K_{p-1}(C^*(M)) \rightarrow
$$
The Index Map

Let $M$ be a manifold. The short exact sequence

$$0 \rightarrow C^*(M) \rightarrow D^*(M) \rightarrow D^*(M)/C^*(M) \rightarrow 0$$

gives rise to a long exact sequence in $K$-theory:

$$\rightarrow K_p(C^*(M)) \rightarrow K_p(D^*(M)) \rightarrow K_{1-p}(M) \rightarrow K_{p-1}(C^*(M)) \rightarrow$$

If $M$ is compact, then

$$K_*(C^*(M)) \cong K_*(C^*(\text{point})) \cong K_*(k(H)).$$
The Index Map

Let $M$ be a manifold. The short exact sequence

$$0 \to C^*(M) \to D^*(M) \to D^*(M)/C^*(M) \to 0$$

gives rise to a long exact sequence in K-theory:

$$\to K_p(C^*(M)) \to K_p(D^*(M)) \to K_{1-p}(M) \to K_{p-1}(C^*(M)) \to$$

If $M$ is compact, then

$$K_*(C^*(M)) \cong K_*(C^*(\text{point})) \cong K_*(k(H)).$$

The boundary map $K_0(M) \to K_0(k(H)) \cong \mathbb{Z}$ is the index map described above (in even dimensions).
Let $M$ be a $n$-manifold with fundamental group $G$ and universal cover $\tilde{M}$. 

**The Analytic Surgery Exact Sequence**

The short exact sequence:

$$0 \to C^* G(\tilde{M}) \to D^* G(\tilde{M}) \to D^* G(\tilde{M}) / C^* G(\tilde{M}) \to 0$$

gives rise to a long exact sequence in K-theory:

$$\cdots \to K^{n+1}(C^* G(\tilde{M})) \to K^{n+1}(D^* G(\tilde{M})) \to K^n(M) \to K^n(C^* G(\tilde{M})) \to \cdots$$

This is the analytic surgery exact sequence.
The Analytic Surgery Exact Sequence

Let $M$ be a $n$-manifold with fundamental group $G$ and universal cover $\tilde{M}$.

$G$ acts on $D^*(\tilde{M})$ and $C^*(\tilde{M})$, and we have

$$K_p(D^*_G(\tilde{M})/C^*_G(\tilde{M})) \cong K_p(M)$$
The Analytic Surgery Exact Sequence

Let $M$ be a $n$-manifold with fundamental group $G$ and universal cover $\tilde{M}$.

$G$ acts on $D^*(\tilde{M})$ and $C^*(\tilde{M})$, and we have

$$K_p(D^*_G(\tilde{M})/C^*_G(\tilde{M})) \cong K_p(M)$$

The short exact sequence:

$$0 \to C^*_G(\tilde{M}) \to D^*_G(\tilde{M}) \to D^*_G(\tilde{M})/C^*_G(\tilde{M}) \to 0$$

gives rise to a long exact sequence in K-theory:
The Analytic Surgery Exact Sequence

Let $M$ be a $n$-manifold with fundamental group $G$ and universal cover $\tilde{M}$.

$G$ acts on $D^*(\tilde{M})$ and $C^*(\tilde{M})$, and we have

$$K_p(D^*_G(\tilde{M})/C^*_G(\tilde{M})) \cong K_p(M)$$

The short exact sequence:

$$0 \to C^*_G(\tilde{M}) \to D^*_G(\tilde{M}) \to D^*_G(\tilde{M})/C^*_G(\tilde{M}) \to 0$$

gives rise to a long exact sequence in $K$-theory:

$$\cdots \to K_{n+1}(C^*_G(\tilde{M})) \to K_{n+1}(D^*_G(\tilde{M})) \to K_n(M) \to K_n(C^*_G(\tilde{M})) \to \cdots$$
The Analytic Surgery Exact Sequence

Let $M$ be a $n$-manifold with fundamental group $G$ and universal cover $\tilde{M}$.

$G$ acts on $D^*(\tilde{M})$ and $C^*(\tilde{M})$, and we have

$$K_p(D^*_G(\tilde{M})/C^*_G(\tilde{M})) \cong K_p(M)$$

The short exact sequence:

$$0 \to C^*_G(\tilde{M}) \to D^*_G(\tilde{M}) \to D^*_G(\tilde{M})/C^*_G(\tilde{M}) \to 0$$

gives rise to a long exact sequence in $K$-theory:

$$\to K_{n+1}(C^*_G(\tilde{M})) \to K_{n+1}(D^*_G(\tilde{M})) \to K_n(M) \to K_n(C^*_G(\tilde{M})) \to$$

This is the analytic surgery exact sequence.
Let $N$ be a closed even dimensional manifold, and consider the partitioned manifold $M = N \times \mathbb{R}$. 
Let $N$ be a closed even dimensional manifold, and consider the partitioned manifold $M = N \times \mathbb{R}$.

We have an index map $K_0(N) \to K_0(C^*(N)) \cong \mathbb{Z}$ which sends $[D_N]$ to its index.
Partitioned Manifolds Revisited

Let $N$ be a closed even dimensional manifold, and consider the partitioned manifold $M = N \times \mathbb{R}$.

We have an index map $K_0(N) \to K_0(C^*(N)) \cong \mathbb{Z}$ which sends $[D_N]$ to its index.

We also have a “coarse index map” $K_1(M) \to K_1(C^*(M))$. 
Let $N$ be a closed even dimensional manifold, and consider the partitioned manifold $M = N \times \mathbb{R}$.

We have an index map $K_0(N) \to K_0(C^*(N)) \cong \mathbb{Z}$ which sends $[D_N]$ to its index.

We also have a “coarse index map” $K_1(M) \to K_1(C^*(M))$.

Computation: $K_1(C^*(M)) \cong K_1(C^*(\mathbb{R})) \cong \mathbb{Z}$, and the coarse index map sends $[D_M]$ to the index of $U_+$ (in the notation above).
To prove the partitioned manifold index theorem, we will construct a commuting diagram:

\[
\begin{array}{ccc}
K_1(M) & \longrightarrow & K_1(C^*(M)) \\
\downarrow & & \downarrow \\
K_0(N) & \longrightarrow & K_0(C^*(N))
\end{array}
\]

such that \( K_1(M) \to K_0(N) \) sends \([D_M]\) to \([D_N]\).
To prove the partitioned manifold index theorem, we will construct a commuting diagram:

\[
\begin{array}{ccc}
K_1(M) & \longrightarrow & K_1(C^*(M)) \\
\downarrow & & \downarrow \\
K_0(N) & \longrightarrow & K_0(C^*(N))
\end{array}
\]

such that \( K_1(M) \rightarrow K_0(N) \) sends \([D_M]\) to \([D_N]\).

The vertical maps will be boundary maps in K-theoretic Mayer-Vietoris sequences.
The Mayer-Vietoris Principle in K-theory

Let $A$ be a C*-algebra, and let $I$ and $J$ be ideals such that $I + J$ is dense in $A$. Then there is a Mayer-Vietoris sequence:

$$
\to K_p(I \cap J) \to K_p(I) \oplus K_p(J) \to K_p(A) \to K_{p-1}(I \cap J) \to
$$
The Mayer-Vietoris Principle in K-theory

Let $A$ be a C*-algebra, and let $I$ and $J$ be ideals such that $I + J$ is dense in $A$. Then there is a Mayer-Vietoris sequence:

$$
\rightarrow K_p(I \cap J) \rightarrow K_p(I) \oplus K_p(J) \rightarrow K_p(A) \rightarrow K_{p-1}(I \cap J) \rightarrow
$$

It is “natural” in the sense that it is induced by a certain short exact sequence of C*-algebras.
Let $A$ be a C*-algebra, and let $I$ and $J$ be ideals such that $I + J$ is dense in $A$. Then there is a Mayer-Vietoris sequence:

$$
\cdots \rightarrow K_p(I \cap J) \rightarrow K_p(I) \oplus K_p(J) \rightarrow K_p(A) \rightarrow K_{p-1}(I \cap J) \rightarrow \cdots
$$

It is “natural” in the sense that it is induced by a certain short exact sequence of C*-algebras.

(Warning: NOT $0 \rightarrow I \cap J \rightarrow A \rightarrow I \oplus J \rightarrow 0$)
Let $A$ be a C*-algebra, and let $I$ and $J$ be ideals such that $I + J$ is dense in $A$. Then there is a Mayer-Vietoris sequence:

$$
\rightarrow K_p(I \cap J) \rightarrow K_p(I) \oplus K_p(J) \rightarrow K_p(A) \rightarrow K_{p-1}(I \cap J) \rightarrow
$$

It is “natural” in the sense that it is induced by a certain short exact sequence of C*-algebras.

(Warning: NOT $0 \rightarrow I \cap J \rightarrow A \rightarrow I \oplus J \rightarrow 0$)

For geometric applications, we need a correspondence between subsets of a metric space and ideals in the associated C*-algebras.
Ideals in K-Homology and Coarse Geometry

Let $X$ be a proper metric space and let $Y \subseteq X$ be a closed subspace.

- $\mathcal{D}(Y \subseteq X)$ is the ideal in $\mathcal{D}(X)$ of pseudolocal operators which are “locally compact relative to $Y$”.

- $\mathcal{C}^*(Y \subseteq X)$ is the ideal in $\mathcal{C}^*(X)$ generated by locally compact controlled operators which are supported in a neighborhood of $Y$.

- $\mathcal{D}^*(Y \subseteq X)$ is the ideal in $\mathcal{D}^*(X)$ generated by pseudolocal controlled operators which are locally compact relative to $Y$ and supported in a neighborhood of $Y$.

Fact: The inclusion $Y \hookrightarrow X$ induces isomorphisms $K^p(F(Y \subseteq X)) \simeq K^p(F(Y))$ for $F = \mathcal{D}$, $\mathcal{C}^*$, and $\mathcal{D}^*$.
Let $X$ be a proper metric space and let $Y \subseteq X$ be a closed subspace.

- $\mathcal{D}(Y \subseteq X)$ is the ideal in $\mathcal{D}(X)$ of pseudolocal operators which are “locally compact relative to $Y$”.
- $C^*(Y \subseteq X)$ is the ideal in $C^*(X)$ generated by locally compact controlled operators which are supported in a neighborhood of $Y$. 
Ideals in K-Homology and Coarse Geometry

Let $X$ be a proper metric space and let $Y \subseteq X$ be a closed subspace.

- $\mathcal{D}(Y \subseteq X)$ is the ideal in $\mathcal{D}(X)$ of pseudolocal operators which are “locally compact relative to $Y$”.
- $\mathcal{C}^*(Y \subseteq X)$ is the ideal in $\mathcal{C}^*(X)$ generated by locally compact controlled operators which are supported in a neighborhood of $Y$.
- $\mathcal{D}^*(Y \subseteq X)$ is the ideal in $\mathcal{D}^*(X)$ generated by pseudolocal controlled operators which are locally compact relative to $Y$ and supported in a neighborhood of $Y$. 
Ideals in K-Homology and Coarse Geometry

Let $X$ be a proper metric space and let $Y \subseteq X$ be a closed subspace.

- $\mathcal{D}(Y \subseteq X)$ is the ideal in $\mathcal{D}(X)$ of pseudolocal operators which are “locally compact relative to $Y$”.
- $\mathcal{C}^*(Y \subseteq X)$ is the ideal in $\mathcal{C}^*(X)$ generated by locally compact controlled operators which are supported in a neighborhood of $Y$.
- $\mathcal{D}^*(Y \subseteq X)$ is the ideal in $\mathcal{D}^*(X)$ generated by pseudolocal controlled operators which are locally compact relative to $Y$ and supported in a neighborhood of $Y$.

Fact: The inclusion $Y \hookrightarrow X$ induces isomorphisms $K_p(F(Y \subseteq X)) \cong K_p(F(Y))$ for $F = \mathcal{D}$, $\mathcal{C}^*$, and $\mathcal{D}^*$.
Excision Conditions

Again, let $F = \mathcal{D}$, $C^*$, and $D^*$. There is a Mayer-Vietoris sequence in the K-theory of $F(X)$ for a decomposition $X = Y_1 \cup Y_2$ if the decomposition is $F$-excisive, meaning:

• $F(X) = F(Y_1 \subseteq X) + F(Y_2 \subseteq X)$
• $F(Y_1 \cap Y_2 \subseteq X) = F(Y_1 \subseteq X) \cap F(Y_2 \subseteq X)$

Here are the appropriate excision conditions:

• $X = Y_1 \cup Y_2$ is $\mathcal{D}$-excisive if $Y_1$ and $Y_2$ are closed.
• $X = Y_1 \cup Y_2$ is $C^*$-excisive if for every $S$ there exists $R$ such that the $S$-neighborhood of $Y_1 \cap Y_2$ contains the intersection of the $R$-neighborhoods for $Y_1$ and $Y_2$.
• $X = Y_1 \cup Y_2$ is $\mathcal{D}^*$-excisive if it is $\mathcal{D}$- and $C^*$-excisive.
Excision Conditions

Again, let $F = \mathcal{D}, C^*$, and $D^*$. There is a Mayer-Vietoris sequence in the K-theory of $F(X)$ for a decomposition $X = Y_1 \cup Y_2$ if the decomposition is $F$-excisive, meaning:

- $F(X) = F(Y_1 \subseteq X) + F(Y_2 \subseteq X)$
- $F(Y_1 \cap Y_2 \subseteq X) = F(Y_1 \subseteq X) \cap F(Y_2 \subseteq X)$

Here are the appropriate excision conditions:
Excision Conditions

Again, let $F = \mathcal{D}$, $C^*$, and $D^*$. There is a Mayer-Vietoris sequence in the K-theory of $F(X)$ for a decomposition $X = Y_1 \cup Y_2$ if the decomposition is $F$-excisive, meaning:

- $F(X) = F(Y_1 \subseteq X) + F(Y_2 \subseteq X)$
- $F(Y_1 \cap Y_2 \subseteq X) = F(Y_1 \subseteq X) \cap F(Y_2 \subseteq X)$

Here are the appropriate excision conditions:

- $X = Y_1 \cup Y_2$ is $\mathcal{D}$-excisive if $Y_1$ and $Y_2$ are closed.
Excision Conditions

Again, let \( F = \mathcal{D}, \ C^*, \text{ and } D^* \). There is a Mayer-Vietoris sequence in the K-theory of \( F(X) \) for a decomposition \( X = Y_1 \cup Y_2 \) if the decomposition is \( F \)-excisive, meaning:

- \( F(X) = F(Y_1 \subseteq X) + F(Y_2 \subseteq X) \)
- \( F(Y_1 \cap Y_2 \subseteq X) = F(Y_1 \subseteq X) \cap F(Y_2 \subseteq X) \)

Here are the appropriate excision conditions:

- \( X = Y_1 \cup Y_2 \) is \( \mathcal{D} \)-excisive if \( Y_1 \) and \( Y_2 \) are closed.
- \( X = Y_1 \cup Y_2 \) is \( C^* \)-excisive if for every \( S \) there exists \( R \) such that the \( S \)-neighborhood of \( Y_1 \cap Y_2 \) contains the intersection of the \( R \)-neighborhoods for \( Y_1 \) and \( Y_2 \).
Excision Conditions

Again, let $F = D$, $C^*$, and $D^*$. There is a Mayer-Vietoris sequence in the K-theory of $F(X)$ for a decomposition $X = Y_1 \cup Y_2$ if the decomposition is $F$-excisive, meaning:

- $F(X) = F(Y_1 \subseteq X) + F(Y_2 \subseteq X)$
- $F(Y_1 \cap Y_2 \subseteq X) = F(Y_1 \subseteq X) \cap F(Y_2 \subseteq X)$

Here are the appropriate excision conditions:

- $X = Y_1 \cup Y_2$ is $D$-excisive if $Y_1$ and $Y_2$ are closed.
- $X = Y_1 \cup Y_2$ is $C^*$-excisive if for every $S$ there exists $R$ such that the $S$-neighborhood of $Y_1 \cap Y_2$ contains the intersection of the $R$-neighborhoods for $Y_1$ and $Y_2$.
- $X = Y_1 \cup Y_2$ is $D^*$-excisive if it is $D$- and $C^*$-excisive.
Mayer-Vietoris for a Partitioned Manifolds

If $M$ is a manifold partitioned by $N$, the metric on $M$ can be chosen so that the decomposition $M = M^+ \cup M^-$ is excisive in each of the above senses.
Mayer-Vietoris for a Partitioned Manifolds

If $M$ is a manifold partitioned by $N$, the metric on $M$ can be chosen so that the decomposition $M = M^+ \cup M^-$ is excisive in each of the above senses.

Hence the “baby surgery exact sequences” (ignoring the fundamental group) for $N$, $M^+$, $M^-$, and $M$ are linked by Mayer-Vietoris sequences.
If $M$ is a manifold partitioned by $N$, the metric on $M$ can be chosen so that the decomposition $M = M^+ \cup M^-$ is excisive in each of the above senses.

Hence the “baby surgery exact sequences” (ignoring the fundamental group) for $N$, $M^+$, $M^-$, and $M$ are linked by Mayer-Vietoris sequences.

In particular, we get the square we wanted:

$$
\begin{array}{ccc}
K_1(M) & \longrightarrow & K_1(C^*(M)) \\
\downarrow & & \downarrow \\
K_0(N) & \longrightarrow & K_0(C^*(N))
\end{array}
$$
Mayer-Vietoris for a Partitioned Manifolds

If $M$ is a manifold partitioned by $N$, the metric on $M$ can be chosen so that the decomposition $M = M^+ \cup M^-$ is excisive in each of the above senses.

Hence the “baby surgery exact sequences” (ignoring the fundamental group) for $N$, $M^+$, $M^-$, and $M$ are linked by Mayer-Vietoris sequences.

In particular, we get the square we wanted:

$$
\begin{array}{c}
K_1(M) \longrightarrow K_1(C^*(M)) \\
\downarrow \quad \downarrow \\
K_0(N) \longrightarrow K_0(C^*(N))
\end{array}
$$

That $[D_M]$ maps to $[D_N]$ follows from the fact that $D_M$ restricts to $D_N$. 
Applications

- The proof generalizes to non-compact $N$
Applications

• The proof generalizes to non-compact $N$
• Also generalizes to “higher indices”, using the full analytic surgery exact sequence
Applications

- The proof generalizes to non-compact $N$
- Also generalizes to “higher indices”, using the full analytic surgery exact sequence
- Positive scalar curvature obstructions
Applications

- The proof generalizes to non-compact $N$
- Also generalizes to “higher indices”, using the full analytic surgery exact sequence
- Positive scalar curvature obstructions
- Secondary index theory?