Taut to Tight: shining light on the bigger picture

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Contact Geometry has been a key tool in the following recent results in low-dimensional topology.

(1) **Kronheimer and Mrowka's** proof that non-trivial knots satisfy property p. (i.e. non-trivial surgery on non-trivial knots yields non-simply connected manifolds.)
(2) **Ozsváth and Szabó's** proof that the unknot, trefoil, and figure eight knot are determined by surgeries on them. (i.e. Let K =unknot, trefoil, or figure eight knot. Let K' be a knot. If $\exists\ r \in \mathbb{Q}$ such that $S^3_r(K) \cong S^3_r(K')$ (orientation preserving), then $K = K'$.)
(3) **Ozsváth and Szabó's** proof that Heegaard-Floer invariants detect the Thurston norm of a manifold and the seifert genus of a knot.

The following are some of the key ideas that went into these results. (All definitions will be given later.)

Start with a closed 3-manifold M and a surface $\Sigma \subset M$ such that Σ is minimal genus among surfaces homologous to it (assume the genus of Σ is positive).

(1) **Gabai** gives a taut foliation $\mathcal{F}$ that contains Σ as a leaf.
(2) **Eliashberg - Thurston** give a positive and negative contact structure $\xi_\pm$ on M that is C^0 close to $\mathcal{F}$.
(3) They also give a symplectic structure on $M \times [-\epsilon, \epsilon]$ that "dominates" (or "fills") $(M, \xi_+) \coprod (M, \xi_-)$.

(4) **Eliashberg, Etnyre** find a closed symplectic manifold X that $M \times [-\epsilon, \epsilon]$ embeds into

these "caps" are constructed using

(a) **Giroux's** correspondence between open books and contact structures.

(b) **Eliashberg, Weinstein's** ideas on contact surgery and symplectic handle attachment.

(5) Uses Seiberg-Witten or Heegaard-Floer to conclude something about M and Σ based on the existence of X. (e.g. HF invariant of $X \neq 0 \quad \Rightarrow$ $HF^+(M, \mathfrak{s}_{\mathcal{F}}) \neq 0 \quad \Rightarrow \quad |\langle c_1(\mathfrak{s}_{\mathcal{F}}), [\Sigma] \rangle| \leq 2g-2$; but since Σ is a leaf of $\mathcal{F}$ we have $|\langle c_1(\mathfrak{s}_{\mathcal{F}}), [\Sigma] \rangle| = 2g-2$; thus the Heegaard-Floer basic classes detect the Thurston norm).

Then contact geometry part of the above is steps (2) $\rightarrow$ (4). The goal of this talk is to explain the ideas behind (2), and perhaps (3).

Basic Ideas and Definitions

Let M be an oriented 3-manifold.

A **plane field** ξ on M can (locally) be given as the kernel of a 1-form α.

$$\xi_x = \ker(\alpha_x), \quad x \in M$$

Examples:

(1) $\mathbb{R}^3$, $\xi_1 = \ker(\alpha_1)$, $\alpha_1 = dz$

(2) $\mathbb{R}^3$, $\xi_2 = \ker(\alpha_2)$, $\alpha_2 = dz - y\,dx$, or
$\mathbb{R}^3$, $\xi_3 = \ker(\alpha_3)$, $\alpha_3 = dz + y\,dx$

Definition

(1) ξ is a **foliation** if $\alpha \wedge d\alpha \equiv 0$.
(2) ξ is a positive (resp. negative) **contact structure** if $\alpha \wedge d\alpha$ is never zero and induces the given (resp. opposite) orientation on M (i.e. $\alpha \wedge d\alpha > 0$ (resp. $\alpha \wedge d\alpha < 0$)).
(3) ξ is a positive (resp. negative) **confoliation** if $\alpha \wedge d\alpha \geq 0$ (resp. $\alpha \wedge d\alpha \leq 0$).

Frobenius Theorem "If a plane filed ξ is closed under Lie bracket, then $M = \coprod S_\lambda$ (S_λ a surface with $\xi_x = T_x S_\lambda$)."

Examples:

(1) $\mathbb{R}^3$, $\xi_1 = \ker(\alpha_1)$, $\alpha_1 = dz$. In this case, $d(dz) = 0$, so α_1 is a foliation. Letting $S_{z_0} = \{(x, y, z_0)\}$, we have that $T_(x, y, z_0) S_{z_o} = \xi_{(x,y,z_0)}$.

(2) Let $M = S^1 \times \Sigma_g$, and let $\xi = \ker(d\theta)$. Then $d(d\theta) = 0$, and $\xi_{(\theta,p)} = T_p(\{\theta\} \times \Sigma_g)$.
(3) ξ_2 is a positive contact structure on $\mathbb{R}^3$ since $\alpha_2 \wedge d\alpha_2 = dx \wedge dy \wedge dz$. Similarly, ξ_3 is a negative contact structure on $\mathbb{R}^3$ since $\alpha_3 \wedge d\alpha_3 = -dx \wedge dy \wedge dz$.
(4) $S^3 \subset \mathbb{C}^2$ with $\xi_4 = \ker(\alpha_4)$, where $\alpha_4 = r_1^2\,d\theta_1 + r_2^2\,d\theta_2$.

One natural question to ask at this point is the following. "How prevalent are contact structure and foliations?"

Answer: All oriented 3-manifolds have foliations and positive (negative) contact structures.

Lemma Given a plane field ξ one can find local coordinates (x, y, z) such that α can be written

$$\alpha = dz - a(x, y, z)\, dx$$

Lemma

(1) ξ is a positive (resp. negative) contact structure if and only if $\partial a/\partial y > 0$ (resp. $\partial a/\partial y < 0$).
(2) ξ is a foliation if and only if $\partial a/\partial y \equiv 0$

Theorem (Darboux, Pfaff)

(1) If ξ is a foliation, then we can take $\alpha = dz$.
(2) If ξ is a positive (resp. negative) contact structure, then we can take $\alpha = dz - y\, dx$ (resp. $\alpha = dz + y\, dx$).

This tells us that, locally all foliations and positive/negative contact structures "look the same". So unlike in Riemannian geometry, nothing interesting happens locally. Instead, contact structures and foliations give us global information about our manifold.

The above theorem is one example of the many similarities between contact structures and foliations. There is, however, one very big difference. Foliations have non-trivial deformations, whereas contact structure do not.

Theorem (Gray's Theorem) Let ξ_t, $t \in [0, 1]$ be a family of contact structure on M^3, then there exists a family of diffeomorphisms $\psi_t : M \to M$ such that $(\psi_t)_*(\xi_t) = \xi_0$.

Example:
Let $\mathcal{F}_s$ = the foliation of T^2 by lines of slope s, and let $\xi_s = \mathcal{F}_s \times S^1$ be a foliation of T^3.

To see more similarities between contact structures and foliations, we need a few more definitions and examples.

Examples:

(1) The **Reeb foliation** of $S^1 \times D^2$ is the following.

(2) A **Lutz tube** is a contact structure on $S^1 \times D^2$ obtained as follows. Let $S^1 \times D^2 = \{(r, \theta, z) \mid \; r \leq \pi\} \subset \mathbb{R}^3$, and let $\xi_{ot} = \ker(\cos(r)\, dz - r\sin(r)\, d\theta)$.

If D is a miridional disk, then the singular foliation induced on D is.

This is called an **overtwisted disk**.

It is easy to construct foliations with Reeb components and contact structures with Lutz tubes, but it is harder to construct foliations and contact structures without them.

A contact structure without overtwisted disks (or equivalently without Lutz tubes) is called **tight**. Otherwise it is called **overtwisted**.

Recall an oriented 2-dimensional bundle (like ξ) has an Euler class $e \in \mathrm{H}^2(M\,;\mathbb{Z})$.

Theorem

(1) (Thurston) If ξ is a Reebless foliation, then for any surface Σ embedded in M,

$$\begin{aligned} |\langle e(\xi), [\Sigma]\rangle| &\leq -\chi(\Sigma) \quad \text{if } \Sigma \neq S^2 \\ &= 0 \qquad\quad\ \ \text{otherwise} \end{aligned}$$

(2) (Eliashberg) If ξ is a tight contact structure, then the same inequality/equality holds.

This theorem and a result of Gabai (see below) imply that the Euler classes of Reebless foliations characterize the Thurston norm. Then Eliashberg-Thurston tell us that the same is true for tight contact structures.

How do we find tight contact structure?

Recall that a 4-manifold X is **symplectic** if there exists a 2-form ω such that $d\omega = 0$, and $\omega \wedge \omega$ is never zero.

If $M = \partial X$, and ξ is a contact structure on M, then we say ω **dominates** ξ if $\omega|_\xi > 0$.

If (M, ξ) is one component of a contact manifold (Y, ξ'), and (X, ω) is a compact symplectic 4-manifold such that ω dominates ξ', then we say (X, ω) is a **week semi-filling** of (M, ξ) (if Y is connected, then we call this a **weak-filling**).

Theorem (Eliashberg-Gromov) If (M, ξ) is weakly (semi-)fillable, then ξ is tight.

Example:
Consider $S^3 \subset \mathbb{C}^4$, with $\xi = \ker(\alpha)$, where $\alpha = r_1^2\, d\theta_1 + r_2^2\, d\theta_2$. Then we have that $\omega = d\alpha = 2r_1\, dr_1 \wedge d\theta_1 + 2r_2\, dr_2 \wedge d\theta_2$ is a symplectic form on $\mathbb{C}^2$ (and therefore also on B^4).

Now $S^3 = \partial B^4$, and $\omega|_\xi = d\alpha|_\xi > 0$ (since $\alpha \wedge d\alpha > 0$). This tells us that (S^3, ξ) is tight (i.e. has no Lutz tubes).

The above is in stark contrast with a theorem of Novikov, showing that any foliation of S^3 has Reeb components.

(Con-)Foliations Into Contact Structures

Consider the interesting foliation of $S^2 \times S^1$.

$$\zeta_{(p,\theta)} = T_p(S^2 \times \{\theta\})$$

Theorem (Eliashberg-Thurston) Any oriented C^2-foliation ξ on an oriented 3-manifold M, other than the above foliation of $S^2 \times S^1$, may be C^0-approximated by positive and negative contact structures.

We say ξ can be $\mathbb{C}^k$**-deformed** into a contact structure if there is a C^k-family ξ_t such that ξ_t is contact for $t > 0$ and $\xi_0 = \xi$.

ξ is C^k**-approximated** by a contact structure if in any C^k-neighborhood of ξ, there is a contact structure (need not be a deformation).

Example: On T^3 consider $\alpha_n^t = dz + t\,(\cos(2\pi nz)\ dx + \sin(2\pi nz)\ dy)$. At $t = 0$, we get a foliation of T^3 by T^2's. For $t > 0$, we get contact structure ξ_n^t.

Note that Gray's theorem tells us that ξ_n^t is independent of t, so we denote it ξ_n.

Kanda, Giroux tell us that ξ_n are all distinct and give all the tight contact structures on T^3.

Remarks:

(1) A foliation can be approximated by (deformed into) infinitely many different contact structures!
(2) The above theorem implies only that a foliation can be approximated by contact structure, not deformed (though it still may be true for deformations).
(3) We loose smoothness on C^0 (C^2 is still maybe true).

Why is ξ on $S^2 \times S^1$ special?

Reeb Stability (for confoliations) Suppose a confoliation ξ on M admits an embedded integral 2-sphere S (i.e. $\forall x \in S,\ \ T_xS = \xi_x$). Then (M, ξ) is diffeomorphic to $(S^2 \times S^1, \zeta)$.

Similarly,

Theorem Any confoliation of $S^2 \times S^1$, C^0-close to the foliation ζ is a foliation diffeomorphic to ζ. (Here we just need that the S^1 factor is transverse to the plane field.)

Not only do these theorems explain the unique nature of ζ on $S^2 \times S^1$, they also allow us to see you can't "locally" perturb a foliation into a contact structure.

The proof of the Eliashberg-Thurston theorem involves two steps. Given a foliation on M.

(1) Perturb ξ into a confoliation ξ' such that ξ' is contact on a "sufficiently large" portion of M.
(2) Perturb ξ' into a contact structure.

Let's think about step 2 first, so we can figure out what "sufficiently large" in step 1 means.

Given a confoliation ξ' let

$$H(\xi') = \{x \in M \mid \ \xi_x \text{ is contact at } x \ \text{ (i.e. } (\alpha \wedge d\alpha)_x > 0)\}.$$

This is called the **hot region** (or **contact region**). Now let

$$G(\xi') = \{x \in M \mid \ \exists \text{ a path } \gamma \text{ from } x \text{ to } y \text{ with } y \in H(\xi') \text{ and } \gamma \text{ tangent to } \xi'\}.$$

Theorem If $G(\xi') = M$ then ξ' can be C^∞-deformed into a contact structure.

To prove this, we first observe we have a neighborhood V,

and in V $\xi' = \ker(\alpha)$, $\alpha = dz - a(x, y, z)dx$. Suppose ξ' is contact near $y = 1$. Then we will show there is a C^k-small deformation of ξ' in V to a contact structure that agrees with ξ' along ∂V.

Since ξ' is a confoliation, we know that

$$\begin{aligned} \frac{\partial a}{\partial y} &\geq 0 \quad \text{in } V \text{ and} \\ \frac{\partial a}{\partial y} &> 0 \quad \text{near } y = 1, \text{ since } \xi' \text{ is contact here.} \end{aligned}$$

So for fixed x_0 and z_0, $a(x_0, y, z_0)$ is of the form

We can clearly replace this with a new a whose graph looks like

In particular, $\partial a/\partial y(x_0, y, z_0) > 0$ for this new a.

Exercise 1: Show you can do this for all x, z simultaneously.

Now pick arcs $\gamma_1, \ldots, \gamma_n$ such that γ_i is tangent to ξ', each γ_i has a neighborhood V_i as above, and the V_i's cover $M - H(\xi')$.

We can fix ξ' on one V_i at a time. We needs to be cautious of the fact that as you change ξ' on V_i, we might mess up the modles on the other V_j's.

Exercise 2: Convince yourself that if the perturbation on V_i is sufficiently small, then you can slightly modify the V_j's so that they still have the appropriate form.

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Now we know that in step 1, we need to perturb ξ to a confoliation ξ' such that $G(\xi') = M$ (i.e. every point in M is connected to a contact region by a path tangent to ξ'.)

For this we need to consider the **holonomy** of the foliation ξ.

Let γ be a closed curve in M tangent to ξ. Let $A = (-\epsilon, \epsilon) \times S^1$ be an embedded annulus in M such that

(1) $\{0\} \times S^1 = \gamma$
(2) A is transverse to ξ

ξ induces a **line field** on A.

Considering this foliation of A, pick $p \in S^1$ and set $I = (-\epsilon, \epsilon) \times \{p\}$.

When defining a return map

$$\phi_\gamma : I \to I,$$

we note that ϕ_γ might not be defined on all of I, but will be defined in a neighborhood of 0 (since $\phi_\gamma(0) = 0$).

ϕ_γ is called the **holonomy** of ξ at p along γ.

Note: ϕ_γ only depends on the homotopy class of γ (through curves tangent to ξ).
Definition
Holonomy is called

non-trivial if $\phi_\gamma \neq \mathrm{id_I}$
non-trivial linear holonomy if $\phi'_\gamma(0) \neq 1$
attracting if $|\phi_\gamma(x)| < |x|$
repelling if $|\phi_\gamma(x)| > |x|$
weakly attracting if $|\phi_\gamma(x)| < |x|$ on intervals arbitrarily close to 0.
weakly repelling if $|\phi_\gamma(x)| > |x|$ on intervals arbitrarily close to 0.

Recall that we're trying to prove the Eliashberg-Thurston theorem. To finish things off, we need to show that ξ can be perturbed into a confoliation ξ' such that every point in M can be connected to a point where ξ' is a contact structure. To prove this, we use holonomy?

Theorem (M, ξ) a C^k-foliation

(1) If Γ is a curve in a leaf with non-trivial linear holonomy, then ξ can be C^k-deformed into a positive (negative) contact structure in a neighborhood U of Γ, leaving ξ fixed outside a larger neighborhood.
(2) If Γ has weakly attracting holonomy, then ξ can be C^0-approximated by a positive (negative) contact structure in a neighborhood V of Γ, leaving ξ fixed outside a large neighborhood.

proof
(1) Let $U = \Gamma \times [-1, 1] \times [-1, 1]$ be a neiborhood such that $\xi = \ker(\alpha)$, where $\alpha = dz - a(x, z)dx$.

Claim: We can choose coordinates (x, y, z) such that

$$\frac{\partial a}{\partial z} \geq c \quad \text{for some} \quad c > 0.$$

Consider the vertical annuli $A_{y_0} = \{(x, y, z) \mid y = y_0\}$.

In the region R between A_{-1} and $A_{-1/2}$ apply a diffeomorphism that is the identity on $\partial U \cap R$ and so that on $A_{-1/2}$ all the tangents to the foliation are rotated clockwise from where they were. For example $A_{-1/2}$ should look something like the following after the diffeomorphism.

Now as y goes from $A_{-1/2}$ to A_1 rotate these tangents back to where they were. This turns the region between $A_{-1/2}$ and A_1 into a contact region.

The moral of the story is that holonomy is good. If we can arrange that every leaf in a foliation is arbitrarily close to a leaf with holonomy, then we would be done.

With that in mind, let ξ be a foliation on M, and define a **minimal set** in M to be a non-empty closed union of leaves that contains no such smaller set. (i.e. a closed union of leaves that is the closure of any leaf in it.)

Fact Any leaf limits to a minimal set.

Thus we just need to see we can perturb ξ so that every minimal set has curves with holonomy.

Theorem In a C^2-foliation on a 3-manifold M, every minimal set is one of the following.

(1) All of M. (In this case the foliation is called **minimal**.)
(2) A closed, compact leaf.
(3) An **exceptional minimal set**. (There are only finitely many of these.)

Theorem(Stacksteder) An exceptional minimal set contains leaves with linear holonomy.

We now consider the two remaining possibilities. If ξ is minimal, then either it has has holonomy or it doesn't. If ξ has holonomy, then a theorem of Ghys tells us that it has linear holonomy, and we're done. If it has no holonomy, then a theorem of Tishler implies that ξ can be C^0-approximated by a fibration over a circle (this is one of the places "C^0" and "approximate" enter the picture).

Now perturb the picture as shown below.

This perturbation gives us linear holonomy, finishing the minimal case.

In the case where the minimal set is a closed leaf Σ, we first perturb our foliation ξ to get a new foliation with only finitely many closed leaves. Now we have three basic cases (this three case list is actually not exhaustive, but is "essentially" complete). In the first case, our leaf has linear or weakly attracting holonomy, and we're done.

In the second case, our closed leaf has only trivial holonomy. If this happens, we can apply a different form of Reeb stability to conclude that our foliation is $\Sigma \times (-1, 1)$ in a neighborhood of σ. This is a contradiction since ξ has only finitely man closed leafs.

The final case to consider is where ξ has holonomy, but it is neither weakly attracting, nor weakly repelling. This tells us that our holonomy is weakly attracting on one side, and weakly repelling on the other. If this is the case, we cut out Σ, and replace it as follows.

Taut foliations and fillability

Given a foliation ξ on M we can perturb it into a contact structure ξ', but what can we say about ξ'? Is ξ' tight, fillable, overtwisted,...?

Here I should mention that if ξ is Reebless, then ξ' will be tight. However, in our case, we're looking for something slightly stronger than just being tight.

Definition A foliation is called **taut** if each leaf is intersected by a closed transverse curve.

An equivalent definition of taut is that there exists a vector field v transverse to ξ that preserves some volume for Ω on M.

Note: If ξ has a Reeb component, then ξ is not taut.

This tells us that if a foliation is taut, then it is also Reebless.

Theorem (Eliashberg-Thurston) Suppose that ξ' is a contact structure that is C^0-close to ξ. Then if ξ is taut, ξ' is symplectically fillable.

Proof
Let $X = M \times [-1, 1]$, and let $\xi = \ker(\alpha)$. Let ξ_+, ξ_- be positive, negative contact structures C^0 close to ξ. Set $\widetilde{\omega} = \iota_v \Omega$ (from above), and note that

(1) $\widetilde{\omega}|_\xi > 0$ (If $\widetilde{\omega}|_\xi < 0$, then reverse v.)
(2) $d\widetilde{\omega} = d\iota_v\Omega = d\iota_v\Omega + \iota_v d\Omega = \mathfrak{L}_v\Omega = 0$

Set $\omega = \widetilde{\omega} + \epsilon\, d(t\alpha) = \widetilde{\omega} + \epsilon\, dt \wedge \alpha$. Then $d\omega = 0$, and $\omega \wedge \omega = 2\epsilon\, \widetilde{\omega} \wedge dt \wedge \alpha$.

This tells us that ω is a symplectic form for X, and that $\omega|_{\xi \times \{\pm 1\}} > 0$. Thus, since ξ' is C^0-close to ξ, we have that $\omega|_{\xi' \times \{\pm 1\}} > 0$. Similarly $\omega|_{\xi_\pm} > 0$. Therefore ω fills $(M, \xi_+) \coprod (-M, \xi_-)$, (note ξ_- is a positive contact structure on $-M$).

We can now construct lots of tight contact structures using the following theorem.

Theorem (Gabai) Let M be an irreducible 3-manifold, and let $F \subset M$ be an orientable surface representing a non-trivial homology class. Suppose F has minimal genus among all representatives of this class. Then there exists a taut foliation ξ on M with F as a leaf.

Corollary Let M and F be as above. Then there is a fillable contact structure ξ' on M such that $\langle e(\xi'), [\Sigma] \rangle = \pm(2 - 2g(\Sigma))$.

Actually we need to argue a little more to get this corollary. The ξ from Gabai's theorem is only C^2 if the genus of Σ is ≥ 2. If the genus of Σ is equal to one, then the only non-C^2-part of the foliation is along Σ, and this will have linear holonomy. So the above proof still works.