KHOVANOV HOMOLOGY AND THE SLICE GENUS

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1. Why do we care?

- Khovanov homology has proven to be a very useful tool in the classification of knot types.
- It's strictly stronger than the Jones polynomial, and has the Jones polynomial as its graded Euler characteristic.
- It's computationally tractable. You can write a computer program to effectively compute this invariant (see Bar-Naton's website for a mathematica package).
- It's been used to prove some important results (Milnor conjecture, Lenny Ng's-bound on TB).

2. Overview

We have several goals for these lectures.

- Define two versions Khovanov homology (Khovanov and Lee), and discuss how to compute them.
- Define Rasmussen's s-invariant, discuss some of its properties.
- Use the s-invariant to prove the Milnor conjecture.

3. Definitions

Given a link $L \subset S^3$, we want to compute its Khovanov homology. We do this using the following scheme.

$$L \longrightarrow D \longrightarrow C_D \xrightarrow{\mathcal{A}} CKh(D) \longrightarrow HKh(D) = HKh(L)$$

(1) Choose a planar diagram D for your link.
(2) Construct the "cube of resolutions" C_D using a scheme to be discussed shortly.
(3) Apply a functor $\mathcal{A}$ (which happens to be a $(1+1)$-dimensional TQFT) to this cube to obtain the chain complex $CKh(D)$.
(4) Take the (co)homology of this complex to get $HKh(L)$.
(5) Rejoice...have some beer...Yay beer!

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Step (1) is easy, we're all capable of doing that. For step (2), we must construct the "cube" of all possible resolutions of our link.

For any given crossing there are two possible ways of resolving it.

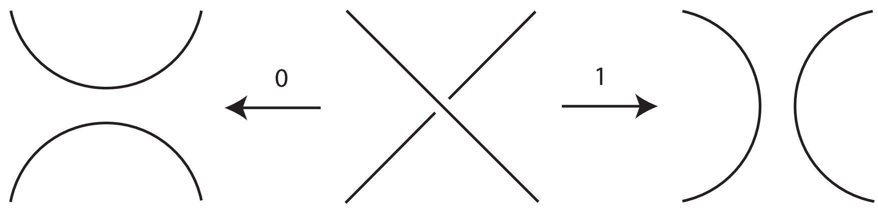
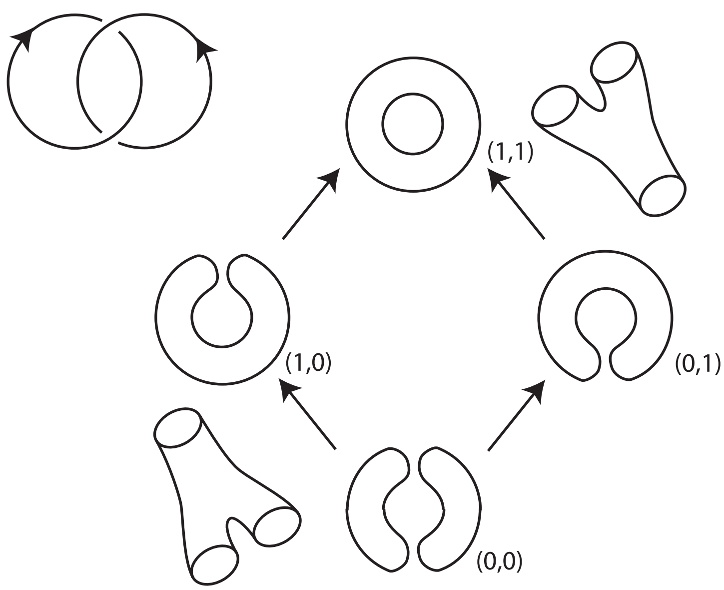


FIGURE 1. The 0- and 1-resolutions of a crossing.

This is an n-dimensional cube $[0,1]^n$, where n is the number of crossings of D. To each vertex v we associate the corresponding 1-manifold gotten by resolving the i^{th}-crossing of D according to the i^{th}-coordinate of v. To each edge we assign a canonical cobordism which is the product, except at a crossing change, where we get saddle cobordism.

Example 3.1. The Hopf link.



Remark 3.1. Notice that the coordinates of two vertices, which share a common edge, can differ in at most one component. This means that exactly one component of each edge cobordism is a saddle cobordism while the rest are products.

As we've constructed it, our "cube" is really just a diagram in Cob − 2, the category of 2-dimensional cobordisms between 1-manifolds.

This brings us to step (3). Let's go ahead and list off the relevant properties of $\mathcal{A}$ so that we can compute $CKh(D)$.

- $\mathcal{A}$ is a functor from Cob − 2 to the category of graded vector spaces over $\mathbb{Q}$.
- $\mathcal{A}$ takes disjoint unions of circles to the tensor product of vector spaces. In our case,

  $$\mathcal{A}\left(\coprod_n S^1\right) = V^{\otimes n}$$

  where $V = \langle 1, x \rangle$.
- The maps induced by the edge cobordisms are defined as follows (recall that each edge cobordism either merges two circles or splits one circle into two).

  $$\begin{aligned} m(1 \otimes 1) &= 1 \\ m(1 \otimes x) &= m(x \otimes 1) = x \\ m(x \otimes x) &= 0 \\ \Delta(1) &= 1 \otimes x + x \otimes 1 \\ \Delta(x) &= x \otimes x \end{aligned}$$

  For what follows, we also need to know how $\mathcal{A}$ behaves under 0- and 2-handle attachments. To this end, we define maps $\iota : \mathbb{Q} \to V$ and $\epsilon : V \to \mathbb{Q}$ respectively.

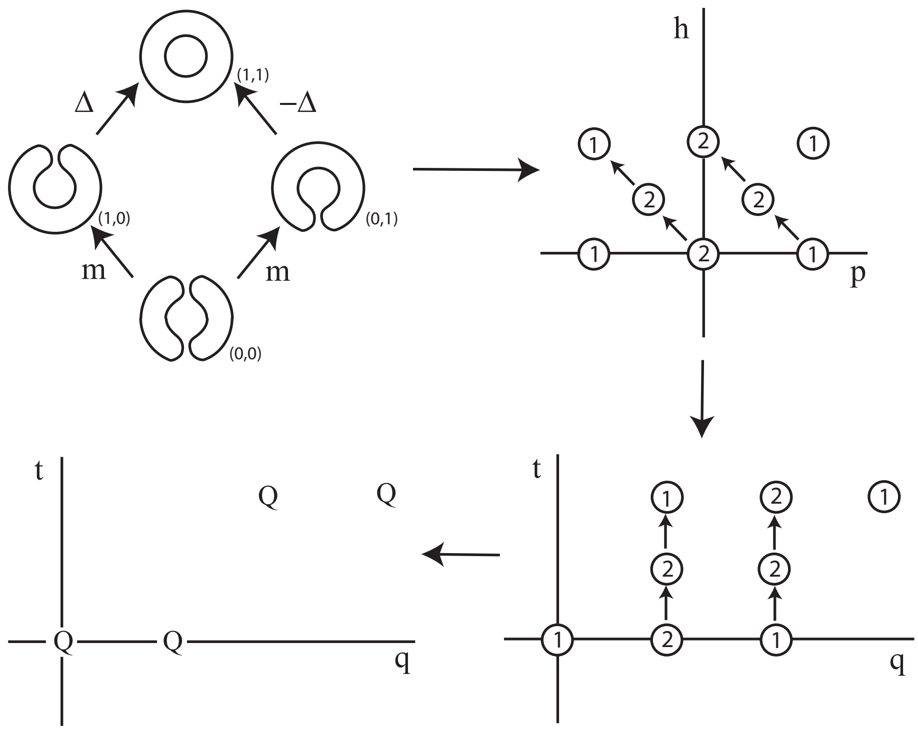
  $$\begin{aligned} \iota(1) &= 1 \\ \epsilon(x) &= 1 \\ \epsilon(1) &= 0 \end{aligned}$$

  - Notice that $m(m(v \otimes w) \otimes z) = m(v \otimes m(w \otimes z))$. Properties/Relations like these reflect the fact that $\mathcal{A}$ is a functor.
- Gradings and the Chain Complex
  - The chain complex $CKh(D)$ is the vector space spanned by all the "states" of D (i.e. all the possible states of all the possible resolutions of D).
  - We define $|\mathbf{v}|$ to be the "height" $\mathbf{v}$.
  - We define a p-grading on V by declaring that "1" has grading $(+1)$, and that "x" has grading (-1). This extends naturally to tensor powers of V by the formula

    $$p(v_1 \otimes \cdots \otimes v_n) = p(v_1) + \cdots + p(v_n).$$

– The differential d is then defined by summing across the various heights, and decorating the m's and Δ's with plus and minus signs so that $d^2 = 0$.

Example 3.2. Consider $\mathcal{A}$ applied to the cube we constructed for the Hopf link.



Remark 3.2. Notice that $\mathcal{A}$ does **NOT** play well with the p-grading. That is, for homogeneous elements, the edge maps reduce the p-grading by 1. To fix this, we (re)define our gradings.

$$\begin{aligned} q(\mathbf{v}) &= p(\mathbf{v}) + |\mathbf{v}| + n_+ - 2n_- \\ t(\mathbf{v}) &= |\mathbf{v}| - n_- \end{aligned}$$

Where the $n_\pm$ (the # pos/neg crossings of D) terms in the t- and q-gradings give us the $(-1)^{n_+}$ and the $q^{n_+ - 2n_-}$ terms in $\hat{J}$ respectively.

To complete the process simply take the (co)homology of $HKh(L)$ of this complex. This is an invariant of our Link, and its graded euler characteristic is $\hat{J}$, the unnormalized Jones polynomial.

4. Lee's TQFT

Lee considers the the same situation as Khovanov, but defines a slightly different maps in her TQFT.

$$\begin{aligned} m'(1 \otimes 1) &= m'(x \otimes x) = 1 \\ m'(1 \otimes x) &= m'(x \otimes 1) = x \\ \Delta'(1) &= 1 \otimes x + x \otimes 1 \\ \Delta'(x) &= 1 \otimes 1 + x \otimes x \end{aligned}$$

The maps ϵ and ι remain unchanged.

$$\begin{aligned} \iota(1) &= 1 \\ \epsilon(x) &= 1 \\ \epsilon(1) &= 0 \end{aligned}$$

Remark 4.1. It might seem like the situation has gotten much worse (notice that these maps no longer respect the q-grading), however hope is not totally lost. Since $q(d(\mathbf{v})) \geq \mathbf{v}$, we now get a filtered complex

$$C^{\geq r,s} = \{\mathbf{x} \in CKh'(D) \mid q(\mathbf{x}) \geq r, \;\; t(\mathbf{x}) = s\}.$$

Furthermore, after making the basis change

$$\begin{aligned} \mathbf{a} &= 1 + x \\ \mathbf{b} &= -1 + x \end{aligned}$$

the maps m' and Δ' become particularly simple.

$$\begin{aligned} m'(\mathbf{a} \otimes \mathbf{a}) &= 2\mathbf{a} \\ m'(\mathbf{a} \otimes \mathbf{b}) &= m'(\mathbf{b} \otimes \mathbf{a}) = 0 \\ m'(\mathbf{b} \otimes \mathbf{b}) &= -2\mathbf{b} \\ \Delta'(\mathbf{a}) &= \mathbf{a} \otimes \mathbf{a} \\ \Delta'(\mathbf{b}) &= \mathbf{b} \otimes \mathbf{b} \\ \epsilon(\mathbf{a}) &= 1 \\ \epsilon(\mathbf{b}) &= 1 \\ \iota(1) &= (\mathbf{a} - \mathbf{b})/2 \end{aligned}$$

Lee used this new complex to prove the following theorem.

Theorem 4.1 (Lee, '02)**.** *$HKh'(L)$ has rank 2^n, where n is the number of components of L.*

Lee proved this theorem by constructing an explicit isomorphism between the set of orientations of L, and a set of *canonical generators* for $HKh'(L)$.

5. Knots and the s-invariant

By Lee's theorem we know that, for a knot K,

$$HKh'(K) = \mathbb{Q} \oplus \mathbb{Q}.$$

Furthermore, sine $CKh'(K)$ was a filtered complex, we get an induced filtration on its homology $HKh'(K)$. From this induced filtration, we then get an induced grading s on $HKh'(K)$.

Definition 5.1.

$$\begin{aligned} s_{max}(K) &= \max\{s(\mathbf{x}) \mid x \in HKh'(K)\} \\ s_{min}(K) &= \min\{s(\mathbf{x}) \mid x \in HKh'(K)\} \end{aligned}$$

Lemma 5.1 (Rasmussen)**.** *$s_{min}(K) + 2 = s_{max}(K)$.*

Thus,

Definition 5.2. $s(K) = s_{max}(K) - 1 = s_{min}(K) + 1$.

Properties:

(1) $s(\overline{K}) = -s(K)$
(2) $s(K_1 \# K_2) = s(K_1) + s(K_2)$
(3) $s(K)$ is always even
(4) $s(U) = 0$ (can be deduced from property (1))
(5) s defines a homomorphism

$$s : Conc(S^3) \to \mathbb{Z}$$

from the concordance group of knots in S^3 to $\mathbb{Z}$.

Theorem 5.2 (Rasmussen, '04)**.**

$$|s(K)| \leq 2g_s(K)$$

Proof Sketch

- Consider an arbitrary cobordism between two links L_0 and L_1 in $S^3 \times I$.
- Wiggle this cobordism so that at each critical level you're either doing a Reidemeister move, or a Morse move.

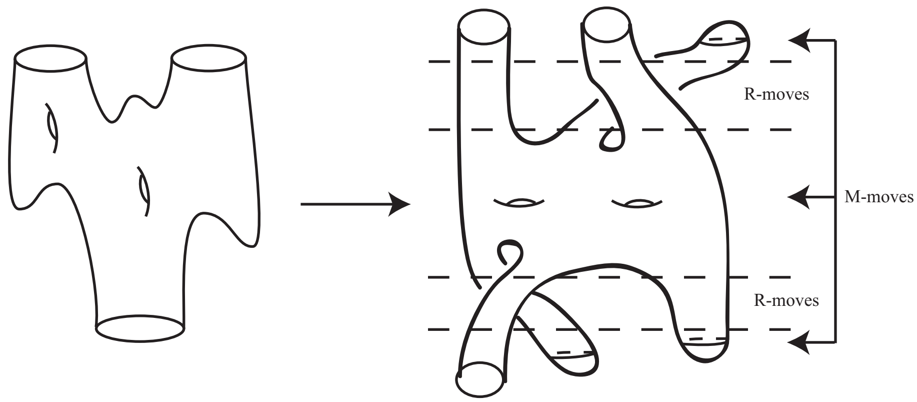


FIGURE 2. Wiggling the cobordism into position

- Rasmussen's idea is to apply the functor $\mathcal{A}'$ to get a chain map
$$\Phi_\Sigma : CKh'(L_0) \to CKh'(L_1)$$
of filtered degree $\chi(\Sigma)$ (+1 for each 0- or 2-handle, -1 for each 1-handle).

Recall: In dimension 2, Morse moves are 0-, 1- and 2-handle attachments.

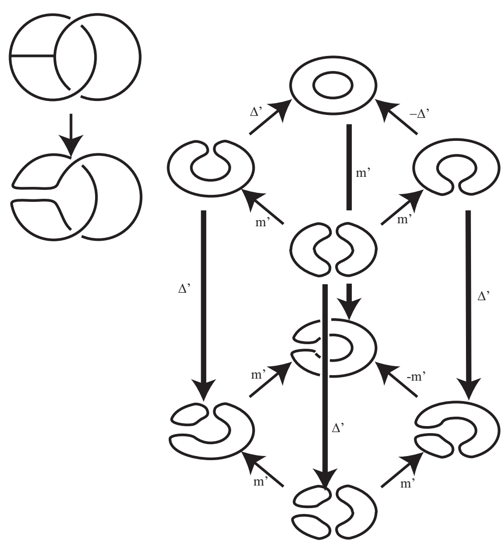


FIGURE 3. Applying the chain map

Theorem 5.3 (Rasmussen, '04). *If Σ is a connected cobordism between knots K_1 and K_2, then $\Phi_\Sigma : HKh'(K_0) \to HKh'(K_1)$ is an isomorphism of filtered degree $\chi(\Sigma) = -2g(\Sigma)$.*

To prove that $|s(K)| \leq 2g_s(K)$, we let $\Sigma \subset S^3 \times I$ be a genus g_s cobordism between K and the unknot. Letting $x \in HKh'(K) - \{0\}$ be an element of maximal grading, we have

$$s(\Phi_\Sigma(x)) \geq s(x) + \chi(\Sigma) = s(x) - 2g_s.$$

On the other hand, $s(U) = 0$, since $U = \overline{U}$. Thus $s_{max}(U) = 1$, and we have that

$$1 + 2g_s \geq s(x) = s_{max}(K),$$

implying,

$$s(K) \leq 2g_s.$$

Now if we run this same argument, but with $\overline{K}$ in place of K, we arrive at the inequality $-s(K) \leq 2g_s$, finishing the proof.

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Theorem 5.4 (Rasmussen,'04). *If K is a positive knot, then*

$$s(K) = 2g_s(K) = 2g(K).$$

Corollary 5.5 (Milnor Conjecture). *The slice genus of a (p, q)-torus knot is equal to its slice genus is equal to $(p-1)(q-1)/2$.*

Idea of the Proof:

- Rasmussen begins by looking at Lee's "canonical generators".
- Using these, he can explicitly computes $s(K)$ via a lemma stating that
$$s(\mathfrak{s}_{\mathbf{o}}) = s(\mathfrak{s}_{\bar{\mathbf{o}}}) = s_{min}(K),$$
where $\mathfrak{s}_{\mathbf{o}}$ is the canonical generator corresponding to the orientation $\mathbf{o}$.
  - Note that $\mathfrak{s}_{\mathbf{o}}$ is determined by assigning **a**'s and **b**'s according to some scheme. Thus
$$\begin{aligned} s_{min}(K) = s(\mathfrak{s}_{\mathbf{o}}) &= q(\mathfrak{s}_{\mathbf{o}}) \\ &= p(\mathfrak{s}_{\mathbf{o}}) + |\mathfrak{s}_{\mathbf{o}}| + n_+ - 2n_- \\ &= -k + 0 + n - 0 \end{aligned}$$
  - k is the number of circles coming from the "oriented resolution" ant $n = n_+$ is the number of positive crossings since K is a positive knot
- He compares the value he gets from this computation to the bound he gets on $g(K)$ via Seifert's algorithm.
  - Recall that Seifert's algorithm is going to give you a disk for each circle and a 1-handle for each resolved crossing.
- Life is good!

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