The Lickorish-Wallace Theorem

Shea Vick

Q:What's the basic problem in n-manifold topology?

A: Classify all n-manifolds (closed & connected) up to homeomorphism.

Let's get started!

$n = 0$:

$n = 1$:

$n = 2$:

Theorem 1.1 (Rado 1926)**.** *Every closed connected 2-manifold is homeomorphic to one of these.*
Furthermore, we can distinguish each of these by their first homology groups. That is,

$$\Sigma_1 \cong \Sigma_2 \leftrightarrow H_1(\Sigma_1) \cong H_1(\Sigma_2)$$

That is, we have an algorithmic method for deciding if two surfaces are homeomorphic.

$n = 3$: Still unknown!

$n \geq 4$: The problem is unsolvable!

Theorem 1.2 (Markov). *For $n \geq 4$, there does not exist an algorithm to decide whether or not two closed connected n-manifolds are homeomorphic or not.*

If nothing else, this is probably a good indication that the problem for 3-manifolds is pretty darn hard. We do get some help though. Here's a handy little theorem.

Theorem 1.3 (n=2 Rado 1926, n=3 Morse 1952). *For $n \leq 3$, any n-manifold is triangulable.*

Q: So why on earth is this so helpful?

A: It gives us a way to "see" 3-manifolds.

§ Heegaard Splittings

Definition 1.1. A **Heegaard Splitting** of a closed 3-manifold is a decomposition of M,

$$M = V_1 \cup_\Sigma V_2$$

where V_1, and V_2 are genus g handlebodies and $\Sigma = V_1 \cap V_2 = \partial V_1 = \partial V_2$.

In this view, we have V_1, V_2 as subsets of M and think of splitting M along their common boundary to get V_1 and V_2.

We can think of this in a slightly different way. We could think of M as being built from two arbitrary genus g handlebodies, glued together along their boundaries using some (orientation reversing) homeomorphism.

$$M = V_1 \cup V_2$$

where

$$V_1, V_2 = \text{standard genus g handlebodies}$$

$$h : \partial V_1 \to \partial V_2 \text{ some homeomorphism}$$

Theorem 1.4. *Every closed (orientable) 3-manifold has a Heegaard splitting.*

Proof. (Sketch)

Let K be a triangulation of M. Then M is built by gluing a bunch of tetrahedron along their faces.

Now let V_1 be a small tubular neighborhood of the 1-skeleton.

Then you can check that V_1 is a handlebody of genus $g = (\#edges) - (\#verticies)$. Furthermore, $\overline{M \backslash V_1}$ is also a handlebody of genus g (look at dual 1-skeleton).

This tells you one method of splitting M up, finishing the proof. □

To get a feel for this, let's do a few examples. Note that if we let one of our handlebodies lie still, then to specify h, it suffices to tell you where the meridians of the second handlebody get mapped.

Example 1.1. $S3$

Example 1.2. $S1 \times S2$

Example 1.3. L(3,1)

Example 1.4. Funny S^3

So there can be lots of potentially very complicated diagrams for any given 3-manifold. In light of this fact, it would be nice to have an alternative way of picturing 3-manifolds.

§ **Dehn Surgery**

Definition 1.2 (Dehn Surgery on a link)**.** Let L be a link in$S3$, and $N(L)$ a small tubular neighborhood. Then if $X(L) = \overline{S3 - N(L)}$, $X(L)$ is a compact 3-manifold with boundary

$$\partial X(L) = \coprod_1^k T^2.$$

Choose a homeomorphism (orientation reversing)

$$h : \coprod_1^k \partial(S1 \times D2) \to \partial X(L)$$

Let $M = X(L) \cup_h \coprod_1^k (S1 \times D2)$, we say M is obtained by **Dehn Surgery on L**.

So what picture should we have in our mind?

Example 1.5. The lens-space $L(p, q)$ is $-p/q$-surgery on the unknot.

Example 1.6. The Poincaré manifold as 1-surgery on the right-handed trefoil.

Example 1.7. The Poincaré manifold as 1-surgery on the Borromean link.

Theorem 1.5 (Lickorish-Wallace)**.** *Let $M3$ be a 3-manifold, then $M3$ can be obtained by integral surgery on a link in $S3$.*

Now to prove this, we're going to introduce a special type of surface automorphism. We preform this homeomorphism as follows.

Let γ be some simple closed curve on our surface Σ, and let $N(\gamma) = S1 \times [0, 1]$ be a small tubular neighborhood of γ.

Definition 1.3. A **positive Dehn twist** about γ is an automorphism

$$\tau_\gamma : \Sigma \to \Sigma$$

defined by $\tau_\gamma = id$ on $\overline{\Sigma \backslash N(\gamma)}$, and $\tau_\gamma(\theta, t) = (\theta + 2\pi t, t)$ on $N(\gamma)$.

A **Dehn twist** about γ is $\tau_\gamma^{\pm 1}$

Now if γ_1 and γ_2 are paths in X, we say $\gamma_1 \sim_D \gamma_2$ if there exists a sequence of Dehn twists $h_1, h_2, \ldots, h_n$, and an isotopy n such that $nh_1 \ldots h_n(\gamma_1) = \gamma_2$.

Lemma 1.6. *If γ, $\gamma' \subset int(\Sigma)$ are simple closed curves with $\gamma \cdot \gamma' = 1$, then $\gamma \sim_D \gamma'$*

Proof. We have $\tau_\gamma^{-1}(\gamma')$ and $\tau_{\gamma'}(\gamma)$ are isotopic. Therefore $\tau_\gamma \tau_{\gamma'}(\gamma)$ is isotopic to γ'. □

Lemma 1.7. *Let γ, $\gamma' \subset int(\Sigma)$ be non-separating simple closed curves, then there exist simple closed curves $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n = \gamma'$, such that $\gamma_i \cdot \gamma_{i+1} = 1$ for $1 \leq i \leq n - 1$.*

Corollary 1.8. *Let γ, $\gamma' \subset int(\Sigma)$ be non-separating simple closed curves, then $\gamma \sim_D \gamma'$*

Proof. Proof by induction on the number of intersection points $\gamma \cdot \gamma'$.
Base Case: $\gamma \cdot \gamma' = 0$

subcase (i): $\gamma \cup \gamma'$ separate the surface Σ, $\sigma \backslash (\gamma \cup \gamma') = \Sigma_1 \cup \Sigma_2$

Then there exist arcs $\beta_i \in \Sigma_i$, $i = 1, 2$, joining γ to γ' in Σ_i with $\partial\beta_1 = \partial\beta_2$.
Let $\beta = \beta_1 \cup \beta_2$, then we have that $\gamma \cdot \beta = 1$, and $\gamma' \cdot \beta = 1$. Thus, $\gamma \sim_D \gamma'$.

subcase (ii): $\gamma \cup \gamma'$ does not separate Σ.

Then consider the following picture.

Where γ_+, γ_- (resp. γ'_+, γ'_-) are the boundary curves for a small tubular neighborhood of γ (resp γ'). Then we have

$$\partial(\overline{\Sigma \backslash (N(\gamma) \cup N(\gamma'))}) = \gamma_+ \cup \gamma_- \cup \gamma'_+ \cup \gamma'_- \cup \partial\Sigma$$

Then since $\overline{\Sigma \backslash (N(\gamma) \cup N(\gamma'))}$ is connected, we have embedded arcs, $\beta_\pm$ joining $\gamma_\pm$ to $\gamma'_\pm$.

Unwanted intersections between β_+ and β_-, can be taken care of by "piping over γ_-".

After preforming this operation for each of the intersections between β_+ and β_-, we get new curves (again called β_+ and β_-) which no longer intersect one another.

Now, letting $\beta = \beta_+ \cup \beta_-$, we have that β is an embedded loop in Σ with $\beta \cdot \gamma = 1$, $\beta \cdot \gamma' = 1$.

Inductive Step: Assume the lemma is true for $\gamma \cdot \gamma' \leq n$.

Now, suppose $\gamma \cdot \gamma' = n$, then we can assume $n \geq 2$, since $n = 1$ was taken care of in a previous lemma. Let a,b be two points of $\gamma \cap \gamma'$ which occur consecutively on γ'. Then we have two cases.

Case 1: a and b both have the same sign (see picture below).

In this case, consider the curve β below. Then, $\beta \cdot \gamma = 1$, and $\beta \cdot \gamma' < n$.

Case 2: a and b have opposite signs (see picture below).

In this case, we have that $a \cup b$, splits γ into two arcs α_1 and α_2. Letting α' be the segment of γ' between a and b, we have the following

$$[\alpha_1 \cup \alpha'] + [\alpha_2 \cup -\alpha'] = [\alpha_1 \cup \alpha_2] = [\gamma] \neq 0$$

This tells us that one of $[\alpha_1 \cup \alpha']$, $[\alpha_2 \cup -\alpha']$ is nonzero in homology and is therefore nonseparating. Letting β be the nonseparating one, we have that $\beta \cdot \gamma = 0$, and $\beta \cdot \gamma' < n$.

□

Definition 1.4. A complete curve system for Σ is a maximal disjoint union of simple closed curves $\gamma_1, \ldots, \gamma_n$ in Σ such that $\Sigma \backslash \bigcup_i \gamma_i$ is connected.

Example 1.8.

Remark 1.1. If $\{\gamma_i\}$ is a complete curve system and $h : \Sigma \to \Sigma$ is any automorphism of Σ, then $\{h(\gamma_i)\}$ is also a complete curve system.

Theorem 1.9. *Let Γ, Γ' be two complete curve systems for a compact surface Σ, then $\Gamma \sim_D \Gamma'$.*

Proof. We have that $\gamma = \coprod_1^n \gamma_i$, and $\gamma' = \coprod_1^m \gamma_i'$ where $m \geq n$ (as a corollary to this proof, we'll ge that $m = n$). Then by a previous corollary, we have that there is a sequence of Dehn twists, h_1 such that $h_1(\gamma_1) = \gamma_1'$.

Now, we put this first accomplishment in the bank by cutting our surface along γ_1', forcing any future Dehn twists to leave γ_1' fixed. We call this new surface $\Sigma_{\gamma_1'}$.

Now by the same corollary, now applied to $\Sigma_{\gamma_1'}$, we have a sequence of Dehn twists, h_2 such that $h_2 h_1(\gamma_2) = \gamma_2'$.

Again, we put this second accomplishment in the bank by cutting our surface along γ_2', forcing future Dehn twists to now miss γ_1', and γ_2'.

Continuing this process, we get that $\Gamma \sim_D \Gamma'$, and that $m = n$.

□

Now we're on the home stretch. All that remains to be shown is how this story ties together with the one that we told last week about Heegaard splittings and Dehn surgery.

Suppose we're given some 3-manifold M, and told it has a genus-g Heegaard splitting. Then we know that M is built from two copies of the standard genus g handlebody glued together along their boundaries.

As a visual aid, we picture a genus g handlebody sitting comfortably in S^3. We can think of this as a genus-g splitting of S^3, where the first handlebody is the guy sitting in space, and the second is "everything else.

Now we know that M has a genus-g splitting, so if we remove V_1 and glue it back in the appropriate way, we will recover our manifold M.

From our previous discussion, we know that this homeomorphism of Σ can be broken down into a sequence of Dehn twists and isotopies.

Now, an isotopy of Σ can be extended to an isotopy of V_1; so this begs the following question:

Q: How does a Dehn twist along some curve γ in Σ effect V_1?

It's precisely equivalent to doing $\pm 1/1$ surgery on some link component in $V_1 \subset S^3$.

Preforming this operation for each of the required Dehn twists (requiring that each solid torus is cut out closer to Σ than its predecessor), we see that M can be obtained by removing a bunch of solid tori and gluing them in differently.

Thus, every closed, connected, orientable 3-manifold can be obtained by surgery on a link with all surgery coefficients equal to ± 1.