On the equivalence of Legendrian and transverse invariants in Heegaard Floer homology

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Motivation

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A knot $K \subset (Y, \xi)$ is Legendrian or transverse if its tangents either lie within, or transversally intersect ξ respectively.

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If we knew GRID $=$ LOSS, we could combine the intrinsic advantages of both invariants to prove great theorems.

The GRID invariant

In 2006, Ozsváth, Szabó and Thurston defined an invariant of Legendrian knots.

$$\lambda(L) \in \mathrm{HFK}^-(-Y, L)$$

The invariant λ is unchanged under negative Legendrian stabilization. Thus, if K is transverse and L is a Legendrian approximation, OST define:

$$\theta(K) := \lambda(L).$$

Legendrian $\leftrightarrow$ Transverse

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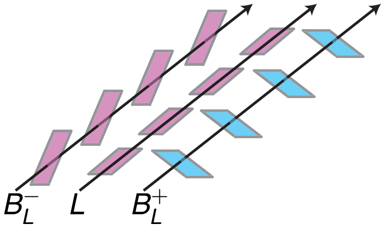

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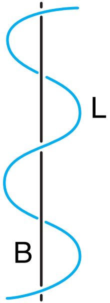

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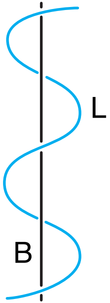

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Two Legendrian approximations related by "negative stabilization".

The LOSS invariant

In 2008, Lisca, Ozsváth, Stipsicz and Szabó defined an invariant of Legendrian knots.

$$\mathcal{L}(L) \in \mathrm{HFK}^{-}(-Y, L).$$

Like the GRID invariant, $\mathcal{L}$ is unchanged under negative Legendrian stabilization. Thus, if K is transverse and L is a Legendrian approximation, LOSS define:

$$\mathcal{T}(K) := \mathcal{L}(L).$$

The correspondence theorem

Theorem (with Baldwin and Vértesi)

Let K be a transverse knot in the standard contact 3-sphere. There exists a isomorphism of bigraded $(\mathbb{Z}/2\mathbb{Z})[U]$-modules,

$$\psi : \mathrm{HFK}^-(-S^3, K) \to \mathrm{HFK}^-(-S^3, K),$$

which sends $\mathcal{T}(K)$ to $\theta(K)$.

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Since both λ and $\mathcal{L}$ can be defined via transverse pushoff, if follows immediately from this result that there exists a similar isomorphism identifying the Legendrian invariants $\mathcal{L}$ and λ.

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Our BRAID invariant t takes values in HFK^- and is defined for transverse knots $K \subset (Y, \xi)$ which are braided with respect to an open book decomposition (B, π) of the ambient space.

Note that our BRAID invariant is manifestly a transverse invariant – it is defined in terms of transverse knots rather than in terms of Legendrian knots via Legendrian approximation.

Right-Veering braids

Our invariant also lends itself more naturally to understanding the connections between transverse links, braids and mapping class groups, a rich area of exploration even for knots in (S^3, ξ_{std}). As a preliminary step in this direction, we prove the following:

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Theorem (with Baldwin and Vértesi)

Suppose K is a transverse braid with respect to some open book for (Y, ξ). If K is not right-veering, then $\widehat{t}(K) = 0$.

Transverse braids

Definition

Suppose (B, π) is an open book compatible with the contact structure (Y, ξ). A transverse knot K in (Y, ξ) is said to be a *braid with respect to* (B, π) if K is positively transverse to the pages of (B, π).

Pavelescu's Thesis

Theorem (Pavelescu)

Suppose (B, π) *is an open book compatible with* (Y, ξ)*. Then every transverse link in* (Y, ξ) *is transversely isotopic to a braid with respect to* (B, π)*.*

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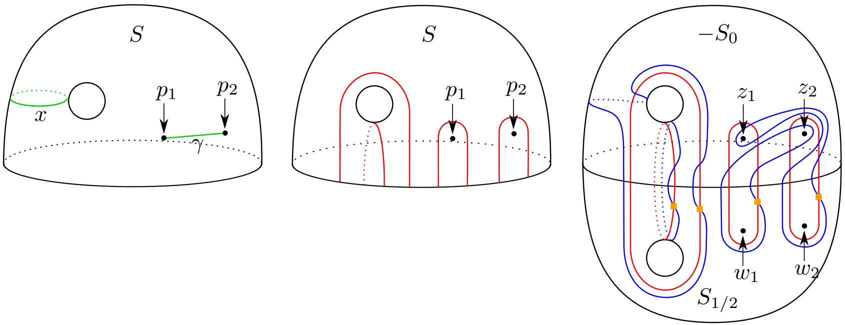
Theorem (Pavelescu)

Suppose K_1 and K_2 are braids with respect to an open book (B, π) compatible with (Y, ξ). Then K_1 and K_2 are transversely isotopic if and only if there exist positive Markov stabilizations K_1^+ and K_2^+ around the binding components of (B, π) such that K_1^+ and K_2^+ are transversely isotopic with respect to (B, π).

The BRAID invariant

Here's how to construct our BRAID invariant in one picture.



The monodromy map is $D_x \circ d_\gamma^{-1}$.

The BRAID invariant

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By Pavalescu's results, it suffices to show that, for each of the following operations, the element $t(K)$ is fixed under the induced isomorphisms on knot Floer homology:

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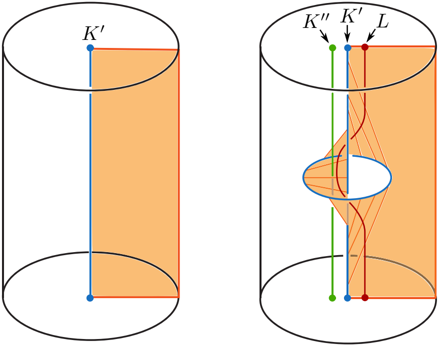
By Pavalescu's results, it suffices to show that, for each of the following operations, the element $t(K)$ is fixed under the induced isomorphisms on knot Floer homology:

1. change of the basis $\{a_1, \ldots, a_k\}$,
2. isotopy of $\widehat{\varphi}$ fixing $\{p_1, \ldots, p_n\}$ point-wise,
3. positive open book stabilization,
4. conjugation of $\widehat{\varphi}$,
5. positive Markov stabilization.

BRAID = LOSS

A proof of BRAID = LOSS in two pictures.



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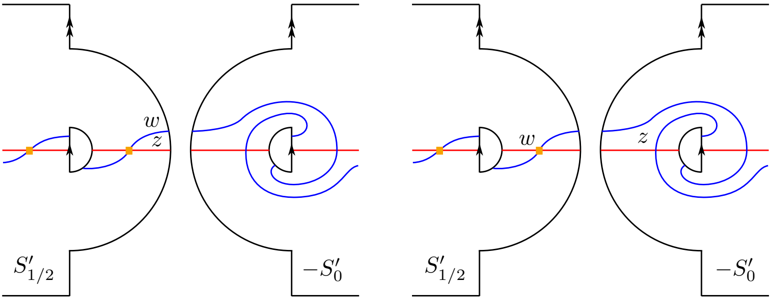

Characterizing BRAID

Let $K \subset (S^3, \xi_{std})$ be a transverse knot. Then K can be braided with respect to (U, π), the standard open book decomposition of (S^3, ξ_{std}) whose pages are disks.

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The unknot $-U \subset -S^3$ induces an Alexander filtration on $\mathrm{CFK}^-(-S^3, K)$

$$\emptyset = \mathcal{F}_m^{-U} \subset \mathcal{F}_{m+1}^{-U} \subset \cdots \subset \mathcal{F}_n^{-U} = \mathrm{CFK}^-(-S^3, K).$$

Characterizing BRAID

Let

$$b = \min\{j \mid H_*(\mathcal{F}_j^{-U}) \neq 0\},$$

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Theorem (with Baldwin and Vértesi)

The summand $H_t(\mathcal{F}_b^{-U})$ has rank one and that $t(K)$ can be characterized as the image of the generator of $H_t(\mathcal{F}_b^{-U})$ under the map

$$H_*(\mathcal{F}_b^{-U}) \to \mathrm{HFK}^-(-S^3, K)$$

induced by inclusion.

Characterizing GRID

We use grading an non-vanishing results proved by Ozsváth-Szabó-Thurston for the GRID invariant to establish the following characterization:

Theorem (with Baldwin and Vértesi)

The element $\theta(K)$ can be characterized as the image of the generator of $H_t(\mathcal{F}_b^{-U})$ under the map

$$H_*(\mathcal{F}_b^{-U}) \to \mathrm{HFK}^-(-S^3, K)$$

induced by inclusion.

LOSS $=$ GRID

The fact that the filtered quasi-isomorphism type of the filtration

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Since we already showed that the LOSS = BRAID, it follows that LOSS = GRID.

Thanks for listening!