Arnold's dream: a search for higher-order helicity invariants

David Shea Vela-Vick

Joint with Dennis DeTurck, Herman Gluck,
Rafal Komendarczyk, Paul Melvin
and Clayton Shonkwiler

Department of Mathematics
Columbia University

Rice University Colloquium

Helicity.

Definition

The **helicity** of a vector field V defined on a bounded domain $\Omega \subset \mathbb{R}^3$ is define by

$$\begin{aligned} \mathrm{Hel}(V) &= \frac{1}{4\pi}\int_{\Omega\times\Omega} V(x) \times V(y) \bullet \frac{x-y}{|x-y|^3}\, dx\, dy \\ &= -\int_{\Omega\times\Omega} V(x) \times V(y) \bullet \nabla_x \phi(x-y)\, dx\, dy \end{aligned}$$

Helicity.

Definition

The **helicity** of a vector field V defined on a bounded domain $\Omega \subset \mathbb{R}^3$ is define by

$$\begin{aligned}\mathrm{Hel}(V) &= \frac{1}{4\pi}\int_{\Omega\times\Omega} V(x) \times V(y) \bullet \frac{x-y}{|x-y|^3}\, dx\, dy \\ &= -\int_{\Omega\times\Omega} V(x) \times V(y) \bullet \nabla_x \phi(x-y)\, dx\, dy\end{aligned}$$

- $\phi(x) = 1/(4\pi|x|)$ is the fundamental solution of the scalar Laplacian on $\mathbb{R}^3$.

Fluid mechanics and plasma physics.

Helicity was invented by Woltjer in 1958 while studying the Crab Nebula.

Fluid mechanics and plasma physics.

Helicity was invented by Woltjer in 1958 while studying the Crab Nebula.

- Woltjer showed that helicity remains unchanged if a field is allowed to evolve according to the laws of MHD and gives a lower bound for the L_2-field energy during such evolution.

Fluid mechanics and plasma physics.

Helicity was invented by Woltjer in 1958 while studying the Crab Nebula.

- Woltjer showed that helicity remains unchanged if a field is allowed to evolve according to the laws of MHD and gives a lower bound for the L_2-field energy during such evolution.
- The term "helicity" was coined by Moffatt in 1969, who derived the formula on the previous slide.

Gauss's Linking Integral

Gauss defined the linking number between two curves $K = \{x(s)\}$ and $L = \{y(t)\}$ by the integral formula

$$\mathrm{Lk}(K, L) = -\int_{K\times L} \frac{dx}{ds} \times \frac{dy}{dt} \bullet \nabla_x \phi(x - y)\, ds\, dt.$$

Gauss's Linking Integral

Gauss defined the linking number between two curves $K = \{x(s)\}$ and $L = \{y(t)\}$ by the integral formula

$$\mathrm{Lk}(K, L) = -\int_{K\times L} \frac{dx}{ds} \times \frac{dy}{dt} \bullet \nabla_x \phi(x - y)\, ds\, dt.$$

Gauss's integral has two important features

Gauss's Linking Integral

Gauss defined the linking number between two curves $K = \{x(s)\}$ and $L = \{y(t)\}$ by the integral formula

$$\mathrm{Lk}(K, L) = -\int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \bullet \nabla_x \phi(x - y)\, ds\, dt.$$

Gauss's integral has two important features

- It computes a topological quantity – the linking number.

Gauss's Linking Integral

Gauss defined the linking number between two curves $K = \{x(s)\}$ and $L = \{y(t)\}$ by the integral formula

$$\mathrm{Lk}(K, L) = -\int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \bullet \nabla_x \phi(x - y)\, ds\, dt.$$

Gauss's integral has two important features

- It computes a topological quantity – the linking number.
- It's geometric in nature – the integrand is invariant under orientation-preserving rigid motions of R^3.

Comparison with helicity.

Let's compare Gauss's formula for the linking number

$$\mathrm{Lk}(K, L) = -\int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \bullet \nabla_x \phi(x - y)\, ds\, dt,$$

Comparison with helicity.

Let's compare Gauss's formula for the linking number

$$\mathrm{Lk}(K, L) = -\int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \bullet \nabla_x \phi(x - y)\, ds\, dt,$$

to the Woltjer-Moffatt formula for helicity

$$\mathrm{Hel}(V) = -\int_{\Omega \times \Omega} V(x) \times V(y) \bullet \nabla_x \phi(x - y)\, dx\, dy.$$

Comparison with helicity.

Let's compare Gauss's formula for the linking number

$$\mathrm{Lk}(K, L) = -\int_{K\times L} \frac{dx}{ds} \times \frac{dy}{dt} \bullet \nabla_x \phi(x-y)\, ds\, dt,$$

to the Woltjer-Moffatt formula for helicity

$$\mathrm{Hel}(V) = -\int_{\Omega\times\Omega} V(x) \times V(y) \bullet \nabla_x \phi(x-y)\, dx\, dy.$$

We see, therefore, that helicity measures the extent to which integral field lines of V link one-another in Ω.

Arnold's dream.

Our study is motivated by a problem proposed by Arnold and Khesin in 1998 regarding a search for higher-order helicities for divergence-free vector fields.

Arnold’s dream.

Our study is motivated by a problem proposed by Arnold and Khesin in 1998 regarding a search for higher-order helicities for divergence-free vector fields.

The dream is to define a hierarchy of invariants for generic vector fields such that, whereas all f the invariants of order $\leq k$ *have zero value for a given field and there exists a nonzero invariant of order* $k+1$*, this nonzero invariant provides a lower bound for the field energy.*

– Arnold and Khesin

Arnold's dream.

Our study is motivated by a problem proposed by Arnold and Khesin in 1998 regarding a search for higher-order helicities for divergence-free vector fields.

The dream is to define a hierarchy of invariants for generic vector fields such that, whereas all f the invariants of order $\leq k$ *have zero value for a given field and there exists a nonzero invariant of order* $k+1$*, this nonzero invariant provides a lower bound for the field energy.*

– Arnold and Khesin

Since ordinary helicity is a field-analogue of the linking between simple, closed curves, higher-order helicities should correspond to higher-order linking invariants.

Link homotopy

Definition

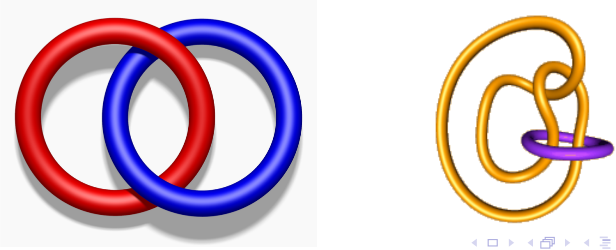
A **link homotopy** of an *n*-component link is a deformation, during which each component may cross itself, but distant components must remain disjoint.

Link homotopy

Definition

A **link homotopy** of an *n*-component link is a deformation, during which each component may cross itself, but distant components must remain disjoint.

When $n = 2$, the linking number provides a complete invariant. Two two-component links are link-homotopic if and only if they have the same linking number.

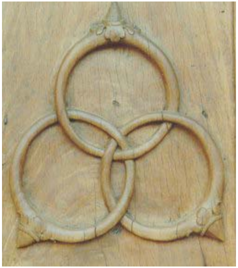


Milnor’s senior thesis.

Three component links were classified up to **link homotopy** by Milnor in his senior thesis.

Milnor's senior thesis.

Three component links were classified up to **link homotopy** by Milnor in his senior thesis.



A complete set of invariants is given by the pairwise linking numbers p, q and r, together with a third integer μ, defined modulo $\mathrm{gcd}(p,q,r)$.

Thinking in analogy with ordinary helicity, we seek an integral formula for the Milnor triple-linking number which is both topological and geometric in nature

Thinking in analogy with ordinary helicity, we seek an integral formula for the Milnor triple-linking number which is both topological and geometric in nature

Theorem (with DeTurck et. al.)

For three-component links in S^3 *with* $p = q = r = 0$*, there exists an explicit integral formula for Milnor's triple linking number* μ *whose integrand is invariant under orientation-preserving rigid motions of* S^3*.*

Thinking in analogy with ordinary helicity, we seek an integral formula for the Milnor triple-linking number which is both topological and geometric in nature

Theorem (with DeTurck et. al.)

For three-component links in S^3 *with* $p = q = r = 0$*, there exists an explicit integral formula for Milnor's triple linking number* μ *whose integrand is invariant under orientation-preserving rigid motions of* S^3*.*

The remainder of this talk will be devoted to deriving this formula and understanding where it comes from.

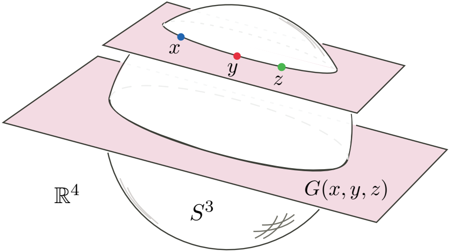
The Grassman map.

Let x, y and z be distinct points on $S^3 \subset \mathbb{R}^4$.

The Grassman map.

Let x, y and z be distinct points on $S^3 \subset \mathbb{R}^4$.

They cannot lie on a straight line and therefore span a 2-plane in $\mathbb{R}^4$



Translate this plane so that it passes through the origin and orient it so that $x - z$ and $y - z$ form a positive basis. The result is an element $G(x, y, z)$ of the Grassman manifold oriented 2-planes in $\mathbb{R}^4$.

The Grassman map (cont.).

Definition

The **Grassman Map**

$$G : \mathrm{Conf}_3(S^3) \to G_2(\mathbb{R}^4)$$

takes the ordered triple of distinct points (x, y, z) to the element $G(x, y, z)$ of the Grassman manifold $G_2(\mathbb{R}^4)$.

The Grassman map (cont.).

Definition

The **Grassman Map**

$$G : \mathrm{Conf}_3(S^3) \to G_2(\mathbb{R}^4)$$

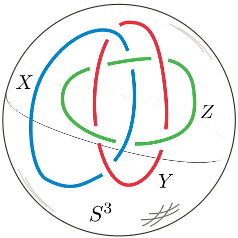
takes the ordered triple of distinct points (x, y, z) to the element $G(x, y, z)$ of the Grassman manifold $G_2(\mathbb{R}^4)$.

- The Grassman map is equivariant with respect to the $SO(4)$-actions on $\mathrm{Conf}_3(S^3)$ and $G_2(\mathbb{R}^4)$.

The Grassman map (cont.).

Given a 3-component link L with components

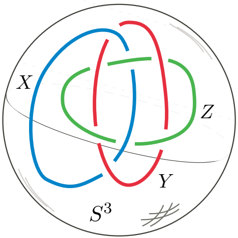
$$X = \{x(s)\}, \qquad Y = \{y(t)\}, \qquad Z = \{z(u)\},$$



The Grassman map (cont.).

Given a 3-component link L with components

$$X = \{x(s)\}, \qquad Y = \{y(t)\}, \qquad Z = \{z(u)\},$$



we get a **Characteristic Map** $g_L : S^1 \times S^1 \times S^1 \to S^2$, defined by

$$g_L(s, t, u) = \pi \circ G(x(s), y(t), z(u)),$$

where $\pi : G_2(\mathbb{R}^4) \cong S^2 \times S^2 \to S^2$ is projection onto either factor.

Pontryagin's invariants

The map $g_L : T^3 \to S^2$ is constructed so that link-homotopies of L translate to homotopies of the map g_L.

Pontryagin's invariants

The map $g_L : T^3 \to S^2$ is constructed so that link-homotopies of L translate to homotopies of the map g_L.

Pontryagin classified maps $f : M^3 \to S^2$ from a 3-complex to a 2-sphere in 1941.

Pontryagin's invariants

The map $g_L : T^3 \to S^2$ is constructed so that link-homotopies of L translate to homotopies of the map g_L.

Pontryagin classified maps $f : M^3 \to S^2$ from a 3-complex to a 2-sphere in 1941.

For maps of a 3-torus, a complete set of invariants is given by the degrees p, q, r of the map restricted to the three faces, together with one further integer ν, defined modulo $2\mathrm{gcd}(p, q, r)$.

Correspondence theorem.

One cannot help but notice the similarity with Milnor's classification theorem for 3-component links up to link-homotopy

Correspondence theorem.

One cannot help but notice the similarity with Milnor's classification theorem for 3-component links up to link-homotopy

Theorem (with DeTurck et. al.)

Let L be a 3-component link in S^3. *Then the pairwise linking numbers p, q and r of L are equal to the degrees of the characteristic map* $g_L : T^3 \to S^2$ *on the two-dim'l coordinate subtori, while twice Milnor's* μ*-invariant for L is equal to Pontryagin's* ν*-invariant for* g_L *modulo* $2\mathrm{gcd}(p, q, r)$.

Integral formulas.

If the pairwise linking numbers *p*, *q* and *r* are all zero, both the μ- and ν-invariants are ordinary integers.

Integral formulas.

If the pairwise linking numbers *p*, *q* and *r* are all zero, both the μ- and ν-invariants are ordinary integers.

In this case, we use J.H.C. Whitehead's integral formula for the Hopf invariant, adapted to maps of the 3-torus to the 2-sphere, together with a Fourier series for the Laplacian on the 3-torus to obtain an explicit formula for $\nu(g_L)$, and hence $\mu(L)$.

Integral formulas (cont.).

Definition

Let ω be the unit normalization of the standard area 2-form on S^2.

$\omega_L = g_L^*\omega =$ **characteristic 2-form** of L on T^3

$\mathbf{v}_L =$ conversion of ω_L to a divergence-free vector field on T^3

$=$ **characteristic vector field** of L

Integral formulas (cont.).

Definition

Let ω be the unit normalization of the standard area 2-form on S^2.

$\omega_L = g_L^*\omega =$ **characteristic 2-form** of L on T^3

$\mathbf{v}_L =$ conversion of ω_L to a divergence-free vector field on T^3

$=$ **characteristic vector field** of L

When p, q and r are all zero, ω_L is exact and $\mathbf{v}_L$ is in the image of curl.

Integral formulas (cont.).

Theorem (with DeTurck et. al.)

If the pairwise linking numbers of a three-component link L are all zero, then Milnor's μ-invariant of L is given by the following formulas.

- $\mu(L) = \frac{1}{2}\int_{T^3} \delta(\phi * \omega_L) \wedge \omega_L.$
- $\mu(L) = -\frac{1}{2}\int_{T^3 \times T^3} \mathbf{v}_L \times \mathbf{v}_L \bullet \nabla_x \phi(x-y)\, dx\, dy.$

Integral formulas (cont.).

Theorem (with DeTurck et. al.)

If the pairwise linking numbers of a three-component link L are all zero, then Milnor's μ-invariant of L is given by the following formulas.

- $\mu(L) = \frac{1}{2} \int_{T^3} \delta(\phi * \omega_L) \wedge \omega_L.$
- $\mu(L) = -\frac{1}{2} \int_{T^3 \times T^3} \mathbf{v}_L \times \mathbf{v}_L \bullet \nabla_x \phi(x - y)\, dx\, dy.$

As desired, the integrands in both formulas are $SO(4)$-equivariant, attesting to their naturality.

Fourier series formula.

Because our integrals are taking place over the 3-torus, it is possible to obtain one further formula for $\mu(L)$

$$\mu(L) = 8\pi^3 \sum_{\mathbf{n}\neq\mathbf{0}} \mathbf{a_n} \times \mathbf{b_n} \bullet \frac{\mathbf{n}}{|\mathbf{n}|^2}.$$

Here, $\mathbf{a_n}$ and $\mathbf{b_n}$ are the real and imaginary parts of the vector-valued Fourier series expansion of ω_L (or equivalently $\mathbf{v}_L$).

Fourier series formula.

Because our integrals are taking place over the 3-torus, it is possible to obtain one further formula for $\mu(L)$

$$\mu(L) = 8\pi^3 \sum_{\mathbf{n} \neq \mathbf{0}} \mathbf{a_n} \times \mathbf{b_n} \bullet \frac{\mathbf{n}}{|\mathbf{n}|^2}.$$

Here, $\mathbf{a_n}$ and $\mathbf{b_n}$ are the real and imaginary parts of the vector-valued Fourier series expansion of ω_L (or equivalently $\mathbf{v}_L$).

We have used this formula to successfully compute the Milnor μ-inviariant via Matlab and Maple.

Configuration spaces.

Our goal now is to prove the correspondence theorem. To do this, we need to better understand the underlying topological problem.

Configuration spaces.

Our goal now is to prove the correspondence theorem. To do this, we need to better understand the underlying topological problem.

Let L be an ordered, oriented link in $Y = S^3$ or $\mathbb{R}^3$, with n-components $X = \{x(s)\}$, $Y = \{x(t)\}$, ... Then there is an **evaluation map**

$$e_L : T^n \to \mathrm{Conf}_n(Y), \qquad (s, t, \dots) \mapsto (x(s), y(t), \dots)$$

from the n-torus to the configuration space of ordered n-tuples of distinct points in Y.

Configuration spaces (cont.).

Link homotopies of L induce homotopies of e_L. Thus, we have a map

Configuration spaces (cont.).

Link homotopies of L induce homotopies of e_L. Thus, we have a map

$$e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)],$$

where

$$\begin{aligned} \mathcal{L}_n(Y) &= \text{link-homotopy classes of } n\text{-component links in } Y \\ [T^n, \mathrm{Conf}_n(Y)] &= \text{homotopy classes of maps from } T^n \text{ to } \mathrm{Conf}_n(Y) \end{aligned}$$

Configuration spaces (cont.).

Link homotopies of L induce homotopies of e_L. Thus, we have a map

$$e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)],$$

where

$$\begin{aligned} \mathcal{L}_n(Y) &= \text{link-homotopy classes of } n\text{-component links in } Y \\ [T^n, \mathrm{Conf}_n(Y)] &= \text{homotopy classes of maps from } T^n \text{ to } \mathrm{Conf}_n(Y) \end{aligned}$$

We think of e as a representation from the world of link-homotopy to the world of homotopy, and the fundamental question is whether or not this representation is faithful, that is, one-to-one.

$e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$

Is $e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$ a faithful representation?

$e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$

Is $e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$ a faithful representation?

- **Yes** if $n = 2$ and $Y = \mathbb{R}^3$, since $\mathrm{Conf}_2(\mathbb{R}^3)$ has the homotopy type of S^2. (This is the classical Gauss map)

$e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$

Is $e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$ a faithful representation?

- **Yes** if $n = 2$ and $Y = \mathbb{R}^3$, since $\mathrm{Conf}_2(\mathbb{R}^3)$ has the homotopy type of S^2. (This is the classical Gauss map)
- **No** if $n = 2$ and $Y = S^3$, since $\mathrm{Conf}_2(S^3)$ has the homotopy type of S^3.

$e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$

Is $e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$ a faithful representation?

- **Yes** if $n = 2$ and $Y = \mathbb{R}^3$, since $\mathrm{Conf}_2(\mathbb{R}^3)$ has the homotopy type of S^2. (This is the classical Gauss map)
- **No** if $n = 2$ and $Y = S^3$, since $\mathrm{Conf}_2(S^3)$ has the homotopy type of S^3.
- **Yes** if $n = 3$ and $Y = S^3$, as a consequence of the correspondence theorem $\mathrm{Conf}_3(S^3)$.

$e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$

Is $e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$ a faithful representation?

- **Yes** if $n = 2$ and $Y = \mathbb{R}^3$, since $\mathrm{Conf}_2(\mathbb{R}^3)$ has the homotopy type of S^2. (This is the classical Gauss map)
- **No** if $n = 2$ and $Y = S^3$, since $\mathrm{Conf}_2(S^3)$ has the homotopy type of S^3.
- **Yes** if $n = 3$ and $Y = S^3$, as a consequence of the correspondence theorem $\mathrm{Conf}_3(S^3)$.
- **Yes** if $n = 3$ and $Y = \mathbb{R}^3$, as a consequence of the above.

$e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$

Is $e : \mathcal{L}_n(Y) \to [T^n, \mathrm{Conf}_n(Y)]$ a faithful representation?

- **Yes** if $n = 2$ and $Y = \mathbb{R}^3$, since $\mathrm{Conf}_2(\mathbb{R}^3)$ has the homotopy type of S^2. (This is the classical Gauss map)
- **No** if $n = 2$ and $Y = S^3$, since $\mathrm{Conf}_2(S^3)$ has the homotopy type of S^3.
- **Yes** if $n = 3$ and $Y = S^3$, as a consequence of the correspondence theorem $\mathrm{Conf}_3(S^3)$.
- **Yes** if $n = 3$ and $Y = \mathbb{R}^3$, as a consequence of the above.
- **Yes** for arbitrary n and $Y = \mathbb{R}^3$ if we restrict to links which are link-homotopically trivial when any one component is removed (Koschorke 1997).

Proving the correspondence theorem.

We have two proofs of the correspondence theorem – the first algebraic, and the second topological.

Proving the correspondence theorem.

We have two proofs of the correspondence theorem – the first algebraic, and the second topological.

The topological proof turns the map g_L into a “twisted” Gauss map, and was presented in this same colloquium by Paul Melvin. The algebraic proof centers around the following commutative diagram.

Proving the correspondence theorem.

We have two proofs of the correspondence theorem – the first algebraic, and the second topological.

The topological proof turns the map g_L into a “twisted” Gauss map, and was presented in this same colloquium by Paul Melvin. The algebraic proof centers around the following commutative diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & \mathcal{H}(3) \arrow[l] \arrow[d, "G"] \\
{[S^1 \times S^1 \times S^1, S^2]} & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[l]
\end{tikzcd}$$

Understanding the diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & \mathcal{H}(3) \arrow[l] \arrow[d, "\mathcal{E}"] \\
{[S^1 \times S^1 \times S^1, S^2]} & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[l]
\end{tikzcd}$$

Understanding the diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & & \mathcal{H}(3) \arrow[ll] \arrow[d, "\mathcal{E}"] \\
{[S^1 \times S^1 \times S^1, S^2]} & & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[ll]
\end{tikzcd}$$

- $\mathcal{H}(3)$ is the collection of link-homotopy classes of 3-component string-links – a group via “stacking”

Understanding the diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & & \mathcal{H}(3) \arrow[ll] \arrow[d, "\mathcal{E}"] \\
{[S^1 \times S^1 \times S^1, S^2]} & & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[ll]
\end{tikzcd}$$

- $\mathcal{H}(3)$ is the collection of link-homotopy classes of 3-component string-links – a group via “stacking”
- The top horizontal map is obtained by closing up the string-link to a normal link.

Understanding the diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & & \mathcal{H}(3) \arrow[ll] \arrow[d, "\mathcal{E}"] \\
{[S^1 \times S^1 \times S^1, S^2]} & & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[ll]
\end{tikzcd}$$

- $\mathcal{H}(3)$ is the collection of link-homotopy classes of 3-component string-links – a group via “stacking”
- The top horizontal map is obtained by closing up the string-link to a normal link.
- The bottom horizontal map is obtained by identifying conjugacy classes.

Understanding the diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & \mathcal{H}(3) \arrow[l] \arrow[d, "\mathcal{E}"] \\
{[S^1 \times S^1 \times S^1, S^2]} & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[l]
\end{tikzcd}$$

Understanding the diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & \mathcal{H}(3) \arrow[l] \arrow[d, "\mathcal{E}"] \\
{[S^1 \times S^1 \times S^1, S^2]} & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[l]
\end{tikzcd}$$

- While $\mathcal{H}(3)$ is a group, $\pi_1(\mathrm{Maps}(S^1 \times S^1, S^2))$ is a union of groups.

Understanding the diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & \mathcal{H}(3) \arrow[l] \arrow[d, "\mathcal{E}"] \\
{[S^1 \times S^1 \times S^1, S^2]} & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[l]
\end{tikzcd}$$

- While $\mathcal{H}(3)$ is a group, $\pi_1(\mathrm{Maps}(S^1 \times S^1, S^2))$ is a union of groups.
- We can reinterpret $\mathcal{H}(3)$ as a union of groups in a natural way, making $\mathcal{E}$ into a collection of group homomorphisms.

Understanding the diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & \mathcal{H}(3) \arrow[l] \arrow[d, "\mathcal{E}"] \\
{[S^1 \times S^1 \times S^1, S^2]} & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[l]
\end{tikzcd}$$

- While $\mathcal{H}(3)$ is a group, $\pi_1(\mathrm{Maps}(S^1 \times S^1, S^2))$ is a union of groups.
- We can reinterpret $\mathcal{H}(3)$ as a union of groups in a natural way, making $\mathcal{E}$ into a collection of group homomorphisms.
- By computing explicit presentations for these groups, one can determine the effects of these maps on generators.

Understanding the diagram.

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[d, "e"] & & \mathcal{H}(3) \arrow[ll] \arrow[d, "\mathcal{E}"] \\
{[S^1 \times S^1 \times S^1, S^2]} & & \pi_1(\mathrm{Maps}(S^1 \times S^1, S^2)) \arrow[ll]
\end{tikzcd}$$

- While $\mathcal{H}(3)$ is a group, $\pi_1(\mathrm{Maps}(S^1 \times S^1, S^2))$ is a union of groups.
- We can reinterpret $\mathcal{H}(3)$ as a union of groups in a natural way, making $\mathcal{E}$ into a collection of group homomorphisms.
- By computing explicit presentations for these groups, one can determine the effects of these maps on generators.
- This, plus the commutativity of the diagram prove the correspondence theorem.

What's next?

Derive a 2^{nd}-order helicity invariant.

What's next?

Derive a 2^{nd}-order helicity invariant.

- Sign properties of the Milnor μ-invariant appear to be an obstruction to doing this naively.

What's next?

Derive a 2^{nd}-order helicity invariant.

- Sign properties of the Milnor μ-invariant appear to be an obstruction to doing this naively.
- Rafal Komendarczyk has used our work to compute such an invariant for vector-fields in disjoint, unlinked flux-tubes in S^3.

What’s next?

Derive a 2^{nd}-order helicity invariant.

- Sign properties of the Milnor μ-invariant appear to be an obstruction to doing this naively.
- Rafal Komendarczyk has used our work to compute such an invariant for vector-fields in disjoint, unlinked flux-tubes in S^3.

Prove that the representation e is faithful.

What's next?

Derive a 2^{nd}-order helicity invariant.

- Sign properties of the Milnor μ-invariant appear to be an obstruction to doing this naively.
- Rafal Komendarczyk has used our work to compute such an invariant for vector-fields in disjoint, unlinked flux-tubes in S^3.

Prove that the representation e is faithful.

- Koschorke showed how to do this when one restricts to links which become link-homotopically trivial when any single component is removed.

What's next?

Derive a 2^{nd}-order helicity invariant.

- Sign properties of the Milnor μ-invariant appear to be an obstruction to doing this naively.
- Rafal Komendarczyk has used our work to compute such an invariant for vector-fields in disjoint, unlinked flux-tubes in S^3.

Prove that the representation e is faithful.

- Koschorke showed how to do this when one restricts to links which become link-homotopically trivial when any single component is removed.
- Is this true in general?

What's next?

Derive a 2^{nd}-order helicity invariant.

- Sign properties of the Milnor μ-invariant appear to be an obstruction to doing this naively.
- Rafal Komendarczyk has used our work to compute such an invariant for vector-fields in disjoint, unlinked flux-tubes in S^3.

Prove that the representation e is faithful.

- Koschorke showed how to do this when one restricts to links which become link-homotopically trivial when any single component is removed.
- Is this true in general?

Compute geometrically natural integral expressions for computing all Milnor's $\overline{\mu}$-invariants in S^3 and $\mathbb{R}^3$

- We have ideas how to do this for the first non-vanishing $\overline{\mu}$-invariants.