# Research Statement

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My research centers on questions in contact geometry, low-dimensional topology, knot theory and Riemannian geometry. Contact geometry is an odd-dimensional analogue of, and is deeply connected with, symplectic geometry. My work addresses classification and structure problems for contact 3-manifolds as well as the Legendrian and transverse knots they contain. I am also interested in questions lying in the intersection between classical knot theory and Riemannian geometry. Specifically, I study connections between invariants coming from maps into configuration spaces, higher-order linking invariants for links in the 3-sphere or Euclidean 3-space, and helicity invariants of fluid-flows.

## Research Themes

Explore connections and correspondences between contact geometry and Heegaard Floer homology. Evidence of deep connections between contact geometry and Heegaard Floer theory has steadily mounted since the latter theory first appeared, approximately ten years ago. In one direction, Heegaard Floer homology supports an invariant which is capable of distinguishing contact structures and detecting tightness. In the other, much of the algebraic structure Heegaard Floer possesses reflects appropriate geometric properties and constructions arising in contact geometry. The first theme of my research is to develop and clarify these connections and correspondences.

Analyze the structure of open book decompositions of contact manifolds. Giroux's correspondence theorem provides a bridge joining contact geometry and classical 3-dimensional topology via a structure called an open book decomposition. Giroux's correspondence underpins many of the results obtained in 3-dimensional contact geometry over the past decade. My research seeks to illuminate connections between these two structures by further developing the "dictionary" which translates between the two worlds.

Investigate and classify Legendrian and transverse knots. Complete Legendrian and transverse classification results are known for only the simplest knot types. To study the geometry of a given contact 3-manifold, it is crucial to thoroughly understand the knot theory supported by that space. Through a combination of geometric, algebraic and analytic techniques, my research aims to develop new Legendrian and transverse invariants and to apply these and other known invariants to classify Legendrian and transverse representatives in a broad class of knot types.

Study applications of topology and geometry to hydrodynamics. Helicity is an invariant of vector fields which provides a lower bound on the $L_2$-energy of a field evolving according the laws of ideal magnetohydrodynamics. The dream expressed by Arnol′d and Khesin is to define a hierarchy of helicity invariants, each providing a lower $L_2$-energy bound when all previous invariant vanish. The final aim of my research is to define a hierarchy of higher-order helicity invariants by first computing geometrically natural integral expressions for the Milnor $\mu$-invariants.

## 1. Contact Structures and Heegaard Floer Invariants

**Background.** Contact structures were introduced by Hamilton, Huygens and Jacobi in their study of geometric optics. Contact structures are analogues of symplectic structures which take the form of totally non-integrable hyperplane distributions on odd-dimensional manifolds. Dormant for nearly a century, contact structures and their invariants have enjoyed a surge in interest due to their many recently discovered applications to questions in physics, symplectic geometry and the smooth topology of three and four-dimensional manifolds.

A pair $(Y, \xi)$ is called a *contact manifold* if $Y$ is an oriented 3-manifold and $\xi$ is a 2-plane field on $Y$ satisfying a certain nonintegrability condition. The most elementary example of a contact 3-manifold is real 3-space with its "standard" contact structure $\xi_{std} = \ker(dz - y\,dx)$. A related example is provided by the 3-sphere $S^3$, viewed as the boundary of the unit ball in complex 2-space. In this case, the set of complex tangencies to $S^3$ provides a natural contact structure, called the "standard" contact structure on $S^3$. *Legendrian* and *transverse* knots are embeddings of the circle $S^1$, all of whose tangent spaces are either contained within, or transverse to the contact planes, respectively.

Given a contact 3-manifold $(Y, \xi)$, Ozsáth and Szabó showed how to identify an invariant $\mathrm{EH}(Y, \xi)$ which takes values in the Heegaard Floer homology of the 3-manifold $-Y$ [OS05]. Honda, Kazez, and Matić later presented an alternate description of the Ozsváth-Szabó contact invariant which is defined explicitly in terms of open book decompositions [HKM09b]. They then used this alternate formulation to produce a relative contact invariant for contact 3-manifolds with nonempty (convex) boundary [HKM09a].

Since the inception of Heegaard Floer theory in [OS04c], contact structures have played an increasingly important role. For example, they appear in proofs that Heegaard Floer invariants detect knot genus and fiberedness [OS04a, Ni07], and that the unknot, trefoil and figure-eight knots are determined by their surgeries [OS04a, OS06]. More recently, Honda, Kazez and Matić used contact structures to define "gluing maps" for Heegaard Floer invariants [HKM08]. The input for this map is a compact 3-manifold $Y$, possibly with sutured boundary, a submanifold $Y' \subset Y$, and a contact structure $\xi$ on the complement of $Y'$ in $Y$. Using this data, Honda, Kazez and Matić define a map,

$$\Phi_\xi : \mathrm{SFH}(-Y', \Gamma') \to \mathrm{SFH}(-Y, \Gamma).$$

If, in addition, the submanifold is endowed with a contact structure $\xi'$, compatible with $\xi$ along points of their common boundary, then $\Phi_\xi$ maps the contact invariant for $(Y', \xi')$ to the contact invariant of $(Y, \xi \cup \xi')$.

In [EVZ11a], Etnyre, Zarev and I use a natural contact geometric construction, together with the Honda-Kazez-Matić gluing maps to give an alternate, geometric characterization of the minus version of knot Floer homology. To a triple $(Y, \xi, L)$ consisting of a contact 3-manifold $(Y, \xi)$ and a Legendrian knot $L$, we associate a graded $\mathbb{Z}/2[U]$-module $\underrightarrow{\mathrm{SFH}}(-Y, L)$ whose isomorphism type depends only on the topological types of $Y$ and $L$. By applying tools from bordered Floer homology [LOT08] and, specifically, bordered sutured Floer homology [Zar09, Zar11], we are able characterize the module $\underrightarrow{\mathrm{SFH}}(-Y, L)$.

**Theorem 1** (with Etnyre and Zarev [EVZ11a])**.** *If $K$ is a null-homologous knot in a 3-manifold $Y$, then there is a graded $\mathbb{Z}/2[U]$-module isomorphism*

$$\Psi : \underrightarrow{\mathrm{SFH}}(-Y, K) \to \mathrm{HFK}^-(-Y, K),$$

*identifying the* $\underrightarrow{\mathrm{SFH}}(-Y,K)$ *with the "minus" version of knot Floer homology.*

By generalizing the above construction, we are able to define Heegaard Floer type invariants for noncompact 3-manifolds with $T^2 \times [0,\infty)$-ends and a specified "slope at infinity", which extends the notion of a suture. The construction allows one to further identify an invariant of contact structures on such 3-manifolds. This invariant takes values in our generalized version of Heegaard Floer homology. Using these tools, we show:

**Theorem 2** (with Etnyre and Zarev [EVZ11b])**.** *Let $Y$ denote any of the following spaces: $S^1\times\mathbb{R}^2$, $T^2\times(0,1)$ or $T^2\times[0,1)$. If $\xi_1$ and $\xi_2$ are two tight, minimally twisting contact structures on $Y$ with the same slope at infinity, then $\xi_1$ is isotopic to $\xi_2$ if and only if $\underrightarrow{\mathrm{EH}}(Y,\xi_1)=\underrightarrow{\mathrm{EH}}(Y,\xi_2)$.*

Little is known about contact structures on noncompact 3-manifolds. Eliashberg showed that $\mathbb{R}^3$ supports a unique tight contact structure [Eli91]. Later, he considered tight contact structures on $S^1\times\mathbb{R}^2$, showing that this space supports uncountably many distinct tight contact structures by defining an invariant equivalent to the "slope at infinity" from above [Eli93]. Tripp completed the classification of tight, minimally twisting contact structures on $S^1\times\mathbb{R}^2$, $T^2\times(0,1)$ and $T^2\times[0,1)$ [Tri06]. In particular, he showed that each of these spaces support uncountably many tight contact structures which cannot be tightly embedded in a compact contact 3-manifold with convex boundary. The invariants we define in [EVZ11b] distinguish all tight, minimally twisting contact structures with a given slope at infinity, and therefore can yield an alternate proof of Tripp's classification.

A byproduct of the techniques used to establish Theorems 1 and 2 is a robust computational framework one can apply to effectively compute the Honda-Kazez-Matić gluing maps. Prior to these results, only elementary computations which relied on formal properties of the gluing maps had ever been performed.

**Description of Plan.** One of the guiding principles of my work moving forward is that geometric properties of contact structures are inexorably tied to the algebraic structure underlying smooth topological invariants like Heegaard Floer homology, embedded contact homology and monopole Floer homology. Theorem 1 shows how a natural contact geometric construction can imprint itself in the algebraic structure of Heegaard Floer theory. Given a null-homologous knot, our process reconstructs one of the two filtrations which are naturally defined on knot Floer homology. One could, therefore, sensibly ask the following:

**Question 1.** Do there exist contact geometric constructions which recover the filtered chain-homotopy types of knot Floer homology and Heegaard Floer homology?

There are good reasons to believe that the answer to this question is "yes". A similar procedure can be used to recover the second filtration defined on knot Floer homology. This lends credence to the belief that similar methods can be applied to recover the entire filtered chain-homotopy types of either knot Floer homology or Heegaard Floer homology. Furthermore, a key feature of the constructions used to establish Theorems 1 and 2 is that the resulting algebraic objects are effectively, combinatorially computable. In joint work with Zarev, I will address the following:

**Question 2.** Do there exist contact geometric constructions which can be used to obtain combinatorial descriptions of $\mathrm{HF}^+$, $\mathrm{HF}^-$ and $\mathrm{HF}^\infty$ (and their knot Floer homology analogues) which are effectively computable?

There is a combinatorial method for computing the Heegaard Floer invariants of closed 3-manifolds [OMT09]. Unfortunately, the computational complexity of this general method makes it ineffective to use in practice. A positive answer to Question 2 should provide a tractable combinatorial method for computing the Heegaard Floer theory of a closed 3-manifold (or knot) and would, therefore, constitute a significant advancement in the field.

In a different direction, one can apply these same ideas and constructions to study other invariants of smooth 3-manifolds and the knots they contain. Specifically, gluing maps similar to those defined by Honda, Kazez and Matić exist in Seiberg-Witten Floer homology and embedded contact homology. Thus, Theorem 1 indicates that analogues of the minus, plus and infinity versions of knot Floer homology should exist within these alternate frameworks. Accordingly, I will investigate the following general question:

**Question 3.** What structures and constructions from Heegaard Floer theory can one mimic in Seiberg-Witten Floer homology and embedded contact homology?

I plan to apply the techniques used to establish Theorem 2 to define Heegaard Floer invariants for noncompact 3-manifolds with cylindrical ends. I further plan to investigate the extent to which these invariants can be used to distinguish and to classify contact structures on other noncompact 3-manifolds. Evidence suggests that a generalized suture in the sense of Theorem 2 for an arbitrary cylindrical ends is equivalent to a measured lamination on the boundary at infinity. This suggests that one can use these general techniques to address:

**Question 4.** What relationships exist between hyperbolic and contact geometry?

In light of Lisca and Stipsicz's recent classification of Seifert-fibered spaces admitting tight contact structures [LS09], bridging the gap between hyperbolic and contact geometry is the last major step in deciphering which closed 3-manifolds admit tight contact structures. I hope to use the study of tight contact structures on non-compact 3-manifolds with cylindrical ends as an avenue towards establishing a connection between hyperbolic and contact geometry.

## 2. Open Book Decompositions of Contact 3-Manifolds

**Background.** The Giroux correspondence theorem [Gir02] is a foundational result establishing a correspondence between contact structures and topological objects called open book decompositions. An *open book decomposition* of a 3-manifold $Y$ is a pair $(B, \pi)$, where $B \subset Y$ is an oriented, fibered link, and $\pi : (Y - B) \to S^1$ is a fibration of the complement of $B$ by oriented surfaces whose oriented boundary is $B$. An open book decomposition $(B, \pi)$ of a 3-manifold $Y$ is said to be *compatible* with a contact structure $\xi$ if $B$ is a transverse link and if, after an isotopy of $\xi$ through contact structures, the contact planes appropriately approximate the tangent spaces to the fibers $\Sigma_\theta = \pi^{-1}(\theta)$.

For a decade, Giroux's theorem has catalyzed a flow of tools, techniques and information between two distinct subfields of mathematics. Despite this, much about how geometric properties of contact structures are reflected in their supporting open books is unknown. Thus, a primary aim in modern contact geometry is to develop a dictionary translating between important contact-geometric concepts or structures and topological properties of supporting open book decompositions. For instance, Akbulut and Ozbagci [AO01] and independently Giroux [Gir02] show that a contact 3-manifold is Stein-fillable if and only if it possesses an open book decomposition whose monodromy can be expressed as a sequence of positive Dehn twists. Another such correspondence is provided by Honda, Kazez and Matić's characterization of tightness in terms of right veering monodromies [HKM07]. Etnyre and I establish the following result, generalizing the main theorem from [Vel11]:

**Theorem 3** (with Etnyre [EV10])**.** *Let $(B, \pi)$ be an open book decomposition for a contact manifold $(Y, \xi)$, then the complement of the binding $(Y - B, \xi|_{Y-B})$ has no Giroux torsion.*

Giroux torsion is the only known mechanism by which a 3-manifold can support more than finitely many tight contact structures [CGH09]. Therefore, understanding how torsion interacts with open books is an important step in deciphering the Giroux correspondence. Theorem 3 tells us that the relationship between torsion and open books is complicated, as the binding of any open book for a contact manifold must intersect all Giroux torsion layers.

Given a contact 3-manifold $(Y, \xi)$, the smallest genus page of an open book supporting $\xi$ is called the *support genus*, and was originally defined by Etnyre and Ozbagci [EO08]. A related invariant, dubbed the *binding number*, is equal to the minimum number of binding components of an open book supporting $\xi$ and realizing the support genus. Etnyre showed that all overtwisted contact structures have support genus equal to zero [Etn04]. Etnyre further proved that support genus is a nontrivial invariant by exhibiting contact structures with support genus equal to one. Since that time, further obstructions to support genus zero have been found through Heegaard Floer theory.

**Description of Plan.** I will continue exploring connections between geometric properties of contact structures and topological properties of open book decompositions. Developing a thorough understanding of the support genus and binding number represents an important step in this program.

**Question 5.** Is the support genus additive under the operation of connected sum? In particular, are there contact structures with support genus greater than one?

If either of the two contact structures involved in the connected sum are overtwisted, then Etnyre's theorem shows that the resulting contact 3-manifold must have support genus zero. Experts expect that overtwisted contact structures provide the only counterexamples to Question 5. I hope to address Question 5 through a combination of convex surface theory and classical 3-dimensional topology. Despite subtle differences in the two problems, it is reasonable to believe that the standard proof of Heegaard genus additivity can be suitably generalized to provide a positive answer to Question 5. Relatedly, in joint work with Baldwin, I hope to address the following related problem:

**Question 6.** Is the binding number of a tight contact structure bounded below by its Giroux torsion number?

One can view Theorem 3 as a first step in establishing a positive answer to Question 6. It provides a concrete link between bindings of open book decompositions and Giroux torsion layers contained in contact 3-manifolds. In a tight contact structure, Giroux torsion manifests along incompressible tori. This places a substantial enough restriction on the underlying geometry that understanding the relationship between torsion and open books is possible.

## 3. Legendrian and Transverse Knots

**Background.** Legendrian and transverse knots feature prominently in the study of contact structures on 3-manifolds. Their study yields subtle information about the spaces they live in, and, like their topological counterparts, provides a rich background structure which one can exploit in various situations. Legendrian and transverse knots also appear in solutions to many seemingly unrelated problems in classical 3-dimensional topology. Therefore, understanding these knots and their invariants is crucial to the advancement of 3 and 4-dimensional geometry and topology.

There are two "classical" invariants of Legendrian knots, the Thurston-Bennequin number and rotation number. Transverse knots possess a single classical invariant, the self-linking number. A smooth knot type is called Legendrian or transversally simple, if two Legendrian (respectively transverse) representatives of that knot type are isotopic if and only if their classical invariants agree. Although there are numerous examples of Legendrian and transversally simple knot types, most are not.

In the past decade, many new Legendrian and transverse invariants have been defined. The first non-classical invariant of Legendrian knots, dubbed Legendrian contact homology (LCH) appeared as an outgrowth of symplectic field theory (SFT) [EGH00]. It was rendered combinatorially computable by Chekanov [Che02] and has proven immensely helpful in discovering knot types which are not Legendrian simple. Further, through its connections with SFT, LCH provides a robust set of tools for understanding contact 3-manifolds in general [EES09]. In joint work with Shonkwiler, I showed:

**Theorem 4** (with Shonkwiler [SV11b])**.** *There exist Legendrian knots with non-maximal Thurston-Bennequin invariant having nonvanishing Legendrian contact homology.*

Prior to this result, it was conjectured that the LCH of a Legendrian knot was nonzero if and only if that knot realized the maximum value for the Thurston-Bennequin number. Our techniques have since been applied by Sivek to compute examples of knot types containing non-destabilizable Legendrian knots with both vanishing and nonvanishing LCH [Siv10b].

In a related direction, Sabloff, Shonkwiler and I have begun studying Ng's abelianized characteristic algebra [Ng01] through the language of bordered Legendrian knots [Siv10a]. Using tools developed by Henry [Hen11], we are able to establish the following result, generalizing prior work of Ng and Sabloff [NS06]:

**Theorem 5** (with Sabloff and Shonkwiler [SSV11])**.** *For a bordered Legendrian knot $K$, there is a surjection from the set of partial Morse complex sequences on $K$ to the set of augmentations of the bordered contact homology $Ch(K)$. In particular, the bordered LCH of $K$ possesses a graded augmentation if and only if $K$ possesses a graded normal ruling.*

Recently, a number of Legendrian and transverse invariants have appeared within the framework of Heegaard Floer theory. The first invariant of Legendrian and transverse knots in $S^3$ (or $\mathbb{R}^3$) was defined using grid diagrams and appeared in the work of Ozsváth, Szabó and Thurston [OST08]. Later, Lisca, Ozsváth, Stipsicz and Szabó used open book decompositions to define an invariant of (null-homologous) Legendrian and transverse knots in arbitrary 3-manifolds [LOSS09]. For several years, it was widely expected that these two invariants agreed where they were simultaneously defined. I recently established this in joint work with Baldwin and Vértesi:

**Theorem 6** (with Baldwin and Vértesi [BVV11])**.** *The Legendrian and transverse invariants defined via grid diagrams* [OST08] *and open book decompositions* [LOSS09] *are equal where they are simultaneously defined.*

To prove Theorem 6, we define yet another invariant of transverse knots which are presented as braids with respect to an open book decomposition. Our invariant can then be characterized in terms of a filtration coming from the pages of this same open book. The key step in the proof of Theorem 6 is to show that both the grid and the LOSS invariant can be similarly characterized. In a different direction, Etnyre, Zarev and I define a Legendrian/transverse invariant $\underrightarrow{\mathrm{EH}}$, which takes values in $\underrightarrow{\mathrm{SFH}}$. Through its construction, it is apparent that the invariant $\underrightarrow{\mathrm{EH}}$ is very closely tied to the geometry of a Legendrian/transverse knot's complement.

**Theorem 7** (with Etnyre and Zarev [EVZ11a])**.** *The Legendrian/transverse invariants* $\underrightarrow{\mathrm{EH}}$ *and* $\mathcal{L}$ *defined in* [EVZ11a] *and* [LOSS09] *respectively are identified under the isomorphism* $\Psi$ *given in Theorem 1.*

Each of the Legendrian/transverse invariants discussed above possess unique attributes which can be applied to address problems in contact geometry. Thus, Theorems 6 and 7 provide a crucial nexus, allowing these characteristics to be applied simultaneously to a given problem.

**Description of Plan.** Moving forward, my research will address a number of questions regarding Legendrian and transverse knots and their relationship with contact structures in general. Over the past several years, we have gained a much clearer picture of what the geography of possible Legendrian and transverse representatives of a given knot type looks like. Despite such progress, there is much work to be done. For instance, the answer to the following question regarding the geography of tb-r mountain ranges is unknown:

**Question 7.** Do there exist topological knot types whose tb-r mountain ranges have low peaks?

Based on computational evidence, Chongchitmate and Ng conjecture the existence of such knot types [CN10]. Although they identify several potential examples, their methods provide no hint to a possible proof. Their examples can all be expressed as negative braids and, therefore, have fibered complements. This fact, together with other geometric restrictions placed on these knots based on their classical invariants, implicates convex surface theory applied to the fiber surface as an effective tool. It further suggests that attention be paid to:

**Question 8.** What are all possible Legendrian representatives of knot types which can be expressed as a negative braids?

Complete classification results are known for only a handful of knot types [EH01, EH03, EF98, ENV10]. Any progress on Question 8 would substantially enlarge the class of knot types for which complete classifications are known. As discussed above, the topological and geometric restrictions placed on Legendrian representatives of negative braids suggest that a complete answer to Question 8 is a reasonable expectation.

Relatedly, one can define Legendrian-theoretic invariants by studying Seifert surfaces. Specifically, let $L \subset (Y, \xi)$ be a null-homologous Legendrian knot and $\Sigma$ a Seifert surface for $L$. This Seifert surface $\Sigma$ can be made convex by an appropriately small perturbation assuming $\mathrm{tb}(L)$ is nonpositive. Kanda first observed that the collection of Legendrian knots which appear as subsets of a given convex surface is, generally speaking, restricted and moderately understandable [Kan98]. Thus, in joint work with Etnyre and Sabloff, I will address the following:

**Question 9.** Can the Legendrian knot theory supported by Seifert surfaces produce effective invariants of Legendrian knots?

We expect to establish a positive answer to Question 9 by studying the Legendrian knot theory supported by convex Seifert surfaces of Legendrian twist-knots.

Additionally, I will address questions in Legendrian knot theory arising from Legendrian contact homology. Ng identified a computable outgrowth of LCH dubbed the "characteristic algebra" [Ng01]. Ng's invariant is useful in studying Legendrian knots and Legendrian contact homology in general. He conjectures that the abelianized version of the characteristic algebra should yield an invariant of topological knots. Though phrased in different language, Theorem 5 and similar results by others can be viewed as small steps toward affirming Ng's conjecture. I will address the following specific version of Ng's conjecture, which generalizes Theorem 5:

**Question 10.** If $L$ is a Legendrian knot with nontrivial abelianized characteristic algebra, must $L$ possess a graded normal ruling?

The techniques used to establish Theorem 5 and, in particular, Sivek's theory of bordered Legendrian knots provide a framework one can apply to this problem. More generally, I hope to characterize Legendrian knots whose Legendrian contact homology is nontrivial (or, equivalently to characterize Legendrian knots with vanishing LCH). Again, Sivek's theory of bordered Legendrian knots can be used to address this problem.

## 4. Helicity and Higher-Order Linking Invariants

**Background.** The Helicity of a vector field $V$ defined on a compact domain $\Omega \subset \mathbb{R}^3$ is computed by the following integral formula:

$$\mathrm{Hel}(V) = \int_{\Omega\times\Omega} V(\mathbf{x}) \times V(\mathbf{y}) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\; d\mathrm{Vol}_{\mathbf{x}}\, d\mathrm{Vol}_{\mathbf{y}}.$$

Helicity was defined by Woltjer as a tool to study the Crab Nebula [Wol58]. Woltjer observed that the quantity $\mathrm{Hel}(V)$ remains constant as $V$ evolves according to the laws of ideal magnetohydrodynamics. It further provides a lower bound for the $L_2$-field energy of $V$ under such evolutions. As a result, helicity has proven an invaluable tool for physicists studying plasma and fluid flows. Unfortunately, there exist vector fields defined on domains $\Omega \subset \mathbb{R}^3$ whose helicity vanishes, but whose $L_2$-energy cannot be made arbitrarily small. Thus, Arnol′d and Khesin posited:

> The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order $\leq k$ have zero value for a given field and there exists a nonzero invariant of order $k+1$, this nonzero invariant provides a lower bound for the field energy. – Arnol′d and Khesin [AK98]

Experts expect that higher helicity invariants correspond to field analogues of higher-order linking invariant. This belief is primarily motivated by a theorem of Arnol′d [Arn86], who showed that ordinary helicity of a field $V$ can be obtained by considering the average asymptotic linking numbers of the flows given by $V$. This belief is further reinforced by considering the following famous formula of Gauß, which computes the linking number of two curves $K = \{x(s)\}$ and $L = \{y(t)\}$ in $\mathbb{R}^3$:

$$\mathrm{Lk}(K,L) = \int_{K\times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x-y}{|x-y|^3}\; ds\, dt.$$

One cannot help but observe the analogy between Gauß's formula for the linking number and the above formula for helicity. It is this analogy that motivates a search for geometrically natural integral expressions which compute higher-order linking invariants. Of these invariants, Milnor's triple-linking number is first in line.

Many have been motivated by this analogy to develop integral expressions to compute Milnor's triple-linking number, and to apply these formulas to the problem of defining higher helicities – most notably in the work of Monastyrsky and Retakh [MR86] and Berger [Ber90]. The principal sources for these formulas are Massey triple products in cohomology, quantum field theory in general, and ChernSimons theory in particular. A common feature of these integral formulas is that choices must be made to fix the domain of integration and the value of the integrand. In [DGK+11], we prove the following:

**Theorem 8** (with DeTurck, et. al. [DGK+11])**.** *If the pairwise linking numbers of a three-component link $L$ in $S^3$ are all zero, then Milnors $\mu$-invariant of $L$ is given by an explicit, geometrically natural integral expression which is evaluated over the 3-torus parametrizing the components of $L$.*

Notably, the formulas obtained in [DGK+11] are explicit, computable and geometrically natural – in contrast with previously obtained integral formulas for $\mu(L)$.

To define our integral expression, we associate to a given 3-component link $L \subset S^3$ a characteristic map $e_L$ from the 3-torus to the configuration space $\mathrm{Conf}(S^3, 3)$ of three points in the 3-sphere. Link-homotopies of $L$ are seen to induce homotopies of the map $e_L$. More generally, if $\mathcal{L}(M, n)$ denotes the set of link-homotopy classes of $n$-component links in a space $M$, then one has an evaluation map

$$e : \mathcal{L}(M, n) \to [T^n, \mathrm{Conf}(M, n)].$$

From the perspective of helicity, it is natural to work with configuration spaces since they remember the geometry of the space $M$. We think of the map $e$ as defining a representation from the world of link-homotopy to the world of homotopy, and one can ask whether or not this representation is faithful.

The evaluation map $e$ was first studied by Koschorke [Kos97] who proved that the map $e$ is faithful in certain very restricted situations. The key step in obtaining Theorem 8 is to establish the following:

**Theorem 9** (with DeTurck, et. al. [DGK+11])**.** *For 3-component links in the 3-sphere and Euclidean 3-space, the evaluation maps $e : \mathcal{L}(S^3, 3) \to [T^3, \mathrm{Conf}(S^3, 3)]$ and $e : \mathcal{L}(\mathbb{R}^3, 3) \to [T^3, \mathrm{Conf}(\mathbb{R}^3, 3)]$ are faithful representations.*

**Description of Plan.** My future research will involve better understanding the topology and geometry of the evaluation map $e : \mathcal{L}(M, n) \to [T^n, \mathrm{Conf}(M, n)]$. When $M = S^3$ and $n = 2$, the configuration space $\mathrm{Conf}(S^3, 2)$ has the homotopy type of $S^3$ and the corresponding evaluation map $e$ is easily seen to be trivial. Experts expect that, other than this single example, the evaluation map is always faithful.

There are two proofs of Theorem 9 which appeare in [DGK+11] (see also [DGK+08]). The first is by way of framed cobordism of links in $T^3$, while the second proof goes by way of link-homotopy of string links and torus homotopy groups. The key idea involved in this second proof is to impose algebraic structures on $\mathcal{L}(S^3, 3)$ and $[T^3, \mathrm{Conf}(S^3, 3)]$ respectively. In this case, the faithfulness of $e$ is equivalent to an algebraic property of an associated collection of group homomorphisms. For general $n$-component links, one expects a similar approach to establish the faithfulness of $e$. Indeed, the same algebraic structures which were used in [DGK+11] in the case $n = 3$ are present for any number of link components. However, for general $n$, these algebraic structures take on a subtle form and establishing the analogue of Theorem 9 requires more delicate analysis. Thus, we plan to address the following:

**Question 10:** Can homotopy string-links and torus homotopy groups be used to establish the effectiveness of the evaluation map?

As more about the effectiveness of the representation $e$ is established, I will turn attention to the development of geometrically natural integral expressions for yet higher-order Milnor $\mu$-invariants and to applying these formulas to the development of higher-order helicity invariants. Komendarczyk has already applied the integral formula from Theorem 8 to obtain a second-order helicity for certain domains in $S^3$ and $\mathbb{R}^3$ [Kom09, Kom10]. There are subtleties in establishing the general form of Theorem 8 which are not present for 3-component links in $S^3$. Thus, we are left with the following:

**Question 11:** Are there geometrically meaningful integral expressions for all of Milnor's $\mu$-invariants, and can these expressions be wielded to obtain higher-order helicity invariants?

Despite its noncompact isometry group, we have already observed how to define a geometric integral expression which compute the triple linking number in Euclidean 3-space. We expect to establish similar formulas for all of Milnor's $\mu$-invariants and to apply these formulas to define higher-order helicity invariants.

## References


[AO01] Selman Akbulut and Burak Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. **5** (2001), 319–334 (electronic).

[Arn86] Vladimir I. Arnold, *The asymptotic Hopf invariant and its applications*, Selecta Math. Soviet. **5** (1986), no. 4, 327–345, Selected translations.

[AK98] Vladimir I. Arnold and Boris A. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences, vol. 125, Springer-Verlag, New York, 1998.

[BVV11] John A. Baldwin, David Shea Vela–Vick, and Vera Vértesi, *On the equivalence of Legendrian and transverse invariants in knot Floer homology*, In Preparation, 2011.

[Ber90] Mitchell A. Berger, *Third-order link integrals*, J. Phys. A **23** (1990), no. 13, 2787–2793.

[Che02] Yuri Chekanov, *Differential algebra of Legendrian links*, Invent. Math. **150** (2002), no. 3, 441–483.

[CN10] Wutichai Chongchitmate and Lenhard L. Ng, *An atlas of Legendrian knots*, `arXiv:1010.3997` [math.SG], 2010.

[Col08] Vincent Colin, *Livres ouverts en géométrie de contact (d'après Emmanuel Giroux)*, Astérisque (2008), no. 317, Exp. No. 969, vii, 91–117, Séminaire Bourbaki. Vol. 2006/2007.

[CGH09] Vincent Colin, Emmanuel Giroux, and Ko Honda, *Finitude homotopique et isotopique des structures de contact tendues*, Publ. Math. Inst. Hautes Études Sci. (2009), no. 109, 245–293.

[DGK+08] Dennis DeTurck, Herman Gluck, Rafal Komendarczyk, Paul Melvin, Clayton Shonkwiler, and David Shea Vela–Vick, *Triple linking numbers, ambiguous Hopf invariants and integral formulas for three-component links*, Mat. Contemp. **34** (2008), 251–283.

[DGK+11] ———, *Pontryagin-Hopf invariants and integral formulas for Milnors triple linking number*, Submitted, `arXiv:1101.3374` [math.GT], 2011.

[EES09] Tobias Ekholm, John B. Etnyre, and Joshua M. Sabloff, *A duality exact sequence for Legendrian contact homology*, Duke Math. J. **150** (2009), no. 1, 1–75.

[Eli91] Yakov Eliashberg, *New invariants of open symplectic and contact manifolds*, J. Amer. Math. Soc. **4** (1991), no. 3, 513–520.

[Eli93] ———, *Classification of contact structures on* $\mathbf{R}^3$, Internat. Math. Res. Notices (1993), no. 3, 87–91.

[EF98] Yakov Eliashberg and Maia Fraser, *Classification of topologically trivial Legendrian knots*, Geometry, topology, and dynamics (Montreal, PQ, 1995), CRM Proc. Lecture Notes, vol. 15, Amer. Math. Soc., Providence, RI, 1998, pp. 17–51.

[EGH00] Yakov Eliashberg, Alexander Givental and Helmut Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal. (2000), no. Special Volume, Part II, 560–673, GAFA 2000 (Tel Aviv, 1999).

[Etn04] John B. Etnyre, *Planar open book decompositions and contact structures*, Int. Math. Res. Not. (2004), no. 79, 4255–4267.

[Etn05] ———, *Legendrian and transversal knots*, Handbook of knot theory, Elsevier B. V., Amsterdam, 2005, pp. 105–185.

[Etn06] ———, *Lectures on open book decompositions and contact structures*, Floer homology, gauge theory, and low-dimensional topology, Clay Math. Proc., vol. 5, Amer. Math. Soc., Providence, RI, 2006, pp. 103–141.

[EH01] John B. Etnyre and Ko Honda, *Knots and contact geometry. I. Torus knots and the figure eight knot*, J. Symplectic Geom. **1** (2001), no. 1, 63–120.

[EH03] ———, *On connected sums and Legendrian knots*, Adv. Math. **179** (2003), no. 1, 59–74.

[ENV10] John B. Etnyre, Lenhard L. Ng, and Vera Vértesi, *Legendrian and transverse twist knots*, Preprint, arXiv:1002.2400v3 [math.SG], 2010.

[EO08] John B. Etnyre and Burak Ozbagci, *Invariants of contact structures from open books*, Trans. Amer. Math. Soc. **360** (2008), no. 6, 3133–3151.

[EV10] John B. Etnyre and David Shea Vela–Vick, *Torsion and open book decompositions*, Int. Math. Res. Not. IMRN (2010), no. 22, 4385–4398.

[EVZ11a] John B. Etnyre, David Shea Vela–Vick, and Rumen Zarev, *Bordered Floer homology and invariants of Legendrian and transverse knots*, In Preparation, 2011.

[EVZ11b] ———, *Contact geometry and Heegaard Floer invariants of noncompact 3-manifolds*, In Preparation, 2011.

[Gau33] Carl Friedrich Gauß, *Integral formula for linking number*, Zur mathematischen theorie der electrodynamische wirkungen (Collected Works, Vol. 5), Koniglichen Gesellschaft des Wissenschaften, Göttingen, 2nd ed., 1833, p. 605.

[Gir02] Emmanuel Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 405–414.

[Hen11] Michael B. Henry, *Connections between Floer-type invariants and Morse-type invariants of Legendrian knots*, Pacific J. Math. **249** (2011), no. 1, 77–133.

[HKM07] Ko Honda, William H. Kazez, and Gordana Matić, *Right-veering diffeomorphisms of compact surfaces with boundary*, Invent. Math. **169** (2007), no. 2, 427–449.

[HKM08] ———, *Contact structures, sutured Floer homology, and TQFT*, Preprint, arXiv:0807.2431 [math.GT], 2008.

[HKM09a] ———, *The contact invariant in sutured Floer homology*, Invent. Math. **176** (2009), no. 3, 637–676.

[HKM09b] ———, *On the contact class in Heegaard Floer homology*, J. Differential Geom. **83** (2009), no. 2, 289–311.

[Kan98] Yutaka Kanda, *On the thurston-bennequin invariant of legendrian knots and nonexactness of bennequin's inequality*, Invent. Math. **133** (1998), no. 2, 227–242.

[Kom09] Rafal Komendarczyk, *The third order helicity of magnetic fields via link maps*, Comm. Math. Phys. **292** (2009), no. 2, 431–456.

[Kom10] ———, *The third order helicity of magnetic fields via link maps. II*, J. Math. Phys. **51** (2010), no. 12, 122702, 16pp.

[Kos97] Ulrich Koschorke, *A generalization of Milnor's $\mu$-invariants to higher-dimensional link maps*, Topology **36** (1997), no. 2, 301–324.

[LOT08] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston, *Bordered Heegaard Floer homology: Invariance and pairing*, Preprint, arXiv:0810.0687v4 [math.GT], 2008.

[LOSS09] Paolo Lisca, Peter Ozsváth, András I. Stipsicz, and Zoltán Szabó, *Heegaard Floer invariants of Legendrian knots in contact three-manifolds*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 6, 1307–1363.

[LS09] Paolo Lisca and András I. Stipsicz, *On the existence of tight contact structures on Seifert fibered 3-manifolds*, Duke Math. J. **148** (2009), no. 2, 175–209.

[Mas58] William S. Massey, *Some higher order cohomology operations*, Symposium internacional de topología algebraica International symposium on algebraic topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 145–154.

[Mas69] ———, *Higher order linking numbers*, Conf. on Algebraic Topology (Univ. of Illinois at Chicago Circle, Chicago, Ill., 1968), Univ. of Illinois at Chicago Circle, Chicago, Ill., 1969, pp. 174–205.

[Mil54] John Milnor, *Link groups*, Ann. of Math. (2) **59** (1954), no. 2, 177–195.

[MR86] Michael I. Monastyrsky and Vladimir S. Retakh, *Topology of linked defects in condensed matter*, Comm. Math. Phys. **103** (1986), no. 3, 445–459.

[Ng01] Lenhard L. Ng, *Invariants of Legendrian links*, Ph.D. thesis, Massachusetts Institute of Technology, 2001.

[NS06] Lenhard L. Ng and Joshua M. Sabloff, *The correspondence between augmentations and rulings for Legendrian knots*, Pacific J. Math. **224** (2006), no. 1, 141–150.

[Ni07] Yi Ni, *Knot Floer homology detects fibred knots*, Invent. Math. **170** (2007), no. 3, 577–608.

[OMT09] Peter Ozsváth, Ciprian Manolescu, and Dylan Thurston, *Grid diagrams and Heegaard Floer invariants*, Preprint, arXiv:0910.0078v2 [math.SG], 2009.

[OS10] Peter Ozsváth and András I. Stipsicz, *Contact surgeries and the transverse invariant in knot Floer homology*, J. Inst. Math. Jussieu **9** (2010), no. 3, 601–632.

[OS04a] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334.

[OS04b] ———, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), no. 1, 58–116.

[OS04c] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158.

[OS05] ———, *Heegaard Floer homology and contact structures*, Duke Math. J. **129** (2005), no. 1, 39–61.

[OS06] ———, *The Dehn surgery characterization of the trefoil and the figure eight knot*, Preprint, arXiv:math/0604079v1 [math.GT], 2006.

[OST08] Peter Ozsváth, Zoltán Szabó, and Dylan Thurston, *Legendrian knots, transverse knots and combinatorial Floer homology*, Geom. Topol. **12** (2008), no. 2, 941–980.

[Pon41] Lev S. Pontryagin, *A classification of mappings of the three-dimensional complex into the two-dimensional sphere*, Rec. Math. [Mat. Sbornik] N. S. **9** (1941), no. 51, 331–363.

[SSV11] Joshua M. Sabloff, Clayton Shonkwiler, and David Shea Vela–Vick, *Rulings invariants and augmentations or bordered Legendrian knots*, In Preparation, 2011.

[SV11a] Clayton Shonkwiler and David Shea Vela-Vick, *Higher-dimensional linking integrals*, Proc. Amer. Math. Soc. **139** (2011), no. 4, 1511–1519.

[SV11b] ———, *Legendrian contact homology and nondestabilizability*, J. Symplectic Geom. **9** (2011), no. 1, 33–44.

[Siv10a] Steven Sivek, *A bordered Chekanov-Eliashberg algebra*, Preprint, arXiv:1004.4929v1 [math.SG], 2010.

[Siv10b] ———, *The contact homology of Legendrian knots with maximal Thurston-Bennequin invariant*, Preprint, arXiv:1012.5038 [math.SG], 2010.

[Siv11] ———, Preprint, arXiv:1107.6028 [math.SG], 2011.

[SV09] András I. Stipsicz and Vera Vértesi, *On invariants for Legendrian knots*, Pacific J. Math. **239** (2009), no. 1, 157–177.

[TW75] William P. Thurston and Horst E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345–347.

[Tor00] Ichiro Torisu, *Convex contact structures and fibered links in 3-manifolds*, Internat. Math. Res. Notices (2000), no. 9, 441–454.

[Tri06] James J. Tripp, *Contact structures on open 3-manifolds*, J. Symplectic Geom. **4** (2006), no. 1, 93–116.

[Vel09] David Shea Vela-Vick, *Ozsváth-Szabó invariants and contact geometry*, Ph.D. thesis, University of Pennsylvania, 2009.

[Vel11] ———, *On the transverse invariant for bindings of open books*, To appear in J. Differential Geom., arXiv:0806.1729 [math.SG], 2011.

[Whi47] John H. C. Whitehead, *An expression of Hopf's invariant as an integral*, Proc. Natl. Acad. Sci. USA **33** (1947), no. 5, 117–123.

[Wol58] Lodewijk Woltjer, *A theorem on force-free magnetic fields*, Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 489–491.

[Zar09] Rumen Zarev, *Bordered Floer homology for sutured manifolds*, Preprint, arXiv:0908.1106v2 [math.GT], 2009.

[Zar10] ———, *Joining and gluing sutured Floer homology*, Preprint, arXiv:1010.3496v1 [math.GT], 2010.

[Zar11] ———, *Equivalence of gluing maps for sutured Floer homology*, In Preparation, 2011.