Weakly étale v.s. pro-étale.

I. Ind/Weakly-étale  II. Statements and consequences  III. pf.

I. A \rightarrow B ring map is called:

- Ind-étale if B is a filtered colimit of étale A-alg.
- Weakly-étale if A \rightarrow B is flat & B \otimes_A B \rightarrow B is also flat.

Defn. A \subset R ideal is called pure if R/I is flat - R-alg.

Lemma. If I is a pure ideal, then \forall J \subset R ideal, I.J = I \cap J.

pf. Consider \[(0 \rightarrow J \rightarrow \mathbb{R} \rightarrow R/J \rightarrow 0) \otimes R \rightarrow \mathbb{R} \rightarrow \frac{R}{I/J} \rightarrow 0\]

Cor. Apply. If A \rightarrow B is weakly étale, then \mathcal{S}^1_{B/A} = 0.

Ex: k \rightarrow T_k is weakly étale if k is a finite field.

weakly étale + finite presentation = étale.

Lemma. If f is ind-étale, then it's naïve Ind-étale \implies weakly étale.

pf. Consider: flat map is flat\[\dim B_i \otimes B_i \rightarrow B_i \rightarrow B_i^{\text{flat}}\]

Lemma. If f & g of are ind-étale, then g of is ind-étale.

pf. ind-étale case can be reduced to étale case.

weakly-étale case:

\[A \rightarrow B \otimes A \rightarrow B \rightarrow C \rightarrow D \rightarrow \mathbb{R} \rightarrow 0\]
*Lemma.*

\[
\begin{array}{c}
B \xrightarrow{f} \frac{f\circ f}{C} \\
\text{-flat} \\
\text{A-flat}
\end{array}
\]

*pp.*

commutative diagram:

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{\text{flat}} & \text{C} \\
\text{flat} & \uparrow & \text{flat} \\
\text{B} & \text{flat} & \text{flat} \\
\text{A-flat}
\end{array}
\]

Defn. \(d \leq d\), we say \(A\) has weak dim \(\leq d\) if every \(A\)-module has tor dim \(\leq d\). (equivalently, admits flat resolution of length \(d\))

\[A\text{ has weak dim } \leq d \text{ is also called absolutely flat}
\]

"all the modules are flat". (equivalent to \(A\) being

\[
\begin{array}{c}
\text{1. reduced} \\
\text{2. 0-dim.}
\end{array}
\]

Fact:

\[A \rightarrow B \text{ weakly étale } \Rightarrow B \text{ has weak dim } \leq d.
\]

Main Thm:

\[\text{Thm A: } f: A \rightarrow B \text{ weakly étale. Then } \exists B \rightarrow C \text{ faithfully flat and ind-étale}
\]

s.t. \(A \rightarrow C\) is ind-étale.

\[\text{Cor.}
\]

Weakly étale & pro-étale gives rise to the same topos on the category of affine schemes:

\[\text{U pro-ét } \rightarrow \text{V pro-ét}
\]

\[\text{W pro-ét } \rightarrow \text{W pro-ét}
\]

\[\text{W pro-ét } \rightarrow \text{W pro-ét}
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\]
It's easy to see that Thm A implies:

Thm B: A is a strictly henselian local ring, B is a weakly étale local A-alg. Then \( f: A \to B \) is an isom.

Thm C: Any weakly étale \( K \)-alg. \( A \) is a \( \text{ind-étale}/K \).

Example: \( k \xrightarrow{N} k \) is weakly étale iff \( k \) is a finite field.

Proof of Thm C: It suffices to show any \( f \)-g. \( K \)-sub-alg. \( A' \) is \( \text{étale}/K \).

Go to top of page 2, implies \( \forall f \in A \).

A has weak dim'n 0 \( \implies (f) = (f)^2 \) (since \( f \) is a prime ideal).

Every local ring \( A \) is reduced & \( (f) = (e) \) where \( e \) is an idempotent of \( A \) is a field, every prime is max'll.

\( k \hookrightarrow A \hookrightarrow A', \forall \) minimal prime \( p' \subseteq A' \), \( \exists \) prime \( p \subseteq A \) above it.

Consider \( k \hookrightarrow k(p') \hookrightarrow k(p) \triangleq A_p \) still weakly étale.

Go to top page 2, implies \( k \hookrightarrow k(p') \) is weakly étale but it's also fp. hence étale. (sep. alg. extn).

So every generic pt of \( \text{Spec}(A') \) is closed, now \( A' \) is finite type \( /K \), reduced \( \forall \) generic pt an sep alg. extn of \( K \).
Thm C: Defn. A topological space \( X \) is w-local if
- A ring \( A \) is w-local if every counit component of \( \text{Spec}(A) \) contains a unique closed point.
- The set of closed points \( \text{Spec}(A)^c \) is itself a closed subset of \( \text{Spec}(A) \).
- A ring homomorphism of w-local rings \( A \to B \) is w-local if \( A, B \) w-local, \( A \to B \) \( f \) (max ideals in \( B \)) \( \to \) are max
  \( f \) (max ideal) is always a max ideal in \( A \).
- A w-local ring is w-strictly local if
  1. it's w-local and
  2. all the local rings of closed pts are strictly henselian.

Lemma. If \( A \) is a w-local rings, then
1. any Zariski cover \( U = \bigcup U_i \to \text{Spec}(A) \) admits a section.
2. \( (\text{Spec}(A))^c \to \text{Spec}(A) \to \text{top} (\text{Spec}(A)) \)

Proof:
1. since any open cover of w-local is an open cover of \( \text{top} (\text{Spec}(A)) \), which is profinite.
   (To see \( \text{top} (\text{Spec}(A)) \) is always profinite, use Hochster's Thm: \( \text{Spec}(A) \) is always a cofiltered limit of finite \( \text{Spec}(A) \) s).
2. by assumption: \( \text{Spec}(A)^c \to \text{top} (\text{Spec}(A)) \) is a bijective continuous map between compact Hausdorff spaces.
   (an affine scheme is 0-dim \( \Rightarrow \) Hausdorff), hence a homeo.

Fact: The inclusion of rings \( \{ \text{w-local} \} \to \{ \text{rings} \} \) admits a left adjoint.
\( A^2 \to A \)
where \( A \to A^2 \) counit is an ind (Zariski localization).
Lemma \quad \text{w-local ring } A \text{ is w-strictly local iff all local rings of } A \text{ at closed points are strictly henselian.}

\text{if } A \text{ is w-strictly local, } A_x \xrightarrow{\text{f.f.}} B_x \text{ w.f. invertible at } x \text{ equivalently } (f \neq 0, f \in A).

\text{hence } \exists h \in M \text{ w. } h + fg = 1. \text{ Consider } A_x \xrightarrow{f,f} B_x \times A[\frac{1}{h}].

\text{hence get } A_x \xrightarrow{h} B_x \times A', \text{ think about the image of } (b,h).

Conversely, \quad A \xrightarrow{f,f} B, \text{ at every closed pt } x \in \text{Spec}(A), B_x.

\text{have section locally around each } x \in \text{Spec}(A).

\exists \text{ a Zariski cover of } \text{Spec}(A), \text{ over which } \text{Spec}(B) \xrightarrow{f,f} \text{ has a section. By w-locality, we find a section } B \rightarrow A[\frac{1}{f}]. \quad B \rightarrow \bigoplus A[\frac{1}{f_i}]

\rightarrow A.

Key Lemma: \quad \text{(write on the back or side bound.)}

Let f: A \rightarrow B. Then \exists \begin{array}{c} B \rightarrow B' \quad \text{w.} \\ A \rightarrow A' \end{array}

\item A \rightarrow A' & B \rightarrow B' \text{ are F.f. faithfully flat and ind-étale,}
\item A' & B' \text{ are w-strictly local}
\item A' \rightarrow B' \text{ is w-local, inducing homeomorphism}

\text{Spec}(B')^c \cong \text{Spec}(A')^c.

\text{A and } B \text{ have strictly henselian.
pf of Thm B: Thm A: 

\[ \begin{align*}
  B & \xrightarrow{\text{ind.-ét}} B' \\
  \text{w.-ét} & \uparrow \quad \uparrow \\
  A & \xrightarrow{\text{ind.-ét}} A' \\
  \text{f.f.} & \quad \text{f.f.}
\end{align*} \]

Conditions imply: \( A' \to B' \) is weakly étale, 

\[ \forall m \in A', \quad \frac{B'}{mB'} \text{ is weakly étale over } \frac{A'}{mB} \text{, hence ind.-étale, but having a (Thm C) unique closed pt), hence } mB' \text{ is the unique max'l ideal above } m. \]

Consider \( A'_m \to B'_mB' \), it's an isom by \( \square \text{Thm B} \).

As all the max'l ideal in \( B' \) are of the form \( mB' \), we see that \( B' \xrightarrow{\sim} A' \) is also bijection on pts. Hence \( B'_mB' \xleftarrow{\sim} B'_m \), therefore \( A' \cong B' \). \( \square \)

Now back to page 4.

\[ \begin{align*}
  B & \to B^z \\
  \uparrow & \quad \uparrow \\
  A & \to A^z
\end{align*} \]

these are w-local, w/ Jacobson ideal cutting out the closed set of closed pts.

Constructing \( A' \& B' \):

\[ \begin{align*}
  B^z & \to B^z/J_B = B_0 \\
  \uparrow & \quad \uparrow \\
  A^z & \to A^z/J_A = A_0
\end{align*} \]

Lemma: For \( A_0 \to B_0 \) map of absolutely flat rings, \( \exists \text{ ind.-étale } A_0 \to A_0' \), s.t. \( A_0 \& B_0 \) are w-strictly local, cut out set of closed pts.

pf: \( A_0 = \text{colim } A_f \otimes A_J \), where \( J \) finite subset of \( I := \{ \} \), sf étale \( \Lambda_0 \)-algebra \( \Lambda_0 \).

Do the same construction for \( A_0 \otimes B_0 \).
Lemma 12.4.4. $(\mathbb{A}^2, \mathbb{A})$ is a topos and satisfies all terminal objects.

Proof. No details.

Lemma 12.4.5. $(\mathbb{A}^2_{\mathbb{A}}, \mathbb{A})$ is w-strictly local.

Proof. No details.

A topos is $\mathbb{A}$-local, no further details.

Construction. $A \to \mathbb{A}_{\mathbb{A}}$ sends countable $\mathbb{A}$-algebras to $A$.

This is a functor $A \to \mathbb{A}$ whose domain is indexed by

$\mathbb{A}$.

Follows from the fact that any $\mathbb{A}$-algebra $A$ satisfies

$A \to \mathbb{A}_{\mathbb{A}}$.

follows from the fact that $\mathbb{A}$ is an algebra.

A $\mathbb{A}$-algebra $A$ is unique.

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$\mathbb{A}$.
almost there. Check this satisfies all the conditions except for \( 3 \), \( \text{Spec} (B') \to \text{Spec} (A'') \to \text{Spec} (A'') \text{ is not} \ \\
we have no control so far!

Lemma/Construction: Any \( w \)-local map \( f : A'' \to B' \) of \( w \)-local rings admits a canonical factorization \( A'' \to A' \to h : B' \to w \)

1. \( A' \) \( w \)-local
2. \( g \) is a \( w \)-local ind-(Zariski localization)
3. \( h \) is a \( w \)-local map inducing \( \pi_0 (\text{Spec} (B')) = \pi_0 (\text{Spec} (A'')) \)

\( \text{pf.} \) \( \text{Spec} (B') \to \text{Spec} (B') \to \pi_0 (\text{Spec} (B')) \) any map between profinite sets is \( \text{Spec} (A'') \to \pi_0 (\text{Spec} (A'')) \text{ is not} \) \text{pro}-(Zariski localization)

Overview:

\( B \to B' \)
\( A \to A' \)

OK, too tired, I don't wanna say how to get

\( \text{Thm B from Thm C} \)
Lemma. If \( A \to B \) is weakly étale, then \( \text{LL}_{B/A} \cong 0 \).

pf. \( B \otimes B_A \to B \) with \( \ker = \ker^2 \Rightarrow \text{LL}_{B/B \otimes B_A} \cong 0 \).

A \to B \otimes B_A \to B \) gives: \( \text{LL}_{B/A} \cong \text{LL}_{B \otimes B/A} (B \otimes B_A) \). \( \uparrow \) \( \uparrow \)

A \to B \otimes B_A \text{ gives: } \text{LL}_{B/A} \otimes (B \otimes B_A) \cong \text{LL}_{B \otimes B/A} \). \( \uparrow \)

and \( \text{LL}_{B \otimes B/A} \longrightarrow \text{LL}_{B \otimes B_A} \).

\( \quad \text{triangle } \Rightarrow \text{LL}_{B/A} \otimes \cong 0 \)

\( \text{Claim: } A \to B \) local hom. of local rings, weakly étale and \( A \) is sh.

for all \( \varphi \in A \), \( \exists ! \varphi' \in B \) above it w/ \( \kappa(\varphi) = \kappa(\varphi') \).

This implies \( B \otimes B_A \to B \) is bijective on Spec and flat.

Hence kernel = pure\& nilpotent ideal \( = 0 \).

But then \( B \cong B \otimes B_A \), and \( A \to B \), hence \( A \cong B \).

To prove the claim, by Thm \( C \), \( B \otimes \kappa(\varphi) \) is ind-étale over \( \kappa(\varphi) \).

Suppose the claim is wrong, then \( \exists L/\kappa(\varphi) \) alg. sep. field extn.

s.t. \( B \otimes L \) has a nontrivial idempotent.

Now suppose such \( L \) exists, consider \( A' = \text{int'l closure of } A/\varphi \) in \( L \).

\( A' \) being integral \& over \( A \) and domain, must be sh. also.
And $A \to A'$ induces purely inseparable extension on residue fields.

Hence: $B \otimes A'$ is also a local ring.

Lemma/Fact: $A'$ is a normal ring domain w/ fraction field $L$, and $A' \to B'$ is weakly étale. Then $B'$ is integrally closed in $B' \otimes L$.

We apply this lemma to $A'$, $L$ and $B' = B \otimes A'$.

$\Rightarrow B \otimes A'$ is integrally closed in $B \otimes L$.

$\Rightarrow$ # nontrivial idempotents in $B \otimes L$, as $B \otimes A'$ is a local ring.