Notes on diamonds.

Speaker: David Hansen

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diamonds.

Schemes $\rightsquigarrow$ alg spaces $\longrightarrow$ alg stacks

adic spaces $\rightsquigarrow$ diamonds $\rightsquigarrow$ diamond stacks

$Sh(\widetilde{Perf}^{pro\acute{e}t})$

Defn. • A morphism $f: Y = Spa(B,B^+) \longrightarrow X = Spa(A,A^+)$ of aff pftd is <u>affinoid pro-étale</u> if $\exists$ cofiltered inverse system of aff pftd spaces $Y_i = Spa(B_i, B_i^+) \xrightarrow{f_i} X$ s.t. each f_i is étale and $(B,B^+) \xleftarrow{\sim} \widehat{\varinjlim_i (B_i, B_i^+)}$

$$\begin{tikzcd} Y_i \arrow[r, "f_i"] & X \\ Y \arrow[u, "g_i"] \arrow[ur] & \end{tikzcd}$$

• A morphism $f: Y \longrightarrow X$ of pftd spaces is pro-étale if $\forall y \in Y$ $\exists$ open aff pftd subspaces $V \subseteq Y$ and $U \subseteq X$ s.t. $y \in V$, $f(V) \subseteq U$ and $f|_V: V \longrightarrow U$ is aff pro-étale.

Warning this topology is much finer than the old notion of a pro-étale morphism; cptble in the sense that $Y \longmapsto \hat{Y}$ is fully faithful: $X_{pro\acute{e}t, old} \longrightarrow$ (pro-étale maps $Y \to X$).

However, if X aff pftd and $x \in X$ is a rk 1 pt, then $x \hookrightarrow X$ is aff proét.

Basic Sanity checks:

• If $f: Y \longrightarrow X$ is proétale and $X' \longrightarrow X$ is any map from a pftd space X', then $Y' = Y \times_X X' \longrightarrow X'$ is proétale

• compositions of pro-étale maps are pro-étale

• $\forall$ diagram
$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$
w/ f, f' proétale then g is pro-étale.

Defn. the big ~~étale~~ proétale site has underlying cat. of all pftd spaces w/ $\{U_i \longrightarrow X\}_{i \in I}$ a cover iff $\forall$ qc opens $V \subseteq X$, $\exists$ finite subset $I_V \subseteq I$ & qc opens $W_i \subseteq U_i$ $\forall i \in I_V$, s.t. $f(W_i) \subseteq U_i$ and $\bigcup_{i \in I_V} f_i(|W_i|) = |V|$

Notation $\widetilde{Perf} = pftd/\mathbb{Z}_p$, $\widetilde{Perf}^{pro\acute{e}t}$ $\quad Perf = pftd/\mathbb{F}_p$, $Perf^{pro\acute{e}t}$

Note tilting is not f.f. as a functor $\widetilde{Perf} \longrightarrow Perf$. However, if $S \in \widetilde{Perf}$ is fixed, have an eq. $\widetilde{Perf}_{/S} \cong Perf_{/S^\flat}$

For any analytic adic space $X/\mathbb{Z}_p$, get a presheaf on $\widetilde{Perf}$ sending $S \in \widetilde{Perf}$ to $Hom_{adic}(S, X)$. Call this F_X.

Thm F_X is a proétale sheaf for any X.

Cor. $\widetilde{Perf}^{pro\acute{e}t}$ and $Perf^{pro\acute{e}t}$ are subcanonical: $h_X = Hom(-,X)$ is a sheaf $\forall X \in$ ~~rep~~ respective site

Lemma Suppose X is pftd and $\tilde{X} \longrightarrow X$ is a proétale cover, then $|\tilde{X}| \longrightarrow |X|$ is a quotient map.

pf. this is local on $|X|$, so WLOG X is affinoid. Since $\tilde{X} \longrightarrow X$ is a proétale cover, $\exists$ a qc open $V \subseteq \tilde{X}$ s.t. $V \longrightarrow X$ is a proétale cover. We may write $V = \bigcup_{i \in I, \text{ finite}} V_i$, V_i aff pftd, reduces to $\coprod_{i \in I} V_i \longrightarrow X$

Then apply the following

Lemma if $Y \longrightarrow X$ is any map of aff analytic adic spaces s.t. $|Y| \longrightarrow |X|$ is surj, then $|Y| \longrightarrow |X|$ is a quotient.

pf. $|Y| \to |X|$ is surj, spectral (preimage of qc open is qc open) and generalizing ($\forall x \prec \tilde{x} \in |X|$ and $\forall y \in f^{-1}(x)$, $\exists \tilde{y} \in f^{-1}(\tilde{x})$ s.t. $y \prec \tilde{y}$). Then we reduce to the following

Lemma any surj generalizing spectral map of spectral spaces is a quotient map

pf. Lemma 3.2 of H., "Qts of adic spaces by finite gps."

Defn If X is a pftd space, write $X_{\text{proét}}$ = cat. of pftd spaces proét $/X$ w/ covers = proét covers

We have a natural map
$$\begin{tikzcd} X_{\text{proét}} \arrow[r, "\nu"] \arrow[dr] & X_{\text{ét}} \\ & X_{\text{proét, old}} \arrow[u] \end{tikzcd}$$

Prop. Let X be any pftd space, and $\nu: X_{\text{proét}} \to X_{\text{ét}}$, then for any F abelian sheaf on $X_{\text{ét}}$, $F \to \nu_* \nu^* F$ is an isom., and $R^i \nu_* \nu^* F = 0 \ \forall i \geq 1$. ∎

Prop. Same setup: if $X_0 \subseteq X$ is affinoid and $Y = \varprojlim_i Y_i \to X_0$ is aff. proétale, then $(\nu^* F)(Y) = \varinjlim_i F(Y_i)$.

Prop. if X is pftd, the presheaf on $X_{\text{proét}}$ sending $Y \to X$ to $\mathcal{O}_Y(Y)$ is a sheaf, and likewise for $\mathcal{O}^+$.

pf. WLOG, X is aff. pftd. Consider $X^{\text{aff}}_{\text{proét}} \subseteq X_{\text{proét}}$ (site of $Y \to X$ aff. proét). They define equivalent topoi.

fix $\varpi \in \mathcal{O}^+_X(X)$, then $\forall n \geq 1$, have $\widetilde{\mathcal{O}}^+/\varpi^n \ (Y = \varprojlim Y_i \to X)$
$= \varinjlim_i \mathcal{O}^+_{Y_i}(Y_i)/(\varpi^n)$

Since $\mathcal{O}^+/\varpi^n$ is an almost sheaf on $X_{\text{ét}}$ $\forall n \geq 1$, w/ almost vanishing higher coh. (using some almost version of prop above) we get $\nu^*(\mathcal{O}^+/\varpi^n) \overset{a}{=} \widetilde{\mathcal{O}}^+/\varpi^n$.

So $\mathcal{O}_{X,\text{proét}} := \varprojlim_n (\widetilde{\mathcal{O}}^+/\varpi^n)[\frac{1}{\varpi}]$ is a <u>sheaf</u> w/ vanishing higher cohomology. Then one checks that $\mathcal{O}_{X,\text{proét}}(Y) = \mathcal{O}_Y(Y)$ $\forall Y \in X^{\text{aff.}}_{\text{proét}}$

Thm F_X is a sheaf on $\widetilde{\text{Perf}}^{\text{proét}}$ $\forall$ adic space X

pf Suppose given $T \in \widetilde{\text{Perf}}$, $\widetilde{T} \to T$ a proétale cover. and $f: \widetilde{T} \to X$ s.t. want to construct g s.t. f factors
$$\begin{tikzcd} \widetilde{T} \times_T \widetilde{T} \arrow[d, shift left] \arrow[d, shift right] \arrow[drr, "\widetilde{f}"] & & \\ \widetilde{T} \arrow[rr, "f"] \arrow[d] & & X \\ T \arrow[urr, dashed, "g"'] & & \end{tikzcd}$$

First, assume X is affinoid, then $\text{Hom}(Y, X) = \text{Hom}((\mathcal{O}_X(X), \mathcal{O}^+_X(X)), (\mathcal{O}_Y(Y), \mathcal{O}^+_Y(Y)))$ and because $\mathcal{O}_*, \mathcal{O}^+_*$ are sheaves on proétale site, we're done.

In general, notice that $|g|: |T| \to |X|$ already exists since $\text{Coeq}(|\widetilde{T} \times_T \widetilde{T}| \rightrightarrows |\widetilde{T}|) = \text{Coeq}(|\widetilde{T}| \times_{|T|} |\widetilde{T}| \rightrightarrows |\widetilde{T}|) \cong |T|$ b/c $|\widetilde{T}| \to |T|$ is a quotient map.
Then we can construct g locally on X.

Defn
- A morphism $F \to G$ of sheaves on $\text{Perf}^{\text{proét}}$ is representable if $\forall T \in \text{Perf}$ and any map $h_T \to G$, then $h_T \times_G F = h_S$ for $S \in \text{Perf}$.
- we can define properties of morphisms ∎.

Defn. A proétale equivalence relation on a pftd space $X \in \text{Perf}$ is a triple (R, s, t) where R is a pftd space and s, t are proétale morphisms $R \to X$ s.t. $\forall T \in \text{Perf}$,
$(s,t): R(T) \hookrightarrow X(T) \times X(T)$ is an eq. relation.

Defn. A diamond is a sheaf D on $\text{Perf}^{\text{proét}}$ s.t. $D \simeq \text{Coeq}(h_R \rightrightarrows h_X)$ for some proét eq. $R \rightrightarrows X$ of pftd spaces $\in \text{Perf}$.

Prop. D is a sheaf on $\text{Perf}^{\text{proét}}$ s.t. $\exists X \in \text{Perf}$ & a surj. representable proétale map $h_X \to D$, then D is a <u>diamond</u>.

pf. by surjectivity $D \cong \text{Coeq}(h_X \times_D h_X \underset{t}{\overset{s}{\rightrightarrows}} h_X)$ $\quad h_X \times_D h_X = h_R$.

Defn If D is a diamond w/ surj. repble proétale $h_X \to D$, we call D a <u>effective diamond</u>.

Warning it's possible that there $\exists$ diamond $h_R \rightrightarrows h_X$ s.t. $h_X \to D$ is not representable

Thm ~~Given~~ Given $X, Y \in Perf$, $\exists$ a natural pfd space "$X\times Y$" w/ maps to X & Y s.t. $(X\times Y)(T) \cong X(T)\times Y(T)$.

sketch of pf reduce to X,Y affinoid: $X = Spa(A,A^+)$ $Y = Spa(B,B^+)$.
Set $C = A \underset{\mathbb{F}_p}{\otimes} B$, and let C^+ be the integral closure of $A^+ \underset{\mathbb{F}_p}{\otimes} B^+$ in C

Then choose pseudouniformizers $\varpi_A \in A^+$, $\varpi_B \in B^+$ and let D^+ be $I = (\varpi_A, \varpi_B)$-adic cpltion of C^+
Then $X\times Y := Spa(D^+, D^+) \setminus V(\varpi_A \varpi_B)$.

Cor. If D_1, D_2 are diamonds, then so is $D_1 \times D_2$
pf. Write $D_i = Coeq(h_{R_i} \underset{t_i}{\overset{s_i}{\rightrightarrows}} h_{X_i})$ then set $Y = X_1 \times X_2$,
$S = R_1 \times R_2$ $s' = (s_1, s_2)$, $t' = (t_1, t_2)$.
Then $D_1 \times D_2 = Coeq(h_S \underset{t'}{\overset{s'}{\rightrightarrows}} h_Y)$
to check s' and t' are proétale:
e.g. $R_1 \times R_2 \to X_1 \times R_2 \to X_1 \times X_2$.

Def. if D is a diamond, an <u>open subdiamond</u> is a subfunctor $\mathcal{E} \subseteq D$ s.t. $\mathcal{E} \to D$ is representable open immersion.

Thm open subdiamonds are indeed diamonds.
pf. choose $D \cong Coeq(h_R \rightrightarrows h_X)$ and set $h_U = \mathcal{E} \times_D h_X$.
so U is an open immersion and $h_U \to \mathcal{E}$ is surjective.
$S = R \underset{(X\times X)}{\times} (U \times U)$, then $h_U \times_{\mathcal{E}} h_U = h_S$.
$h_S = \mathcal{E} \times_D h_R$

Prop& Defn. If D is a diamond w/ an isom $D \simeq Coeq(h_R \rightrightarrows h_X)$, we set $|D| = |X|/\sim_{|R|} = Coeq(|R| \rightrightarrows |X|)$ w/ the quotient top.

Given 2 presentations $D = Coeq(h_{R_i} \rightrightarrows h_{X_i})$ $i = 1,2$.
$\exists$ natural homeo. $|X_1|/\sim_{|R_1|} \cong |X_2|/\sim_{|R_2|}$

in particular, $|D|$ is well-defined.

Rmk $D \mapsto |D|$ is the "left Kan extn" of $X \mapsto |X|$ along

$$\begin{tikzcd} Perf \arrow[r, "|\cdot|"] \arrow[d] & top \\ Diam \arrow[ur] & \end{tikzcd}$$

Prop. $\exists$ a natural inclusion preserving bijection $\{$open diamonds $\mathcal{E} \subseteq D\}$ $\cong$ $\{$open subsets $|\mathcal{E}| \subseteq |D|\}$.
pf. $\mathcal{E} \subseteq D$ open, then we find $U \subseteq X$ open & $|R|$ stable
So it descends to an open subspace $|\mathcal{E}| \subseteq |D|$

Conversely, given $W \subseteq |D|$. we see that $\mathrm{preim}(|X| \to |D|)(W)$ is open and $|R|$ stable, so comes from $U \subseteq X$ and $S \subseteq R$.
set $\mathcal{E}_W = Coeq(h_S \rightrightarrows h_U)$. which is a subfunctor.
On the other hand, we see that for $h_T \to D$,
$h_T \times_D \mathcal{E} = h_T \times_D |W|$ and b/c morphisms between diamonds are continuous, we're done

Lemma $D \to \Sigma$ maps of diamonds, then $|D| \to |\Sigma|$ is continuous.
pf It suffices to check on

$$\begin{tikzcd} X \arrow[d] & Y \arrow[d] \\ D \arrow[r] & \Sigma \end{tikzcd}$$

, but then for some $X \to D$, we can find $X \to Y$ which is continuous.

Thm Given $W \subseteq |D|$ open, $\mathcal{E}_W(T) = \{$maps $h_T \to D$ $\mid$ $|T|$ mapped in $|W|\}$

Immersion of diamond.

Defn. Fix a diamond $\mathcal{D}$. A subfunctor $\mathcal{E} \subseteq \mathcal{D}$ is an immersion if $\mathcal{E} \hookrightarrow \mathcal{D}$ factors thru an isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{D} \times_{|\mathcal{D}|} S$ for some locally closed generalizing subset $S \subseteq |\mathcal{D}|$.

Thm If $\mathcal{E} \subseteq \mathcal{D}$ is an immersion, then $\mathcal{E}$ is a diamond, and if $\mathcal{E} \cong \mathcal{D} \times_{|\mathcal{D}|} S$ then $|\mathcal{E}| \to |\mathcal{D}|$ factors thru a homeo $|\mathcal{E}| \cong S$.

In particular, $\mathcal{E}$ determines S uniquely. Moreover, the functor $S \longmapsto \mathcal{D} \times_{|\mathcal{D}|} S = \mathcal{D}_S$ defines inclusion preserving bijection between $\{\text{locally closed generalizing subsets of } |\mathcal{D}|\} \cong \{\text{isom. classes of immersions } \mathcal{D}_S \to \mathcal{D}\}$.

warning If $\mathcal{E} \to \mathcal{D}$ is an immersion, it may not be representable.

Idea of pf: Given $S \subseteq |\mathcal{D}|$ locally closed and generalizing, want to show that $\mathcal{D}_S$ is a diamond w/ $|\mathcal{D}_S| \cong S$.
shrink $|\mathcal{D}|$, we may assume S is closed in $|\mathcal{D}|$ (Unfortunately, closed subsets of pftd spaces might not come from a closed immersion of pftd spaces).
The <u>miracle</u> is that after replacing X by a suitable proétale cover $\tilde{X}$, <u>every</u> locally closed generalizing subset of $|\tilde{X}|$ is pftd.

Defn. A spectral space T is <u>w-local</u> if the set of closed pts is closed and every conn'd component of T has a unique closed pt.

E.g.
- w-local + connected $\iff$ local
- any profinite set is w-local.

Defn. A w-local pftd space is a qcqs pftd space X, such that the spectral space $|X|$ is w-local.

Thm if X is aff. pftd, then X has a canonical w-localization $X^{wl} \to X$. The space X^{wl} is aff. pftd and the map $X^{wl} \to X$ is a pro-(surj. étale w/ aff. source).

- The construction of X^{wl} is "explicit": let $\mathcal{B}$ be a cofinal set of open covers $\mathcal{B} = \{\{U_i\}_{i\in I}, |I| < \infty\}$ of X by rat'l subsets. This is cofiltered, and we set $X^{wl} = \varprojlim_{\{U_i\} \in \mathcal{B}} (\coprod U_i)$

- if X is a w-local pftd space, then the conn'd components of X are all of the form $\mathrm{Spa}(K_i, K_i^+)$ for some pftd fields K_i. Moreover $X^{wl} \to X \to \pi_0(X)$ is a homeomorphism.

Defn. A subset S of the topological space $|X|$ of a pftd space is <u>pftd</u> if $\exists$ a pftd space & a map $\mathcal{S} \to X$ inducing a homeo $|\mathcal{S}| \cong S$ and final for maps from pftd spaces to X factoring thru S on topological spaces.

Miracle Thm If X is w-local pftd space, then any loc. closed generalizing subset $S \subseteq |X|$ is pftd.

pf. read Dave's blog!

back to $X \to D$, replace X by $\coprod X_i$ for some open affinoid cover then $\tilde{X} = \coprod X_i^{wl} \to X$ is proétale. In particular, ~~preim($|\tilde{X}| \to |D|$)~~ $\mathrm{preim}(|\tilde{X}| \to |D|)(S) = |\mathcal{S}|$ is loc. closed & generizing, thus pftd $\mathcal{S}$.
Observe that $\tilde{R} = R \times_{(X \times X)} (\tilde{X} \times \tilde{X})$ is an proétale equivalence relation on $\tilde{X}$, and $D \cong \mathrm{Coeq}(\tilde{R} \rightrightarrows \tilde{X})$.
Moreover, $|\mathcal{S}| \subseteq |\tilde{X}|$ is $\tilde{R}$-stable, so $\tilde{R}_{\mathcal{S}} = \tilde{R} \times_{\tilde{X}} \mathcal{S}$ defines a

proétale equivalence relation on S and $D_S \simeq \mathrm{Coeq}(\tilde{R}_S \rightrightarrows S)$.

Defn: D is qc if $D \simeq \mathrm{Coeq}(h_R \rightrightarrows h_X)$ w/ X qc.
Note: this implies $|D|$ is qc, but converse might fail.

To define qs diamonds, we need:
Prop: if $D \longrightarrow \mathcal{E} \longleftarrow F$ is a diagram of diamonds, then $D \times_{\mathcal{E}} F$ is a diamond.
pf omitted.

Defn. D is qs if $\forall$ qc diamonds $\mathcal{E}_1, \mathcal{E}_2$ w/ maps $\mathcal{E}_1, \mathcal{E}_2 \to D$, $\mathcal{E}_1 \times_D \mathcal{E}_2$ is qc.
Note: this is not the same as $D \xrightarrow{\Delta} D \times D$ is qc, b/c the space $X \times Y$ appearing in the isom. $X \times_D Y \cong (X \times Y) \times_{(D \times D)} D$ is usually not qc.

Defn
- $\mathcal{E} \longrightarrow D$ is qc if $\mathcal{E} \times_D F$ is qc for all qc diamonds F.
- $\mathcal{E} \longrightarrow D$ is qs if $\mathcal{E} \xrightarrow{\Delta} \mathcal{E} \times_D \mathcal{E}$ is qc.
- $\mathcal{E} \longrightarrow D$ is separated if $\mathcal{E} \xrightarrow{\Delta} \mathcal{E} \times_D \mathcal{E}$ is a closed immersion.

Rmk: these satisfy all expected stability properties, under composition, fiber product, base change, etc.

The diamond associated w/ an adic space / pro-ét G-torsors

Defn: Fix $Y \in \mathrm{Perf}$. An <u>untilt</u> of Y is a pair $(Y^\#, \iota)$ where $Y^\# \in \widetilde{\mathrm{Perf}}$ and $\iota: Y^{\#\flat} \xrightarrow{\sim} Y$.

Notation: $\mathrm{Spd}(\mathbb{Z}_p)$ denotes the presheaf sending $Y \in \mathrm{Perf}$ to the set of isom. classes of ~~units~~ untilts of Y. (untilt has no automorphism)

Defn. Given an analytic space $X/\mathbb{Z}_p$ and $Y \in \mathrm{Perf}$, an untilt of Y over X is a triple $(Y^\#, \iota, f)$ where $(Y^\#, \iota)$ an untilt of Y and $f: Y^\# \longrightarrow X$.

Defn. $X^\diamond$ = the presheaf on Perf sending Y to the set of isom. classes of untilts of Y over X.

Lemma: If $X \in \widetilde{\mathrm{Perf}}$, then $X^\diamond \cong h_{X^\flat}$.
pf. this is immediate from the equivalence $\widetilde{\mathrm{Perf}}/X \cong \mathrm{Perf}/X^\flat$.

Thm: $X^\diamond$ is a diamond, w/ $|X^\diamond| = |X|$ & many other cptibilities.
heuristic: take some pfd pro-étale cover $\tilde{X} \longrightarrow X$ (in the old $X_{\text{proét}}$). Then $\tilde{\tilde{X}} = \tilde{X} \times_X \tilde{X}$ is also pfd, and "$X \approx \mathrm{Coeq}(\tilde{\tilde{X}} \rightrightarrows \tilde{X})$", which suggests that $X^\diamond \cong \mathrm{Coeq}(h_{\tilde{\tilde{X}}^\flat} \rightrightarrows h_{\tilde{X}^\flat})$.

Interlude: let G be any locally profinite gp. Get a sheaf $\underline{G}$ on $\widetilde{\mathrm{Perf}}^{\text{proét}}$, sending $S \in \mathrm{Perf}$ to $\mathrm{Cont}(|S|, G)$.

Note: $\underline{G}$ is not representable, but $\underline{G} \times X$ is representable for any pfd X.
e.g. if G is profinite and $X = \mathrm{Spa}(A, A^+)$ then $X \times \underline{G} \cong \mathrm{Spa}(\mathrm{Cont}(G, A), \mathrm{Cont}(G, A^+))$.

Defn. A map of pftd spaces $Y \to X$ is a pro-étale $\underline{G}$-torsor if $\exists$ a $\underline{G}$-action on Y lying over the trivial $\underline{G}$-action on X s.t.
$$Y \times_X Y \cong Y \times \underline{G} \qquad (y, yg) \leftarrow\!\shortmid (y,g)$$
and $Y \to X$ has a section after pullback along a proét cover $X' \to X$.

In parallel, a map of proét sheaves $\mathcal{G} \to \mathcal{F}$ on Perf is a proét G-torsor if blah blah blah...

Thm (H., Weinstein)
- Any proét G-torsor $\mathcal{G} \to \mathcal{F}$ is (representable and) proét. Moreover, if $K \subseteq G$ is open cpt, then $\mathcal{G} \to \mathcal{G}/\underline{K}$ is (rep'ble and) pro-(finite étale), and the map $\mathcal{G}/\underline{K} \to \mathcal{F}$ is (rep'able and) separated and étale.
- If $\mathcal{G} \to \mathcal{F}$ w/ a $\underline{G}$-action on $\mathcal{G}$ lying over trivial $\underline{G}$-action on $\mathcal{F}$, then it's a G-torsor $\Leftrightarrow$ G acts freely & transitively restricted to geometric fibers $\mathcal{G} \times_{\mathcal{F}} (C, \mathcal{O}_C)$.

pf idea
$$\begin{tikzcd} X' \times \underline{G/K} \arrow[r, "\text{f. ét}"] \arrow[d] & X' \arrow[d, "\text{proét}"] \\ \mathcal{G}/\underline{K} \arrow[r] & X \end{tikzcd}$$

Prop. Weinstein Finite étale maps of pftd spaces satisfy effective descent wrt proétale covers.

A small further argument shows that $\mathcal{G} \cong \varprojlim_{\substack{K \subseteq G \\ \text{open}}} \mathcal{G}/\underline{K}$.

Cor. If $\mathcal{F}$ is a proét sheaf w/ $\underline{G}$-action and G acts freely on $\mathcal{F}(C, \mathcal{O}_C)$ $\forall$ $C/\mathbb{F}_p$, cplt alg. closed. then $\mathcal{F} \to \mathcal{F}/\underline{G}$ is a pro-étale G-torsor.

Cor. If $\mathcal{G} \to \mathcal{F}$ is a pro-ét G-torsor, then $\mathcal{G}$ is a diamond iff $\mathcal{F}$ is a diamond.

Back to $X \rightsquigarrow X^\diamond$.

Thm Assume $X = \mathrm{Spa}(R, R^+)$ affinoid. Choose a filtered directed system (R_i, R_i^+) of fin. étale Galois (R, R^+)-alg. s.t. the cplted dir. limit (R_∞, R_∞^+) is pftd. Let $G_i = \mathrm{Gal}(X_i/X)$.
$G = \varprojlim G_i = \mathrm{Gal}(\tilde{X} = \mathrm{Spa}(R_\infty, R_\infty^+)/X)$.
then: (1) $\underline{G}$ acts freely on geom. pts of $h_{\tilde{X}^\flat}$, so $h_{\tilde{X}^\flat}/G$ is a diamond, and $h_{\tilde{X}^\flat} \to h_{\tilde{X}^\flat}/\underline{G}$ is a proét G-torsor.
(2) $\exists$ a nat. identification $X^\diamond \cong h_{\tilde{X}^\flat}/\underline{G}$, so $X^\diamond$ is a diamond.

part (1) follows from previous results.
part (2) let's construct $X^\diamond \to h_{\tilde{X}^\flat}/\underline{G}$.
start w/ $(Y^\#, \iota, f) \in X^\diamond(Y)$.

Defn $\mathrm{Hom}_G(\tilde{T}/T, \tilde{X})/\cong$ denotes isom. classes of diagrams
$$\begin{tikzcd} \tilde{T} \arrow[r] \arrow[d, dashed, "\text{proét } G\text{-torsor}"'] & \tilde{X} \arrow[loop right, "G"] \\ T & \end{tikzcd}$$

& Claim / Lemma $\exists$ a natural bij. $\mathrm{Hom}(T, X) \cong \mathrm{Hom}_G(\tilde{T}/T, \tilde{X})/\cong$ for any pftd T.

pf. Given $T \to X$, set $T_i = T \times_{f,X} X_i$ which by almost purity is pftd & fin. étale over T.
Let $\tilde{T} = \varprojlim_i T_i$ (exists as a pftd space since all trans. maps are fin. étale), this is a G-torsor$/T$, and the maps $T_i \to X_i$ compile to $\tilde{T} \to \tilde{X}$.

the other direction: suppose $\tilde{T} \to \tilde{X}$ G-equiv.,

$\tilde{T}/T$ a G-torsor $\Leftrightarrow$ a G-equiv. map

$$(R_\infty, R_\infty^+) \xrightarrow{\tilde{f}} (\mathcal{O}(\tilde{T}), \mathcal{O}^+(\tilde{T}))$$

by sheaf property $(\mathcal{O}(T), \mathcal{O}^+(T)) \xrightarrow{\simeq} (\mathcal{O}(\tilde{T})^G, \mathcal{O}^+(\tilde{T})^G)$.

thus $\tilde{f}$ restricts to $(R_\infty^G, R_\infty^{+G}) \to (\mathcal{O}(T), \mathcal{O}^+(T))$.

which only depends on isom. class of
$$\begin{array}{c}\tilde{T} \to \tilde{X} \\ \downarrow \\ T\end{array}$$

Since the map $(R, R^+) \to (R_\infty, R_\infty^+)$ factors thru $(R_\infty^G, R_\infty^{+G})$ we get $(R, R^+) \to (\mathcal{O}(T), \mathcal{O}^+(T))$.

So $\{(Y^\#, \iota, f)/\simeq\} \cong \{(\tilde{Y}^\#, \iota, \tilde{f}) \in \mathrm{Hom}_G(\tilde{Y}^\#/Y^\#, \tilde{X})\}/\simeq$

$\xrightarrow[\sim]{\text{tilt}} \mathrm{Hom}_G(\tilde{Y}/Y, \tilde{X}^\flat) \simeq h_{\tilde{X}^\flat}/\underline{G}\,(Y)$.

Why? : given $\tilde{Y}/Y$ & a G-eq map $\tilde{Y} \xrightarrow{\alpha} \tilde{X}^\flat$,

Step 1: we get $\tilde{Y}^\# \xrightarrow{\alpha^\#} \tilde{X}$. $\tilde{\iota}: \tilde{Y}^{\#\flat} \xrightarrow{\sim} \tilde{Y}$.

Step 2. $G \curvearrowright \tilde{Y}$ lifts to a unique $G \curvearrowright \tilde{Y}^\#$ cptible w/ $\tilde{\iota}$ & $\alpha^\#$:

Prop. Let $Y \in$ Perf be a pftd space equipped w/ an action of a gp G. Then for any given untilt $(Y^\#, \iota)$, there is at most 1 G-action on $Y^\#$, s.t. $\iota: Y^{\#\flat} \xrightarrow{\sim} Y$ is G-eq.

Prop Let $X, Y \in$ Perf be pftd spaces w/ G-action, and let $f: Y \to X$ be a G-eq. map. If $(X^\#, \iota)$ is a G-eq. untilt of X, the untilt $(Y^\#, \iota)$ of Y induced by $\mathrm{Perf}_{/X} \cong \mathrm{Perf}_{/X^\#}$ is G-eq., and the map $f^\#: Y^\# \to X^\#$ is G-eq.

Step 3 $(Y^\#, \iota) := (\tilde{Y}^\#/G, \tilde{\iota}/G)$ is a pftd space.

Prop Fix $Y \in$ Perf, and let $\tilde{Y} \to Y$ be a proétale G-torsor for some profinite gp G. Let $(\tilde{Y}^\#, \iota)$ be a G-eq. untilt of $\tilde{Y}$. Then cat. qt. $\tilde{Y}^\#/G = Y^\#$ is a pftd space, and $\tilde{\iota}$ induces a canonical isom. $Y^{\#\flat} = (\tilde{Y}^\#/G)^\flat = \tilde{Y}^{\#\flat}/G = \tilde{Y}/G \cong Y$. in particular, $(Y^\#, \iota)$ is a untilt of Y.

Claim $|X^\diamond| \cong |X|$.

pf $|X^\diamond| = |h_{\tilde{X}^\flat}/G| = |\tilde{X}^\flat|/G \cong |\tilde{X}|/G \cong (\varprojlim_i |X_i|)/G$

$\cong \varprojlim (|X_i|/G) \cong \varprojlim_i |X| = |X|$.

Claim X aff. $U \subseteq X$ is an open subspace, then the open subdiamond of $X^\diamond$ assoc. w/ $|U| \subseteq |X| = |X^\diamond|$ concides w/ $U^\diamond$

pf. go thru all identifications.

Now get $X^\diamond$ in general by gluing $X_i^\diamond$'s for $X = \bigcup X_i$.

The diamond of $\mathbb{Q}_p$

Notation if (A, A^+) is any Tate-Huber pair $/\mathbb{Z}_p$. $\mathrm{Spd}(A, A^+) \triangleq$ $\triangleq \mathrm{Spa}(A, A^+)^\diamond$; $\mathrm{Spd}(A) = \mathrm{Spd}(A, A^\circ)$.

$$\mathrm{Spd}\,\mathbb{Q}_p = h_{\mathrm{Spa}(\mathbb{Q}_p^{cyc})^\flat} / \underline{\mathbb{Z}_p^\times}$$

$$(\mathbb{Q}_p^{cyc})^\flat \simeq \mathbb{F}_p((t^{1/p^\infty})) := (t\text{-adic cpltion of } \mathbb{F}_p[t^{1/p^\infty}])[\tfrac{1}{t}]$$

$$\mathbb{Z}_p[\zeta_{p^n}]/(p) \cong \mathbb{F}_p[t^{1/p^{n-1}}]/(t^{p-1})$$

$$\zeta_{p^n} \longmapsto t^{1/p^{n-1}} + 1$$

$$(p) = (\zeta_p - 1)^{p-1}$$

$$\Rightarrow \mathbb{Z}_p^{cyc, \flat} = (\mathbb{Z}_p^{cyc}/p)^\flat \cong \mathbb{F}_p[[t^{1/p^\infty}]]$$

Prop. $\exists$ a natural equiv. $\mathrm{Perf}/\mathrm{Spd}\,\mathbb{Q}_p \cong \widetilde{\mathrm{Perf}}/\mathrm{Spa}\,\mathbb{Q}_p$.

$$\mathrm{Spd}\,\mathbb{Q}_p \times \mathrm{Spd}\,\mathbb{Q}_p \not\cong (\text{something})^\diamond$$

Prop. ~~Perf/X~~ Let X be an analytic adic space, then $\widetilde{\mathrm{Perf}}/X \cong \mathrm{Perf}/X^\diamond$

pf. construct ~~a~~ the essential inverse:
for simplicity, assume $X = \mathrm{Spa}(R, R^+)$, choose $\tilde{X} = \mathrm{Spa}(\tilde{R}, \tilde{R}^+)$
X_i, G as before w/ $X^\diamond \cong (\tilde{X}^\flat)^\diamond / \underline{G}$, etc.
given

$$\begin{tikzcd} \tilde{Y}^\diamond \arrow[r] \arrow[d] & Y^\diamond \arrow[d] \\ (\tilde{X}^\flat)^\diamond \arrow[r] & X^\diamond \end{tikzcd}$$

for some $Y \in \mathrm{Perf}$.

We get G-eq. map $\tilde{Y} \to \tilde{X}^\flat$, hence a G-eq. untilt $\tilde{Y}^\# \longrightarrow \tilde{X}$. $Y^\# = \tilde{Y}^\#/G$ is pfted and untilts Y.

Def'n & Prop ~~Prop~~ Let X be an analytic adic space, TFAE
① diamonds over $X^\diamond$
② sheaf on $\widetilde{\mathrm{Perf}}/X^{\text{proét}}$ which $\simeq \mathrm{Coeq}(h_R \rightrightarrows h_Y)$ w/ ~~R⇉Y~~ $R \rightrightarrows Y$ proét of pfted spaces $\in \widetilde{\mathrm{Perf}}/X$.

pf. $\mathrm{Sh}(\mathrm{Perf}^{\text{proét}})/X^\diamond \cong \mathrm{Sh}(\mathrm{Perf}^{\text{proét}}/X^\diamond)$
$\cong \mathrm{Sh}(\widetilde{\mathrm{Perf}}^{\text{proét}}/X)$

So it makes sense to talk about diamonds over X for some X analytic $/\mathbb{Z}_p$ (instead of over $\mathbb{F}_p$).

~~Prop~~ Recall $\mathrm{Spd}\,\mathbb{Z}_p$ ($Y \longmapsto \{\text{untilts of } Y\}$), [in K-L, ~~...~~ it's proved that this is a sheaf on $\mathrm{Perf}^{\text{proét}}$]. It's not a diamond but very close~~d~~ to being one.

Prop. $\forall S \in \mathrm{Perf}$, $\exists$ a canonical analytic adic space "$S \times \mathrm{Spa}\,\mathbb{Z}_p$" over $\mathrm{Spa}\,\mathbb{Z}_p$ s.t. $(S \times \mathrm{Spa}\,\mathbb{Z}_p)^\diamond \cong S^\diamond \times \mathrm{Spd}\,\mathbb{Z}_p$.
In particular $S^\diamond \times \mathrm{Spd}\,\mathbb{Z}_p$ is always a diamond and $\mathrm{Spd}\,\mathbb{Z}_p$ is a sheaf. Similarly if one relace S by a diamond D.

idea of pf. Given $S = \mathrm{Spa}(A, A^+) \in \mathrm{Perf}$ w/ $\varpi \in A$ any pseudouniformizer.
let $S \times \mathrm{Spa}\,\mathbb{Z}_p = \mathrm{Spa}(W(A^+)) \setminus V([\varpi])$.
which turns out to be an honest adic space and is indep of ϖ.
$(S \times \mathrm{Spa}\,\mathbb{Z}_p)^\diamond(Y = \mathrm{Spa}(R, R^+)) \Longleftrightarrow (Y^\#, \iota, f)$
where $f: Y^\# = \mathrm{Spa}(R^\#, R^{\#+}) \longrightarrow S \times \mathrm{Spa}\,\mathbb{Z}_p$
$\longleftrightarrow$ (which is eq. to) $f: W(A^+) \longrightarrow R^{\#+}$, w/ $f([\varpi])$ invertible in $R^\#$ and top. nilpotent
$\longleftrightarrow W(A^+)/(p, [\varpi]) \longrightarrow R^{\#+}/(p, [\varpi])$
(universal property of Witt vectors & pfted rings)

$$A^+/\varpi \xleftrightarrow{(-)^\flat} A^+ \longrightarrow R^{\#\flat +} = R^+$$

$$\longleftrightarrow \begin{tikzcd} Y^{\#\flat} \arrow[r] \arrow[d, "\iota"] & S \\ Y & \end{tikzcd} \Longleftrightarrow (Y^\#, \iota, (f \circ \iota^{-1})^\diamond) \in S^\diamond \times \mathrm{Spd}\,\mathbb{Z}_p(Y)$$

B^+_{dR}/Fil^n.

Recall $\forall$ pftd Tate ring $R/\mathbb{Q}_p$, we define $\mathbb{B}^+_{dR}(R) := \varprojlim_n \frac{W(R^{b\circ})[\frac{1}{p}]}{\ker(\theta)^n}$
where $\theta: W(R^{b\circ})[\frac{1}{p}] \twoheadrightarrow R$.
It's a general fact that $\ker\theta$ is principal and generated by some non-zerodivisor ξ.

Defn. Let $B^+_{dR}/\mathrm{Fil}^n \longrightarrow \mathrm{Spd}\,\mathbb{Q}_p$ be the functor $(\mathrm{Spa}(R,R^+)^\diamond \longrightarrow \mathrm{Spd}\,\mathbb{Q}_p) \mapsto \mathbb{B}^+_{dR}(R^\#)/\mathrm{Fil}^n$, where $R^\#$ is the untilt of R determined by the given map to $\mathrm{Spd}\,\mathbb{Q}_p$.

Thm B^+_{dR}/Fil^n is a diamond.

pf. for $n=1$ $B^+_{dR}/\mathrm{Fil}^1 \cong \mathbb{A}^{1,\diamond}$
induction: $0 \longrightarrow \cancel{B^+_{dR}/\mathrm{Fil}^{n-1}}\ \ker\theta \longrightarrow B^+_{dR}/\mathrm{Fil}^n \longrightarrow \mathbb{A}^{1,\diamond} \longrightarrow 0$

$$\begin{tikzcd} B^+_{dR}/\mathrm{Fil}^n \arrow[r] & \mathbb{A}^{1,\diamond} \\ & \widetilde{B}^\diamond \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

want $\widetilde{B}^\diamond \longrightarrow \mathbb{A}^{1,\diamond}$ s.t.
1. ~~admits~~ pftd, proét covering.
2. lives over $\mathbb{Q}_p^{cyc}$.
3. admits lifting $\widetilde{B}^\diamond \longrightarrow B^+_{dR}/\mathrm{Fil}^n$.

Let $B = (\mathrm{Spa}\,\mathbb{Z}_p^{cyc}[\![T]\!])_\eta$, $\widetilde{B} = \varprojlim_\varphi B$ where $\varphi(T) = (1+T)^p - 1$.

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\pi"] \arrow[dr, dashed, "s"'] & \mathbb{A}^1 \\ & B^+_{dR}/\mathrm{Fil}^n \arrow[u, "\theta"] \end{tikzcd}$$

$\widetilde{B} \xrightarrow{\pi} \mathbb{A}^1$ characterized by $\pi^* x = \log(1+T)$.

$s.$ $r = (r_0, r_1, \dots) \in \widetilde{B}(R^\#) \longmapsto \log[1+r^b]$
where $r^b = (\bar{r}_0, \bar{r}_1, \dots) \in (R^{\#\circ}/p)^b$
Then we see immediately $(\theta \circ s)(r) = \log(1+r_0) = \pi(r)$.

$B^+_{dR}/\mathrm{Fil}^n \times_{\mathbb{A}^{1,\diamond}} \widetilde{B}^\diamond \cong \ker\theta \times_{\mathrm{Spd}\,\mathbb{Q}_p} \widetilde{B}^\diamond \cong (\ker\theta \times_{\mathrm{Spd}\,\mathbb{Q}_p} \mathrm{Spd}\,\mathbb{Q}_p^{cyc}) \times_{\mathrm{Spd}\,\mathbb{Q}_p^{cyc}} \widetilde{B}^\diamond$

$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ $\quad \overset{t = \log[\varepsilon]}{\simeq} (B^+_{dR}/\mathrm{Fil}^{n-1} \times_{\mathrm{Spd}\,\mathbb{Q}_p} \mathrm{Spd}\,\mathbb{Q}_p^{cyc}) \times_{\mathrm{Spd}\,\mathbb{Q}_p^{cyc}} \widetilde{B}^\diamond$

$\cong B^+_{dR}/\mathrm{Fil}^{n-1} \times_{\mathbb{Q}_p^\diamond} \widetilde{B}^\diamond$

vector bundles on the Fargues-Fontaine curve

Let E be a finite extn of $\mathbb{Q}_p$, uniformizer π, residue field $\mathbb{F}_q$.
Let $S = \mathrm{Spa}(A, A^+) \in \mathrm{Perf}/\mathbb{F}_q$. Let ϖ be a topologically nilpotent unit of A.

Defn $A_{\inf} = W(A^+) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ $\quad \mathcal{Y}_S = \mathrm{Spa}(A_{\inf}) \setminus \{x \mid |\pi [\varpi]|_x = 0\}$

$X_S = \mathcal{Y}_S / \mathrm{Frob}_q^{\mathbb{Z}}$. $\exists$ cts map $\mathcal{Y}_S \xrightarrow{\varphi} (0, \infty)$
$y \longmapsto \frac{\log|\pi|_y}{\log|[\varpi]|_y}$ if y has rk 1.
inverse image of $[\alpha, \beta]$ is a rat'l subset if $\alpha, \beta \in \mathbb{Q}^+$.
$\varphi(\mathrm{Frob}_q^*(z)) = \frac{\varphi(z)}{\log(q)}$, so Frob_q acts properly discontinuously.

Prop. $X_S^\diamond \cong S^\diamond/\mathrm{Frob}_q^{\mathbb{Z}} \times_{\mathbb{F}_q} \mathrm{Spd}\,E$.

Since $|X_S^\diamond| = |X_S|$ and $|S^\diamond/\mathrm{Frob}_q| \cong |S^\diamond| = |S|$, there is a map $|X_S| \longrightarrow |S|$ so the "affinoid pieces" glue.

Prop. If $S \in \widetilde{\mathrm{Perf}_E}$, then $\exists$ a closed immersion
$$S \longrightarrow X_{S^b}.$$

Defn. Let G be a reductive alg gp$/E$, the de Rham affine Grassmannian Gr_G is the functor that sends a pftd $(E, \mathcal{O}_E)$-alg (R, R^+) to the set of isom. classes of G-torsors over $\mathrm{Spec}(B^+_{dR}(R))$ w/ a trivialization over $\mathrm{Spec}(B_{dR}(R))$.

Defn & Facts A closed subgp of $G_{\bar{E}}$ is called Borel if it's a max'l connected solvable subgp. Equivalently, $G_{\bar{E}}/B$ is projective.

$B = TN$. T = max'l torus N = max'l unipotent.

Notation $X^*(T) = \mathrm{Hom}_{\bar{E}}(T, \mathbb{G}_m)$ $X_*(T) = \mathrm{Hom}_{\bar{E}}(\mathbb{G}_m, T)$
$X_*(G) = \mathrm{Hom}_{\bar{E}}(\mathbb{G}_m, G)$

Prop. $X_*(G)/\text{conjugacy} \cong X_*(T)/W$, $W = N_G(T)/T$ (normalizer of T in G)

Defn. A cochar. $\mu \in X_*(T)$ is dominant if $\forall$ wt λ of $\mathrm{Lie}(N)$, $\langle \mu, \lambda \rangle \geq 0$
wt: $\exists$ $v \in \mathrm{Lie}(N)$, s.t. $t \cdot v = \lambda(t) v$ $\forall t \in T$.

Fact $X_*(T)_{\text{dom.}} \longrightarrow X_*(T)/W$ bijection.

example $G = GL_n$. cochar: $t \longmapsto \begin{pmatrix} t^{\mu_1} & & \\ & \ddots & \\ & & t^{\mu_n} \end{pmatrix}$ Weyl gp $= S_n$ permute (μ_i).

dominant iff $\mu_1 \geq \mu_2 \geq \cdots$

Notation Let $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{cyc}} = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$ and let $b \in GL_n(\breve{\mathbb{Q}}_p)$. Then for any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$, we may define a vector bundle $\mathcal{E}_b$ on X_S. $\mathcal{E}_b$ corresponds to the φ-equiv VB $\tilde{\mathcal{E}}_b$ on $\mathcal{Y}_S$ given by $(\tilde{\mathcal{E}}_b, \varphi_{\tilde{\mathcal{E}}_b}) = (\mathcal{O}_{\mathcal{Y}} \otimes_{\breve{\mathbb{Q}}_p} \breve{\mathbb{Q}}_p^n, \varphi_{\mathcal{Y}} \otimes b\sigma)$.

Observe If $b \in GL_n(\mathbb{Q}_{p^j})$ for some j, this construction of $\mathcal{E}_{b,S}$ works verbatim $\forall S \in \mathrm{Perf}_{\mathbb{F}_{p^j}}$.

Defn For any S and any $\lambda = \frac{d}{h} \in \mathbb{Q}$, $(d,h)=1$, we let $\mathcal{O}(\lambda)$ denote the vector bundle associated w/ elt $\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ p^d & & & 0 \end{pmatrix} \in GL_n(\mathbb{Q}_p)$
In particular, $\mathcal{O}_S(0) = \mathcal{O}_{X_S}$

Thm (Fargues, Fontaine, Kedlaya-Liu) When $S = \mathrm{Spa}(C, C^+)$ is a geom. pt, any VB $\mathcal{E}$ on X_S is $\simeq \mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_j)$ for a uniquely determined sequence $\lambda_1 \geq \cdots \geq \lambda_j$.

Facts
- $\mathrm{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0$ if $\lambda > \mu$.
- $\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0$ if $\lambda < \mu$.
- $\mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_2) \cong \mathcal{O}(\lambda_3)^n$ for the ~~ap~~ appropriate λ_3 & n ($\lambda_3 = \lambda_1 + \lambda_2$)

Defn Fix $S \in \mathrm{Perf}$ and $\mathcal{E}$ a vb on X_S, then we write

$$\begin{aligned} H^0_S(\mathcal{E}) : \mathrm{Perf}/S &\longrightarrow \mathrm{Sets} \\ f: T \to S &\longmapsto H^0(X_T, f^*\mathcal{E}) \end{aligned}$$

$$f \rightsquigarrow X_T \xrightarrow{f} X_S$$

Thm The functor $H^0_S(\mathcal{E})$ is a diamond over $S^\diamond$.

First consider $\mathcal{E} = \mathcal{O}(n)$ for some $n \geq 1$.

Prop. $\forall S$, $H^0(X_S, \mathcal{O}_X(n)) = H^0(\mathcal{Y}_S, \mathcal{O}_{\mathcal{Y}})^{\varphi = p^n}$

pf. obvious

Fact. $\exists$ a SES of gp sheaves on $\mathrm{Perf}/\mathrm{Spd}\,\mathbb{Q}_p$

$$(*) \quad 0 \longrightarrow \underline{\mathbb{Q}_p(n)} \longrightarrow \mathcal{H}^0_{\mathbb{Q}_p}(\mathcal{O}(n)) \xrightarrow{\theta} \mathbb{B}^+_{dR}/\mathrm{Fil}^n \longrightarrow 0.$$

after base change to $\breve{\mathbb{Q}}_p$, $\underline{\mathbb{Q}_p(n)} \cong \underline{\mathbb{Q}_p}$, then $\mathcal{H}^0(\mathcal{O}(n))$ is a $\mathbb{Q}_p$-torsor over $\mathbb{B}^+_{dR}/\mathrm{Fil}^n$, so $\mathcal{H}^0_{\mathbb{Q}_p}(\mathcal{O}(n))$ is a diamond

explanation of $(*)$ ~~[illegible]~~ fix $\mathrm{Spd}(R, R^+) \longrightarrow \mathrm{Spd}\,\mathbb{Q}_p$, corresp. to an untilt $\mathrm{Spa}(R^\#, R^{\#+})$. Then we get a closed immersion $\mathrm{Spa}(R^\#, R^{\#+}) \hookrightarrow \mathcal{Y}_{\mathrm{Spa}(R,R^+)}$ corresp to $W(R^+)[\frac{1}{p}] \longrightarrow R^\#$, then one checks this fits into a diagram

$$\begin{tikzcd}
W(R^+)[\frac{1}{p}] \arrow[rr] \arrow[dr] \arrow[dd] & & R^\# \\
& & \mathbb{B}^+_{dR}(R^\#)/\mathrm{Fil}^n \arrow[u] \\
H^0(\mathcal{Y}_{\mathrm{Spa}(R,R^+)}, \mathcal{O}_{\mathcal{Y}})^{\varphi=p^n} \cong H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})^{\varphi=p^n} \arrow[urr] & &
\end{tikzcd}$$

We just proved $\mathcal{H}^0(\mathcal{O}(n)) \times \mathrm{Spd}\,\mathbb{Q}_p$ is a diamond.

Fact $\mathcal{F}$ a sheaf on Perf, TFAE:

(1) $\mathcal{F} \times \mathrm{Spd}\,\mathbb{Q}_p$ is a diamond.

(2) $\mathcal{F} \times D$ is a diamond for <u>any</u> diamond D.

the property (2) is called "$\mathcal{F}$ is an absolute diamond".

Thus $\mathcal{H}^0(\mathcal{O}(n)) \times S^\diamond \cong \mathcal{H}^0_S(\mathcal{O}(n)) : \mathrm{Perf}_{/S} \longrightarrow \mathrm{Sets}$ is a diamond for any $S \in \mathrm{Perf}$.

Now, general $\mathcal{E}$ (on some X_S).

Fact ~~Idea~~ After possibly replacing S w/ some open aff. pieces, one can find a surjection $\mathcal{O}^{N_1} \longrightarrow \mathcal{E}(n) \overset{\mathrm{def}}{=} \mathcal{E} \underset{\mathcal{O}}{\otimes} \mathcal{O}(n)$ for some $n \gg 0$ and some $N \geq 1$.

So we apply to $\mathcal{E}^\vee$, to get $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(n_1)^{N_1} \longrightarrow \mathcal{O}(n_2)^{N_2}$

now $\mathcal{H}^0_S(\mathcal{E}) \cong \mathcal{H}^0_S((\mathcal{O}(n_1))^{N_1} \underset{\mathcal{H}^0_S((\mathcal{O}(n_2))^{N_2}}{\times} S^\diamond$

where $S \longrightarrow \mathcal{H}^0_S((\mathcal{O}(n_2))^{N_2}$ corresponds to $0 \in H^0(X_S, \mathcal{O}(n_2)^{N_2})$

~~and~~ (fiber)-product exists in cat. of diamonds.

<u>Cor.</u> If $\mathcal{E}/X_S$ is any VB, the functor $\mathrm{Aut}(\mathcal{E}) : \mathrm{Perf}_{/S} \to \mathrm{Gps}$

$(f : T \to S) \longmapsto \mathrm{Aut}_{\mathcal{O}_{X_T}}(f^*\mathcal{E})$ is a diamond$/S^\diamond$

pf. $\mathrm{Aut}(\mathcal{E}) \cong \mathcal{H}^0_S(\mathrm{Hom}(\mathcal{E}, \mathcal{E}))^2 \underset{\alpha, \mathcal{H}^0_S(\mathrm{Hom}(\mathcal{E},\mathcal{E}))^2, \beta}{\times} S^\diamond$

$\alpha((f, g)) = (f \circ g, g \circ f)$. β correspond to $(\mathrm{id}_{\mathcal{E}})^2$.

<u>Cor.</u> Similarly, Isom functors between VBs are diamonds.

The schematic FF curve.

Defn. For any $Spa(R,R^+) \in Perf$. Set $P_R = \bigoplus_{n\geq 0} H^0(\mathcal{X}_{Spa(R,R^+)}, \mathcal{O}_{\mathcal{X}}(n))$
$X_R := Proj\, P_R$ (indep of R^+). $\quad = \bigoplus_{n\geq 0} H^0(Spa\, W(R^+) \setminus V(p[\varpi]), \mathcal{O})^{\varphi = p^n}$

E.g. When $R = K$ pftd field, this is the "original" FF curve $X_K \in Sch_{/Spec\, \mathbb{Q}_p}$ which is Noeth., reg, dim 1.

Thm (Kedlaya, Liu) $\exists$ a natural morphism $\mathcal{X}_{Spa(R,R^+)} \xrightarrow{f^{an}} X_R$, s.t. $f^{an *}$ gives an equivalence of VBs.

Why the FF curve?

Fix K, a p-adic field. Set $C = \widehat{\bar{K}}$ and $X = X_{C^\flat}$, $\mathcal{X} = \mathcal{X}_{C^\flat}$. By functoriality, we have a natural G_K action on X, $\mathcal{X}$.

First series examples:

Defn. Let D be a φ-isocrys over K_0 ($= W(\text{res field of } K)[\frac{1}{p}] \subseteq K$). i.e., D is a fin. free K_0-v.s. w/ a φ-semilinear aut $\varphi_D : D \xrightarrow{\sim} D$.

Defn (Fontaine) A filtered φ-mod. $/K$ is a triple $(D, \varphi_D, Fil^\bullet \subseteq D_K := D \otimes_{K_0} K)$ where (D, φ_D) is a φ-isocrys. and $Fil^\bullet = \{\cdots \subseteq Fil^{i+1} \subseteq Fil^i \subseteq \cdots\}$ an exhausted separated decreasing filtration by K-vs's.

Prop $\exists$ a natural functor {continuous G_K-rep of fin. dim'l $\mathbb{Q}_p$ v.s.}
$\longrightarrow$ {filtered φ-mods over K}
$V \longmapsto \mathbb{D}(V) = (\mathbb{D}_{crys}(V), \varphi_{\mathbb{D}}, Fil^\bullet = fil$ on
$\mathbb{D}_{crys}(V) \otimes_{K_0} K \cong \mathbb{D}_{dR}(V)$ def. by $\{Fil^i \mathbb{D}_{dR}(V)\})$.

where $\mathbb{D}_{crys}(V) = (V \otimes_{\mathbb{Q}_p} B_{crys})^{G_K}$, a v.s. $/ B_{crys}^{G_K} \cong K_0$
$B_{crys} = (\text{p-adic cpl'ted PD-env. of } W(\mathcal{O}_{C^\flat}) \text{ along } \theta : \twoheadrightarrow \mathcal{O}_C)[\frac{1}{t}]$
$t = \log[(1, \zeta_p, \ldots)]$.

Defn. V is crystalline if $\dim_{K_0}(\mathbb{D}_{crys}(V)) = \dim_{\mathbb{Q}_p} V$. ($\leq$ always holds)

Thm (Fontaine) If V is crystalline, then V can be recovered from $\mathbb{D}(V)$ via a canonical isom $V \xrightarrow{\sim} (D \otimes_{K_0} B_{crys})^{\varphi_D \otimes \varphi = 1} \cap Fil^0(D_K \otimes_K B_{dR}) = \mathbb{V}(\mathbb{D})$

"Pf" Given a φ-mod D/K_0, resp. a filtered φ-mod $\mathbb{D}/K$
$\rightsquigarrow$ produce $\mathcal{E}(D)$, a vb. on X,
$\rightsquigarrow$ a modif. $\mathcal{F}|_{X \setminus \infty} \xrightarrow{\sim} \mathcal{E}(D)|_{X \setminus \infty}$.
Here $\mathcal{F}$ is some vb. s.t. $\mathbb{V}(\mathbb{D}) \cong H^0(X, \mathcal{F})$.

Prop. $X \cong Proj(\bigoplus_{n \geq 0} H^0(Spa(W(\mathcal{O}_{C^\flat})) \setminus V(p[\varpi]), \mathcal{O})^{\varphi = p^n}) = Proj(\bigoplus_{n\geq 0} B_{crys}^{+, \varphi = p^n})$

$(0 \to \underset{\cong \mathbb{Q}_p}{\mathbb{Q}_p(n)} \to B_{crys}^{+, \varphi = p^n} \to \underset{\cong C^n}{B_{dR}^+ / Fil^n} \to 0)$.

(Compare w/ $\mathbb{P}^1_C = Proj(\bigoplus_{n\geq 0} C[x,y]^{\deg n})$).

Then $X \setminus \infty = Spec(B_{crys}^{\varphi = 1})$ and $\widehat{X_\infty} = Spf(B_{dR}^+)$.
where ∞ = zero locus of t.
Moreover, $(X \setminus \infty) \times_X Spec(B_{dR}^+) \cong Spec\, B_{dR}$.

Hard thm (F-F) $B_{crys}^{\varphi = 1} = B_e$ is a PID. $\rightsquigarrow$ any VB $\mathcal{E}$ over X gives a fin. free B_e-mod., namely $H^0(X \setminus \infty, \mathcal{E}) := M_e(\mathcal{E})$.
Moreover, get a fin. free B_{dR}^+-mod $M_{dR}^+(\mathcal{E}) = \mathcal{E}_\infty \otimes_{\mathcal{O}_{X,\infty}} \widehat{\mathcal{O}_{X,\infty}}$.
These also come w/ a natural map $M_{dR}^+(\mathcal{E}) \hookrightarrow M_{dR}(\mathcal{E}) = M_{dR}^+(\mathcal{E})[\frac{1}{t}]$
$\cong M_e(\mathcal{E}) \otimes_{B_e} B_{dR}$ realizing M_{dR}^+ as a B_{dR}^+-lattice in $M_{dR}(\mathcal{E})$.

Defn (Berger) A B-pair is a pair $(\underset{\text{fin. free } B_e\text{-mod.}}{M_e}, M_{dR}^+)$ — M_{dR}^+: B_{dR}^+-lattice in $M_e \otimes_{B_e} B_{dR}$.

Thm The functor from VBs/X to B-pairs is an eq.

Defn. Let $(D, \varphi_D, \mathrm{Fil}^\cdot \subseteq D_K)$ be a filtered φ-mod. /K

- $\mathcal{E}(D)$ is the VB w/ $(M_e, M_{dR}^+) = ((D \otimes_{K_0} B_{crys})^{\varphi=1}, D \otimes_K B_{dR}^+)$
- $\mathcal{F}$ = the VB w/ $(M_e, M_{dR}^+) = ((D \otimes_{K_0} B_{crys})^{\varphi=1}, M_{dR}^+ = \mathrm{Fil}^0(D_K \otimes_K B_{dR}))$

Since they have the same M_e-part, the assoc. VBs inherit a can. mero. isom. $\mathcal{F}|_{X \setminus \infty} \xrightarrow{\sim} \mathcal{E}(D)|_{X \setminus \infty}$ and it's G_K-equivariant. if $\mathrm{Fil}^1 D_K = 0$, then $\mathcal{F} \subseteq \mathcal{E}(D)$.

Prop. $H^*(X, \text{any vb. } \mathcal{E}) \simeq H^*([M_{dR}^+(\mathcal{E}) \oplus M_e(\mathcal{E}) \longrightarrow M_{dR}(\mathcal{E})])$
$$(v, w) \longmapsto v - w$$

$\rightsquigarrow H^0(\mathcal{F})$ is $\cong \mathbb{V}(\mathbb{D})$ when $\mathcal{F}$ constr. from $\mathbb{D}$.

Thm. The above gives a bijection $\{\text{filtrations on } D_K, \text{ for } (D, \varphi_D) \text{ fixed}\}$
$\cong \{G_K\text{-equiv. modif. } \mathcal{F}|_{X \setminus \infty} \xrightarrow{\sim} \mathcal{E}(D)|_{X \setminus \infty}\}$.

pf. Given a modification, get $M_{dR}^+(\mathcal{F}) \subseteq M_{dR}(\mathcal{E}(D)) \cong D_K \otimes_K B_{dR}$, so get $\{\mathrm{Fil}^i = (t^i M_{dR}^+(\mathcal{F}))^{\Gamma}\}$ by Shrenik's talk.

Defn. Let $H/\overline{\mathbb{F}}_p$ be a p-div. gp, so we have
D = the φ-isocrystal $/\breve{\mathbb{Q}}_p$, given by $\mathbb{M}(H)$ w/ its Frobenius
 (contr. rat'l Dieudonne mod. of H.)
For any pfd space $S/\overline{\mathbb{F}}_p$
get a VB $\mathcal{E}_S(H) = \mathcal{E}_S(D^\vee)$ on X_S

- $H = \mu_{p^\infty} \rightsquigarrow \mathcal{E} = \mathcal{O}(1)$
- $H = \mathbb{Q}_p/\mathbb{Z}_p \rightsquigarrow \mathcal{E} = \mathcal{O}$
- $H = E[p^\infty]$, $E/\overline{\mathbb{F}}_p$ s.s. $\rightsquigarrow \mathcal{E} = \mathcal{O}(\frac{1}{2})$.

Defn. $M_{H,\infty}^\diamond$ is the sheafification of
$$T/\breve{\mathbb{Q}}_p \longmapsto \{i : \mathcal{O}_{X_{T^\flat}}^h \hookrightarrow \mathcal{E}_{T^\flat}(H) \text{ s.t. } T \overset{j}{\subseteq} X_{T^\flat}\}$$
w/ coker. killed by ideal cut out T
and $\simeq j_*(\text{fin. loc. free } \mathcal{O}_T\text{-mod of rk } d)$

Thm $M_{H,\infty}^\diamond$ is $GL_n(\mathbb{Q}_p)$-equiv'ly $\cong$ the diamond of the RZ space w/ ∞ level str. assoc. w/ H.

Defn Fix H a p-div. gp$/\overline{\mathbb{F}}_p$ (maybe w/ extra structures) of ht h and dim d.
A deformation of H over a $\breve{\mathbb{Z}}_p$-alg. R (p is nilpotent in R) is a pair (G, ρ) where G is a p-div. gp$/R$, $\rho: H \times_{\overline{\mathbb{F}}_p} R/p \to G \times_R R/p$ a quasiisogeny.
$(G_1, \rho_1) \sim (G_2, \rho_2)$ if $\rho_2 \circ \rho_1^{-1}$ lifts to an isom. $G_1 \xrightarrow{\sim} G_2$.

Defn. $\mathcal{M}_H$ is the functor $\mathrm{Nil}_{\breve{\mathbb{Z}}_p} = \{\breve{\mathbb{Z}}_p\text{-alg.}, p \text{ nilpotent}\} \longrightarrow \mathrm{Sets}$
$R \longmapsto \{(G, \rho)/R\}/\sim$

Thm (Rapoport-Zink) $\mathcal{M}_H$ is (rep'ble by) a nice formal scheme over $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$.

Examples Suppose H = conn'd of dim 1 & ht h. Then H is unique up to isom; one can take H = L.T. $\mathcal{O}_K$-module assoc. w/ $K = \mathbb{Q}_{p^h}$.
Then $\mathcal{M}_H \cong \coprod_{\mathbb{Z}} \mathrm{Spf}\, \breve{\mathbb{Z}}_p[[T_1, \cdots, T_{h-1}]]$ (Lubin-Tate).
In this case, the gp of self-quasiisogenies of H is $\cong D_{1/h}^\times$ where $D_{1/h}$ is a division alg. $/\mathbb{Q}_p$ of inv. $1/h$.

Now suppose $K/\breve{\mathbb{Q}}_p$ is a finite extn, $x \in \mathcal{M}_H(\mathcal{O}_K) \cong \mathcal{M}_{H,\eta}^{ad}(K, \mathcal{O}_K)$.
Then x corresponds to a pair (G_x, ρ_x) for some p-div. gp over $\mathcal{O}_K$.

Thm. Let $D = M(H)$ be the rat'l Dieudonné module of H. Then $V_p G_x$ is a crystalline G_K-repn; moreover, ρ_x "induces" a canonical isom. $D_{crys}(V_p G_x) \cong D \otimes_{\mathbb{Q}_p} K_0$. ($K_0 = \breve{\mathbb{Q}}_p^{unr} \cap K$).

The thm above: the H-dR filtration on $D_{dR}(V_p, G_x)$ gives a canonical K-v.s. $\mathrm{Fil}^1_x \subseteq D_{crys}(V_p, G_x) \otimes_{K_0} K \cong D \otimes_{\mathbb{Q}_p} K$ of dim d.

In particular, given a K-pt $x \in \mathcal{M}_{H,\eta}^{ad}(K, \mathcal{O}_K)$, we produced a can. K-pt $\pi(x) \in \mathcal{F}\ell(K, \mathcal{O}_K)$ = d-dim'l lines in D.

Thm (R. Z.) This construction extends in a can. way to all pts of $\mathcal{M}_{H,\eta}^{ad}$ over cplt non-arch. fields extending $\breve{\mathbb{Q}}_p$, and we get a morphism $\pi: \mathcal{M}_{H,\eta}^{ad} \to \mathcal{F}\ell$ of adic spaces over $\mathrm{Spa}(\breve{\mathbb{Q}}_p, \breve{\mathbb{Z}}_p)$.
"Grothendieck-Messing period map".

Thm (1) The map π is étale (hence open), w/ discrete but infinite fibers.
(2) π is partially proper (sep. + univ. specializing).
(3) If $x, y \in \mathcal{M}_{H,\eta}^{ad}(K, \mathcal{O}_K)$, then $\pi(x) = \pi(y) \Leftrightarrow G_x$ and G_y are quasiisogenous over fin. extn of K.

Note (3) has a more suggestive interpretation:
any pt $z \in \mathcal{F}\ell(K, \mathcal{O}_K)$ gives some filtration on $D \otimes_{\mathbb{Q}_p} K$ which is weakly admissible when $z \in \mathrm{im}(\pi)(\mathcal{M}(K, \mathcal{O}_K)^?)$
then get $\mathbb{V}$ = apply Fontaine functor from last time
$\cong V_p G_x$ for any x lifting z
Then lifts z to $\mathcal{M}$ correspond to G_K-inv't lattices in $\mathbb{V}_x$.
$\rightsquigarrow$ π looks like a fibration w/ constant fibers $\simeq$ "$GL_h(\mathbb{Q}_p)/GL_h(\mathbb{Z}_p)$".

$$\to \mathcal{M}_{H,\eta,n}^{ad} \longrightarrow \cdots \longrightarrow \mathcal{M}_{H,\eta}^{ad} \longrightarrow \mathrm{im}(\pi) \subseteq \mathcal{F}\ell$$

$GL_h(\mathbb{Z}/p^n)$ cover where we trivialize $G^{univ}[p^n]$. $\quad$ $GL_h(\mathbb{Q}_p)/GL_h(\mathbb{Z}_p)$ fibration

Thm (Scholze-Weinstein) $\exists$ a (pre)-pfd space $\mathcal{M}_{H,\eta,\infty}^{ad} \sim \varprojlim_n \mathcal{M}_{H,\eta,n}^{ad}$. Moreover, the can. map $\mathcal{M}_\infty \to \mathcal{F}\ell$ induces on $\mathcal{M}_\infty^\diamond \to \mathrm{im}(\pi)^\diamond$ the structure of a pro-étale $GL_n(\mathbb{Q}_p)$-torsor.

Defn. Let (R,R^+) be any pftd $(\breve{\mathbb{Q}}_p,\breve{\mathbb{Z}}_p)$-alg., so an (R,R^+) pt of $\mathcal{Fl}$ corresp to $D\otimes_{\mathbb{Q}_p}R \xrightarrow{q} W$, w/ W fin. loc. free of rk $h-d$.

Defn. $\mathcal{F}_x$ is the VB on $X_{\mathrm{Spa}(R,R^+)^\flat}$ obtained by modifying $\mathcal{E}(H)=\mathcal{E}(D^\vee)$. Given $i: \mathrm{Spa}(R,R^+)\hookrightarrow X$, it is true that $i^*\mathcal{E}(H)\cong$ VB assoc. w/ $R\otimes_{\mathbb{Q}_p}D^\vee$, so in particular, get a surj. $\mathcal{E}(H)\twoheadrightarrow i_*i^*\mathcal{E}(H)$
$$\begin{array}{ccc} \mathcal{E}(H) & \twoheadrightarrow & i_*i^*\mathcal{E}(H) \\ & \searrow & \downarrow \\ & & (\ker q)^\vee \end{array}$$
Then $\mathcal{F}$ is the kernel of $\mathcal{E}(H)\to(\ker q)^\vee$

Thm (S-W) $S=\mathrm{Spa}(R,R^+)\longrightarrow\mathcal{Fl}$ factors thru $\mathcal{Fl}^{adm}\Leftrightarrow\forall$ geom. pt $x\in S$, $\mathcal{F}_x\cong\mathcal{O}^h$. Moreover (if S is a geom. pt) then $H^0(X_S,\mathcal{F})\cong V_pG_x$. G_x any p-div. gp assoc. w/ $x\in\mathcal{M}$ lifting $f(S)\in\mathcal{Fl}^{adm}$.

putting all together, we get that $\mathcal{M}_\infty$ identifies w/ the functor of modifs $\mathcal{F}$ of $\mathcal{E}(H)$ along the "div. at ∞" which are $\simeq\mathcal{O}^h$ at all geom. pts, + an isom. $\mathcal{O}_X^h\xrightarrow{\sim}\mathcal{F}$.

In fact, $\mathcal{M}_\infty: \widetilde{\mathrm{Perf}}_{/\breve{\mathbb{Q}}_p}\longrightarrow\mathrm{Sets}$ sends $S=\mathrm{Spa}(R,R^+)$ to the set of tuples $s_1,\dots,s_h\in H^0(X_{S^\flat},\mathcal{E}(H))$ s.t. $\mathrm{coker}(\mathcal{O}^h_{X_{S^\flat}}\to\mathcal{E}(H))$ is as it should be.

Defn. A local shtuka datum is a tuple (G,μ,b) where
(1) $G/\mathbb{Q}_p$ a reductive gp.
(2) μ is a conjugacy class of cochar. $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}\longrightarrow G_{\overline{\mathbb{Q}}_p}$.
(3) $b\in G(\breve{\mathbb{Q}}_p)$ is an elt, s.t. blablabla...

$b\rightsquigarrow\mathcal{E}_b=$ a G-bundle on any FF curve X_T, $T\in\mathrm{Perf}_{\overline{\mathbb{F}}_p}$. Then one can define a space "$\mathrm{Sht}_{G,\mu,b}$" of local G-shtukas w/ infinite level str., which is $\simeq\mathcal{M}_\infty$ when $(G,\mu,b)=(GL_n,(\underbrace{x,\dots,x}_{d},\underbrace{1,\dots,1}_{h-d}))$, $b=$ action of φ on a basis of $M(H)$.

Local shtuka spaces for GL_n

Fix $n\geq2$, $\mu=(k_1\geq\cdots\geq k_n)\in\mathbb{Z}^n$, $b\in GL_n(\breve{\mathbb{Q}}_p)$
(For simplicity, we will assume that $k_n\geq0$).

Defn $\mathrm{Sht}_{n,\mu,b}$ is the functor on $\mathrm{Perf}/\mathrm{Spd}\,\breve{\mathbb{Q}}_p=\widetilde{\mathrm{Perf}}/\breve{\mathbb{Q}}_p$ sending $S/\breve{\mathbb{Q}}_p$ to the set of injective maps $i: \mathcal{O}^n_{X_{S^\flat}}\hookrightarrow\mathcal{E}_{b,S^\flat}$ s.t. $i|_{X_{S^\flat}\setminus S}$ is an isom. and $\forall$ pts $x: \mathrm{Spa}(K_x,K_x^+)\longrightarrow S$, the coker of $\mathcal{O}^n_{X_{x^\flat}}\hookrightarrow\mathcal{E}_{b,x^\flat}$ which is $\cong\mathrm{coker}(\underbrace{M^+_{dR}(\mathcal{O}^n_{X_{S^\flat}})}_{B^+_{dR}(K_x)^n}\hookrightarrow M^+_{dR}(\mathcal{E}_{b,x^\flat}))$ has elementary divisors (as a $B^+_{dR}(K_x)$-mod.) which are $\leq(k_1\geq\cdots\geq k_n)$
Here $B^+_{dR}(K_x)$ is a DVR, so any fin. tors mod M is $\simeq\bigoplus_{i\geq1}B^+_{dR}(K_x)/\xi^{n_i}$ for some unique $n_1\geq n_2\geq\cdots\geq n_n$.
$(n_1\geq n_2\geq\cdots\geq n_n)\leq(k_1\geq k_2\geq\cdots\geq k_n)$ if $\sum_{1\leq i\leq j}n_i\leq\sum_{1\leq i\leq j}k_i$ w/ $=$ if $j=n$

Prop. $\mathrm{Sht}_{n,\mu,b}$ is a proétale sheaf.
Idea: sections of vbs on curves are proétale sheaves.

Thm (Scholze) $\mathrm{Sht}_{n,\mu,b}$ is a <u>nice</u> (= spatial, proper) diamond / $\mathrm{Spd}\,\breve{\mathbb{Q}}_p$
idea of pf: construct a period map $\pi_{GM}: \mathrm{Sht}_{n,\mu,b}\longrightarrow\mathrm{Gr}_{n,\leq\mu}$ where $\mathrm{Gr}_{n,\leq\mu}$ is a "closed Schubert cell in a B_{dR}^- affine Grass."
Then show that
① $\mathrm{Gr}_{n,\leq\mu}$ is a <u>nice</u> diamond over $\mathrm{Spd}\,\breve{\mathbb{Q}}_p$
② Im of π is an open subdiamond of $\mathrm{Gr}_{n,\leq\mu}$ ($=\mathrm{Gr}^{b\text{-adm}}_{n,\leq\mu}$)
③ Show that $\mathrm{Sht}_{n,\mu,b}\longrightarrow\mathrm{Gr}^{b\text{-adm}}_{n,\leq\mu}$ is a proétale $GL_n(\mathbb{Q}_p)$-torsor.
Thus $\mathrm{Sht}_{n,\mu,b}$ is a diamond.

Def^n of π_{GM}

Defn. $Gr_{n,\leq\mu}$ is the functor sending $Spa(A,A^+)/\mathbb{Q}_p$ to the set of $\mathbb{B}^+_{dR}(A)$-lattices $M \in \mathbb{B}_{dR}(A)^n$ s.t.

(1) $M \subseteq \mathbb{B}^+_{dR}(A)^n$ ("$k_i \geq 0$")

(2) for all $A \longrightarrow K$, the elementary divisors of $(\mathbb{B}^+_{dR}(A)^n/M) \otimes_{\mathbb{B}^+_{dR}(A)} \mathbb{B}^+_{dR}(K)$ are $\leq \mu$, K a pftd field.

Note If $\mu = (\underbrace{1 \geq 1 \geq \cdots \geq 1}_{d} \geq \underbrace{0 \geq \cdots \geq 0}_{n-d})$, then $\nu \leq \mu \Leftrightarrow \nu = \mu$

Moreover (A,A^+)-pts of $Gr_{n,\leq\mu} = \{\mathbb{B}^+_{dR}(A)^n \supseteq M \supseteq (\xi\mathbb{B}^+_{dR}(A))^n$

s.t. $Q = \mathbb{B}^+_{dR}(A)^n/M \twoheadleftarrow (\mathbb{B}^+_{dR}(A)/\xi)^n = A^n$.

Q has rk d, $\forall A \twoheadrightarrow K \Rightarrow Q$ fin. proj. of rk d.

$\rightsquigarrow Gr_{n,\leq\mu} = Gr(d,n)^\diamond$

To define π_{GM}, note that B-pairs still work in relative setting

$\rightsquigarrow$ for any $S = Spa(A,A^+)/\breve{\mathbb{Q}}_p$ and any S-pt

$(i: \mathcal{O}^n_{X_{S^b}} \hookrightarrow \mathcal{E}_{b,S^b})$ of $Sht_{n,\leq\mu,b}$, we get

$$M^+_{dR}(i): \underset{\mathbb{B}^+_{dR}(A)^n}{\overset{\parallel}{M^+_{dR}(\mathcal{O}^n_{X_{S^b}})}} \hookrightarrow M^+_{dR}(\mathcal{E}_{b,S^b}) \cong \mathbb{B}^+_{dR}(A)^n$$

π_{GM} sends this to $M^+_{dR}(im(i)) \subseteq M^+_{dR}(\mathcal{E}_{b,S^b}) \cong \mathbb{B}^+_{dR}(A)^n$

i.e., "forget the basis for $M^+_{dR}(im(i))$"

$(H^0(X_S, \mathcal{O}_X) = \mathbb{Q}_p$ if $|S|$ is conn'd. $0 \to \mathbb{Q}_p \to B_e \oplus B^+_{dR} \xrightarrow{x-y} B_{dR} \to 0$

$Im(\pi_{GM})$ open: interprete $im(\pi_{GM})$ as follows:

$M^+_{dR}(\mathcal{E}_b, \underbrace{Spa(A,A^+)}_{S}{}^b) \cong \mathbb{B}^+_{dR}(A)$, S-pt of $Gr_{n,\leq\mu}$ gives rise to some $\mathbb{B}^+_{dR}(A)$-lattice S in $M_{dR}(\mathcal{E}_b, S^b)$, so we get a modif $\mathcal{F}$ of $\mathcal{E}_{b,S^b}$

$0 \to \mathcal{F} \to \mathcal{E}_{b,S^b} \to Q \to 0$

Prop $Gr^{b\text{-adm}}_{n,\leq\mu}$ is the subfunctor where $\mathcal{F}_x$ has all λ_i's $= 0$ $\forall x = Spa(K,K^+) \to S$

Key result (K-L) λ_i's are upper semi-continuous.

Scattered aspects of diamond etc

1. Smooth morphisms of diamonds.

Defn. Let $Y \xrightarrow{f} X$ be a map of diamonds

- f is sm. of rel. dim $0 \Leftrightarrow f$ is étale
- f is sm. of rel. dim $\leq d$ if (pro-étale locally on X and analytically locally on Y) f can be factored as

$$\begin{tikzcd} Y \arrow[r] & W & \tilde{W} \arrow[l] \arrow[d] \\ & & X \end{tikzcd}$$

where

- $Y \to W$ is sm. of rel. dim $\leq d-1$,
- $\tilde{W} \to W$ is a Γ-torsor for Γ a pro-p gp.
- $\tilde{W} \to X$ is rep'ble sm. of rel. dim 1.
  i.e., factored as $\tilde{W} \xrightarrow{\text{étale}} \mathbb{B}^1_X \xrightarrow{pr} X$
- f is smooth if loc. on Y, it is sm. of some rel. dim $d_i < +\infty$.

Basic Properties

① smoothness of $Y \to X$ is pro-étale local on X

② composition & pullbacks of sm. maps are sm.

③ If $Y \to X$ sm., $|Y| \to |X|$ is open.

(Idea of pf. Devissage to the fact that $\mathbb{B}^1_X \to X$ is always open, this reduces to openness of $Spa(A\langle T\rangle, A^+\langle T\rangle) \to Spa(A,A^+)$)

④ If $Y \to X$ is sm. and $|Y| \to |X|$ surj., then $Y \to X$ admits a section after pullback along a pro-étale cover.

⑤ "smoothness" is smooth-local on the target".

Examples

① If $Y \to X$ is a sm. map of rigid spaces, then $Y^\diamond \to X^\diamond$ is smooth.

② If X any diamond, the projection $X \times Spd\,\mathbb{Q}_p \to X$ is smooth "of pure dim 1".

pf of ②. WLOG, X is pftd. Then $X \times Spd\,\mathbb{Q}_p = X \times h_{Spa\,\mathbb{Q}_p^{cyc,b}}/\mathbb{Z}_p^\times$

$$\cong X \times h_{Spa\,\mathbb{F}_p((t^{1/p^\infty}))}/\mathbb{Z}_p^\times$$

$$\cong (\mathbb{D}^1_X \setminus 0)/\mathbb{Z}_p^\times.$$

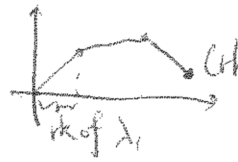
In particular, $\mathrm{Spd}\,\mathbb{Q}_p \longrightarrow \mathrm{Spd}\,\mathbb{F}_p$ is smooth.
One can also show that $\mathrm{Spd}\,\mathbb{Q}_p/\varphi^{\mathbb{Z}} \longrightarrow \mathrm{Spd}\,\mathbb{F}_p$ is smooth and proper(!)

2. Diamond stacks.

Defn. A diamond stack is a cat $X \longrightarrow \mathrm{Perf}^{\text{pro-étale}}$ fibered in gpoids s.t. 1) $X \xrightarrow{\Delta} X \times X$ is rep'ble in diamonds
2) $\exists$ a diamond S + a surj. smooth map $S \to X$.

Ex. & Prop. The fibered cat. $\mathrm{Bun}_n \longrightarrow \mathrm{Perf}$ w/ $\mathrm{Bun}_n(T) = \{$gpoids of rk n VBs over $X_T\}$ is a diamond stack.
- Bun_n is a stack reduces to the fact that X_T is prepftd, and VBs on prepftd spaces can be glued proétale locally.
- diagonal is rep'ble means that for any $T \in \mathrm{Perf}$, and 2 rk n VB on X_T, the sheaf $(S \to T) \longmapsto \mathrm{Isom}_{X_S}(\mathcal{E}_{1,S}, \mathcal{E}_{2,S})$ is a diamond.

To understand Bun_n, recall that when S is a geom. pt, any rk n VB on X_S is $\simeq \mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_m)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$.

Defn. $HN(\mathcal{E}) =$  $(\mathrm{rk}\,\mathcal{E}, \deg \mathcal{E})$.

Thm (K-L). suppose given $\mathcal{E}/X_S$, $\forall x \in S$, can form $HN_x(\mathcal{E})$. then
① rk & deg are loc. constant in families
② $x \rightsquigarrow HN_x(\mathcal{E})$ is upper semicontinuous, i.e., can only jump on closed subset

Ex. If we think of $Gr(2,5)$ as giving some univ. modif. of $\mathcal{O}(\frac{2}{5})$ then the bundles we can get (i.e. bundles F sitting as $0 \to F \to \mathcal{O}(\frac{2}{5}) \to i_* K^2 \to 0$ on (K, K^+)-pts of Gr). are either $\simeq \mathcal{O}^5$ or $\simeq \mathcal{O}(\frac{1}{3}) \oplus \mathcal{O}(\frac{1}{2})$.

$\mathrm{Bun}_n = \coprod_d \mathrm{Bun}_n^d$. Moreover $\mathrm{Bun}_n^d \simeq \mathrm{Bun}_n^{d+n}$
$\mathcal{E} \longmapsto \mathcal{E} \otimes \mathcal{O}(1)$.

Thm (Fargues). $\mathrm{Bun}_n^{d,ss}$ is the classifying stack $BD_{d/n}^\times$.
In particular, $\mathrm{Bun}_n^{0,ss} \simeq BGL_n(\mathbb{Q}_p)$.

Q: Can we see why $B(GL_n(\mathbb{Q}_p))$ is smooth?

Abstract nonsense: If G is a loc profinite gp, then $B\underline{G} = [pt/\underline{G}]$ is smooth
$\Leftrightarrow \exists$ a proétale $\underline{G}$-torsor $Y \to X$ in diamonds which are both smooth$/\mathrm{Spd}\,\mathbb{F}_p$.

$$\Rightarrow \begin{tikzcd} Y \arrow[r, "sm."] \arrow[d, "sm."'] & pt \arrow[d, "sm."] \\ X \arrow[r, "sm"] & {[pt/G]} \end{tikzcd}$$

Set $Y = S \longmapsto \{$inj maps $\mathcal{O}^n \hookrightarrow \mathcal{O}(\frac{1}{n+1})\} \overset{\text{open}}{\subseteq} \mathcal{H}^0(\mathcal{O}(\frac{1}{n+1}))^{\oplus n}$ in fact, it's smooth
have natural $GL_n(\mathbb{Q}_p)$-action on this, $X = Y/\underline{GL_n(\mathbb{Q}_p)}$
$\simeq \{$surj. $\mathcal{O}(\frac{1}{n+1}) \twoheadrightarrow \mathcal{L}$, $\mathcal{L}$ a line bundle of deg $1\}$
So we get $\tilde{X} \to X$ a $\mathbb{Q}_p^\times$ torsor of isom $\mathcal{L} \xrightarrow{\sim} \mathcal{O}(1)$.
So $\tilde{X} =$ surj's $\mathcal{O}(\frac{1}{n+1}) \twoheadrightarrow \mathcal{O}(1)$
$=$ inj's $\mathcal{O} \hookrightarrow \mathcal{O}(\frac{n}{n+1}) \underset{\text{open}}{\subseteq} \mathcal{H}^0(\mathcal{O}(\frac{n}{n+1}))$.