Notes on FGA

Time: Spring 2016.

FGA explained.

The Hilbert & Quot Functors

set up all schemes are loc. Noeth. Write <u>Sch</u> for this cat.

Defn. (Quot functor) Let S be a Noetherian scheme, $X \longrightarrow S$ finite type. $\mathcal{E} \in Coh(X)$. Define $\underline{Quot}_{\mathcal{E}/X/S} : \underline{Sch}/S \longrightarrow \underline{Sets}$.

$$(T \to S) \longmapsto \left\{ (\mathcal{F}, q) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ coh on } X_T, \text{ flat over } T \\ q : \mathcal{E}_T \twoheadrightarrow \mathcal{F}, \ supp(\mathcal{F}) \longrightarrow T \text{ is proper} \end{array} \right\} \Big/ \approx$$

where $(\mathcal{F}, q) \simeq (\mathcal{G}, r)$ iff
$$\begin{tikzcd} & \mathcal{F} \arrow[d, "s"] \\ \mathcal{E}_T \arrow[ur, two heads] \arrow[r, two heads] & \mathcal{G} \end{tikzcd}$$
equivalently $\ker(q) = \ker(r)$.

Defn. For $\mathcal{E} = \mathcal{O}_X$, the functor of $\underline{Quot}_{\mathcal{O}_X/X/S} = \underline{Hilb}_{X/S}$.

Rmk. Note that $\underline{Quot}_{\mathcal{E}/X/S}$ is an fpqc-sheaf. (by descent).

Fact (to be proven) $\underline{Quot}$ is representable if $X \longrightarrow S$ is q-proj, not in general.

Thm. (Snapper's Lemma) Let S be a Noeth. scheme, $X \longrightarrow S$ finite type, $\mathcal{F} \in Coh(X)$, w/ support proper over an Artinian subscheme of S. Let $\mathcal{L}_i \in Pic(X)$ and consider function $\Phi_{\mathcal{F}}(n_1, \cdots, n_m) = \chi(\mathcal{F} \otimes \bigotimes_{i=1}^{n} \mathcal{L}_i^{n_i})$. Then $\Phi_{\mathcal{F}}(n_1, \cdots, n_m) \in \mathbb{Q}[n_1, \cdots, n_m]$ has $\deg \leq \dim(supp\, \mathcal{F})$.

pf. FGA, thm B.7.

Defn. we call Φ the <u>Hilbert polynomial</u> of $\mathcal{F}$ wrt $\mathcal{L}$;

Ex. $S=\mathrm{Spec}(k)$, $X=\mathbb{P}^n_k$, $\mathcal{F}=\mathcal{O}_X$, $n=1$, $\mathcal{L}_1=\mathcal{O}(1)$.

Rmk For $(\mathcal{F},q)\in \underline{\mathrm{Quot}}_{\mathcal{E}/X/S}(T)$, each $t\in T$ gives
$(\mathcal{F}_t, q_t)\in \underline{\mathrm{Quot}}_{\mathcal{E}/X/S}(\kappa(t))$.
$\mathcal{F}_t$ satisfies assumption of snapper's Lemma (over $\mathrm{Spec}(\kappa(t))$)

Defn Let $\mathcal{L}$ be a line bundle on X. Define $\Phi_{\mathcal{F}_t}(m)=\chi(\mathcal{F}_t\otimes\mathcal{L}_t^m)\in\mathbb{Q}[m]$.
This gives $T\longrightarrow \mathbb{Q}[m]$.
$t\longmapsto \Phi_{\mathcal{F}_t}$.

Lemma The function $\Phi_{\mathcal{F}}: T\longrightarrow\mathbb{Q}[m]$ is locally constant.

Cor. The functor $\underline{\mathrm{Quot}}_{\mathcal{E}/X/S}$ decomposes as a coproduct (as sheaves)
$$\underline{\mathrm{Quot}}_{\mathcal{E}/X/S}=\coprod_{\Phi\in\mathbb{Q}[m]}\underline{\mathrm{Quot}}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}.$$

Thm (Grothendieck) Let S be Noeth., let $X\longrightarrow S$ be proj.*, $\mathcal{L}$ be the very ample line bundle of π, $\forall\ \mathcal{E}\in\mathrm{Coh}(X)$, any $\Phi\in\mathbb{Q}[m]$, $\underline{\mathrm{Quot}}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is represented by a proj. S-scheme $\mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$.

$*$: proj. means $X \overset{\text{closed}}{\hookrightarrow} \mathrm{Proj}\,\mathrm{Sym}^*\mathcal{F}$ for $\mathcal{F}\in\mathrm{Coh}(S)$.

Cor. $X\longrightarrow S$ as above, then $\underline{\mathrm{Quot}}_{\mathcal{E}/X/S}$ is represented by a countable disjoint union of proj. schemes.

Rmk. slight generalization to q-proj schemes, whose Quot scheme will be q-proj.
Here q-proj means $X\hookrightarrow\bar{X}\overset{\text{closed}}{\hookrightarrow}\mathrm{Proj}(\mathcal{E})$, where $\mathcal{E}$ is a loc. free rk n sheaf on S.

Examples $X=S$, $\mathcal{L}=\mathcal{O}_S$, then

(1) if $\mathcal{E}=\mathcal{O}_S^{n+1}$, $\Phi=1$, we get
$$\underline{\mathrm{Quot}}^{1,\mathcal{O}_S}_{\mathcal{O}_S^{n+1}/S/S}(T)=\{(\mathcal{F},q)\mid \mathcal{F}\text{ flat over }T,\ \mathcal{O}_S^{n+1}\twoheadrightarrow\mathcal{F},\ \chi(\mathcal{F}_t)=1\}$$
$$=\{(\mathcal{F},q)\mid\mathcal{F}\in\mathrm{Pic}(T),\ \mathcal{O}_T^{n+1}\twoheadrightarrow\mathcal{F}\}$$
$$=(\mathbb{P}^n_T)^\vee=(\mathbb{P}^{n,\vee})(T).$$

(2) S noeth., Grassmanian:
$$\mathrm{Gr}(r,d):\underline{Sch/S}\longrightarrow\underline{Sets}$$
$$(T\to S)\longmapsto\{\mathcal{O}_T^r\twoheadrightarrow\mathcal{F}\mid\mathcal{F}\text{ locally free of rk }d\}$$
$$\| \quad \underline{\mathrm{Quot}}^{d,\mathcal{O}_S}_{\mathcal{O}_S^r/S/S}$$
Similarly $\mathrm{Gr}(\mathcal{E},d)=\underline{\mathrm{Quot}}^{d,\mathcal{O}_S}_{\mathcal{E}/S/S}$
$\mathrm{Gr}(n+1,r+1)=\mathrm{Hilb}^P_{\mathbb{P}^{n+1}_S/S}$, $P(m)=\binom{r+m}{m}$.

Rmk Grassmanian is representable, and <u>Quot</u> will be a subscheme of Grassmanian.

Thm Let S be a Noetherian scheme, $X \subseteq \mathbb{P}(V)$ subscheme, for V a vector bundle on S. Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}(V)}(1)|_X$, $\mathcal{E} \in Coh(X)$, a quotient of $(\pi^* W)(r)$ some vb W on S, $\pi: X \to S$, $r \in \mathbb{Z}$. $\Phi \in \mathbb{Q}[\lambda]$. Then $\underline{Quot}^{\Phi, \mathcal{L}}_{\mathcal{E}/X/S}$ is represented by a scheme $Quot^{\Phi, \mathcal{L}}_{\mathcal{E}/X/S}$ which can be embedded into $\mathbb{P}(F)$, for some vb F on S.

e.g. If S is proj. over a Noetherian ring, then every $\mathcal{E}$ on X satisfies the hypothesis above.

Outline of pf: Step 1: reduction, suffices to take $\mathcal{E} = (\pi^* W)(r)$, $X = \mathbb{P}(V)$. Thus we have to construct $Quot^{\Phi, \mathcal{O}(1)}_{\pi^*(W)/\mathbb{P}(V)/S}$.

Rmk For each fixed quotient $0 \to \mathcal{G} \to \mathcal{E}_T \xrightarrow{q} \mathcal{F} \to 0$ (on X_T), $\forall t \in T$, $\exists\, m \in \mathbb{Z}$, s.t. $\forall\, r \geq m$, we have

- $H^i(X_t, \mathcal{F}_t(r)) = 0 \quad \forall\, i > 0$
- $\mathcal{F}_t(r)$ is globally generated.

Thus $\Phi(r) = h^0(X_t, \mathcal{F}_t(r)) \quad \forall\, r \geq m$ (same for $\mathcal{E}, \mathcal{G}$).

Note that m depends a priori on t, $\langle \mathcal{F}, q \rangle \in \underline{Quot}^{\cdot\cdot}_{\cdot\cdot}(T)$, now we wanna do this ~~exist~~ only depend on Φ.

Main Idea: we may reconstruct $0 \to \mathcal{G} \to \mathcal{E}_T \to \mathcal{F} \to 0$ from

$$0 \to \pi_* \mathcal{G}(r) \to \pi_* \mathcal{E}_T(r) \to \pi_* \mathcal{F}(r) \to 0.$$

Note that we clearly know what $\Phi_{\mathcal{F}}$ and $\Phi_{\mathcal{E}}$ are. So we know $\Phi_{\mathcal{G}}$. So we'll realize $\underline{Quot}^{\cdot\cdot}_{\cdot\cdot}$ as a $\underline{\text{subscheme}}$ of $Gr(W \otimes_{\mathcal{O}_S} Sym^r(V), \Phi(r))$.

Step 2. make m independent of $t, T, \langle \mathcal{F}, q \rangle, \cdots$ (Castelnuovo-Mumford regularity)

Step 3. Prove that you can do this in a relative setting.

Step 4. Prove that $\underline{Quot} \longrightarrow Gr$ is a locally closed immersion.

Step 5. projectivity.

Defn. Let k be a field, $\mathcal{F} \in Coh(\mathbb{P}^n)$, let $m \in \mathbb{Z}$, $\mathcal{F}$ is $\underline{\text{m-regular}}$ if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0 \quad \forall\, i \geq 1$.

Lemma If $\mathcal{F}$ is $\underline{\text{m-regular}}$, and $H \subseteq \mathbb{P}^n$ is a hyperplane not containing any associated pts of $\mathcal{F}$, then $\mathcal{F}|_H$ is m-regular as well. (same m!)

Lemma (Castelnuovo) Let $\mathcal{F}$ be m-regular, then

① $H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{F}(r)) \twoheadrightarrow H^0(\mathbb{P}^n, \mathcal{F}(r+1))$, $\forall\, r \geq m$.

② $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$, $\forall\, r \geq m-i$, $i \geq 1$, i.e., m'-reg, $\forall\, m' \geq m$.

③ $\forall\, r \geq m$, $\mathcal{F}(r)$ is globally generated and has no higher coh.

Lemma. $\mathcal{F}$ coh on $\mathbb{P}^n$, H as before, assume $\mathcal{F}_H$ is m-regular, and for some $r \geq m$, the map $\nu_r : H^0(\mathbb{P}^n, \mathcal{F}(r)) \longrightarrow H^0(H, \mathcal{F}_H(r))$, then ν_{r+p} is surjective, $\forall p \geq 0$.

Thm (Mumford) Let $p, n \in \mathbb{Z}_{\geq 0}$, $\exists F_{p,n} \in \mathbb{Z}[x_0, \cdots, x_n]$, s.t., let k be a field, $\mathcal{F}$ a coh. sheaf on $\mathbb{P}^n_k$ which embeds $\mathcal{F} \subseteq \mathcal{O}^p_{\mathbb{P}^n}$. Write $\chi(\mathcal{F}(r)) = \sum_{i \geq 0}^{n} a_i \binom{r}{i}$, where $a_i \in \mathbb{Z}$, then $\mathcal{F}$ is m-regular, for $m = F_{p,n}(a_0, \cdots, a_n)$.

Flattening Stratification

Lemma. Let $\varphi : T \longrightarrow S$ be a morphism of Noetherian schemes $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^n_S)$, then $\exists r_0$ (depends on φ), s.t. $\forall r > r_0$, $\varphi^* \pi_*(\mathcal{F}(r)) \cong \pi_*(\mathcal{F}_T(r))$.

Lemma $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^n_S)$, S Noetherian. If $\pi_*(\mathcal{F}(m))$ is locally free for all sufficiently large m, then $\mathcal{F}$ is flat.

Thm S Noetherian, $\mathcal{F}$ coherent on $\mathbb{P}^n_S$, let I be the set of Hilbert polynomials occurring for $\mathcal{F}_s$, $\forall s \in S$, then

- I is finite
- $\forall f \in I$, $\exists S_f \subseteq S$ st. $\forall s \in S_f$, $\mathcal{F}|_{\mathbb{P}^n_s}$ has Hilbert poly $= f$.
- $i : \coprod_I S_f \longrightarrow S$, then $i^*\mathcal{F}$ is flat, and i is the universal map with this property.
- $\overline{S_f} \subseteq \bigcup_{I_f} S_g$, where I_f only consists of $g \in I$, where $f(n) \leq g(n)$ for large n.

pf. we may assume $S = \mathrm{Spec}(A)$.

Case 1 $n = 0$ use the fact that every f.g. module M over Noetherian A has a finite filtration, s.t. successive quotient will be of the form A/I.

General case: use the following proof omitted

Thm S Noetherian integral scheme, $p : X \longrightarrow S$ finite type, $\mathcal{F}$ coh on X, then $\exists$ nonempty open $U \subseteq S$, s.t. $\mathcal{F}|_{X_U}$ is flat $/U$.

So we have $V_i \hookrightarrow S$ s.t. $\mathcal{F}$ pull back to V_i is flat and $\bigcup V_i \longrightarrow S$ is set-theoretically bijective

Claim: $\exists N$ s.t. $\forall m > N$: $\begin{cases} H^i(\mathcal{F}_s(m), \mathbb{P}^n_s) = 0 & \forall i > 0, s \in S \\ H^0(\mathcal{F}_s(m), \mathbb{P}^n_s) = \pi_*(\mathcal{F}(m))|_s \end{cases}$

(use the V_i's before, semi-continuity and first lemma in this section)

Consider $E_i = \pi_*(\mathcal{F}(N+i))$ then

$E_0 \rightsquigarrow$ stratification where the restrictions are loc. free

$\vdots$

$E_n \rightsquigarrow$ ―――― " ――――

then we stop. so we get $\pi_*(\mathcal{F}(N+j))$ will have constant rk on each strata $= f(N+j)$, so we get W_f.

For any $j \geq n+1$, define I_j on W_f (a nilpotent ideal) s.t. $\pi_*(\mathcal{F}(N+j))$ is loc. free on $\mathcal{O}_{W_f}/I_j$

Lastly, define $I = \sum I_j$, and $S_f = \mathcal{O}_{W_f}/I$

suppose $i^*\mathcal{F}$ is flat for $i: T \to S$, then

$$\begin{tikzcd} T \arrow[r, "i"] \arrow[dr] \arrow[ddr, bend right] & S \\ & \bigcup W_f \arrow[u, bend right] \\ & \bigcup S_f \arrow[uu, bend right] \end{tikzcd}$$

Constructing Quot Scheme

Step 1. $\mathrm{Quot}^{\Phi, L}_{E/X/S} \longrightarrow \mathrm{Quot}^{\Phi, L}_{\pi^* W(r)/\mathbb{P}(V)/S}$ is a closed immersion.

Lemma
$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$
, $\mathcal{E} \xrightarrow{f} \mathcal{F}$, $\mathcal{F}$ flat over X.
$\exists T' \subseteq T$ closed, s.t. $\forall\, U \xrightarrow{\varphi} T$, s.t. $\varphi^*(f)$ $\mathcal{E}_U \longrightarrow \mathcal{F}_U$ is zero, $\exists!\, U \longrightarrow T' \subseteq T$.

Step 2. by CM regularity $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}_T \longrightarrow \mathcal{F} \longrightarrow 0$ w/ $\Phi(\mathcal{F})$ given,
$\exists m$, s.t.
$$\begin{tikzcd} 0 \arrow[r] & \pi^*\pi_*\mathcal{G}(m) \arrow[r] \arrow[d] & \pi^*\pi_*\mathcal{E}_T(m) \arrow[r] \arrow[d] & \pi^*\pi_*\mathcal{F}(m) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{G}(m) \arrow[r] & \mathcal{E}_T(m) \arrow[r] & \mathcal{F}(m) \arrow[r] & 0 \end{tikzcd}$$

So we only have to keep track of $\pi^*\pi_*\mathcal{E}_T(m) \twoheadrightarrow \pi^*\pi_*\mathcal{F}(m)$.
i.e., we have $\mathrm{Quot}^{\Phi, L}_{E/X/S} \longrightarrow \mathrm{Grass}(\pi_*(\pi^* W \otimes \mathcal{O}_{\mathbb{P}(V)}(n)), \Phi(n))$
$$\pi_*(\pi^* W \otimes \mathcal{O}_{\mathbb{P}(V)}(n)) = W \otimes_{\mathcal{O}_S} \mathrm{Sym}^n(V).$$

Step 3.
$$\begin{tikzcd} X_{Gr} \arrow[r] \arrow[d] & X \arrow[d] \\ Gr \arrow[r] & S \end{tikzcd}$$
on Gr we have
$0 \longrightarrow K \longrightarrow \pi_*\mathcal{E}(n) \longrightarrow \mathcal{G} \longrightarrow 0$
So we have
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[dr] & \pi^*\pi_*\mathcal{E}(n) \arrow[d, two heads] & & \\ & & \mathcal{E}(n) \arrow[r] & \mathcal{F}' \arrow[r] & 0 \end{tikzcd}$$

We find locus of Gr where $\mathcal{F}$ is flat of right hilb poly. (by flattening stratification)

Step 4. valuative criterion

Introduction to deformation theory

Defn A deformation of $V \subseteq X/k$ is a family $\tilde{V} \subseteq \tilde{X} = X \times_k A$, where A is an artinian local k-alg., $\tilde{V}$ is flat over $\operatorname{Spec} A$, and $\tilde{V} \times_A k = V$

Prop. $\{\text{Deformation of smooth } V \subseteq X \text{ over } \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)\} = H^0(V, N_{V/X})$

pf $R \longrightarrow R/I$ $\quad \tilde{R} = R[\varepsilon]/(\varepsilon^2)$ $\quad \tilde{I}$ satisfies:

1. $\tilde{I}|_{\varepsilon=0} = I$
2. $(\varepsilon) \otimes \tilde{R}/\tilde{I} \hookrightarrow \varepsilon\tilde{R}/\tilde{I}$ i.e. if $\varepsilon f \in \tilde{I}$, then $f \in \tilde{I}$

Hence given $f \in I$, $\exists\, g \in R$, s.t. $f + \varepsilon g \in \tilde{I}$, g is unique up to mod I, so this gives map $I/I^2 \longrightarrow R/I$ of R-modules

$$f \longmapsto (g \text{ w/ } f + \varepsilon g \in \tilde{I})$$

things work conversely and glues.

Notation (Art/k) denote cat. of local artin k-alg. w/ residue field k.

Defn. A <u>deformation functor</u> is a covariant functor

$D: (\mathrm{Art}/k) \longrightarrow (\mathrm{Set})$ satisfying $|D(k)| = 1$.

e.g. Fix R local ring$/k$ residue is k.

$D(A) = \operatorname{Hom}_k(R, A)$.

Rmk
1. $h_R \cong h_{\hat{R}}$, because $\hat{A} = A$.
2. $D(A)$ is always nonempty w/ distinguished pt $D(k) \to D(A) \to D(k)$.

Q: $B \longrightarrow A$ in Art/k, $D(B) \longrightarrow D(A)$, surj? inj?

Defn $B \twoheadrightarrow A$ is a <u>small extn</u> if $0 \to M \to B \to A \to 0$ s.t. $M \cdot m_B = 0$.

Lemma $B \twoheadrightarrow A$ factors as sequence of small extensions.

For $D = h_{R = \hat{R}}$, this question has an explicit answer.

<u>Thm</u> Fix
$$\begin{array}{ccccccccc} 0 & \to & M & \to & B & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M' & \to & B' & \to & A' & \to & 0 \end{array}$$
small extn, R complete local w/ finite dim'l tangent space. Then $\exists$ exact sequence of sets:

$$T_1 \otimes_k M \circlearrowright h_R(B) \longrightarrow h_R(A) \xrightarrow{ob} T_2 \otimes_k M, \text{ where } T_1 = (m_R/m_R^2)^\vee.$$

naturality:

$$\begin{array}{ccccccc} T_1 \otimes_k M & \circlearrowright & h_R(B) & \longrightarrow & h_R(A) & \longrightarrow & T_2 \otimes_k M \\ \downarrow & \text{equivariant} & \downarrow & & \downarrow & & \downarrow \\ T_1 \otimes_k M' & \circlearrowright & h_R(B') & \longrightarrow & h_R(A') & \longrightarrow & T_2 \otimes_k M' \end{array}$$

pf. Let $d = \dim_k T_1$, with basis $t_1, \dots, t_d \in m_R$

$k[\![X_i]\!] = S \xrightarrow{\ f\ } R$ isom. on tangent spaces.

$x_i \longmapsto t_i$

Let $n = m_S$ $\quad J = \ker(f)$ $\quad$ Note that $J \subseteq n^2$.

Then we let $T_2 = (J/nJ)^\vee$

Fix $\varphi: R \to A$.

$$\begin{array}{ccccccccc} & & & & S & \xrightarrow{f} & R & & \\ & & & & \tilde{\varphi}\,\vdots & & \downarrow \varphi & & \\ 0 & \to & M & \to & B & \to & A & \to & 0 \end{array}$$

$\tilde{\varphi}$ will descend to $R \to B$ iff $\tilde{\varphi}|_J = 0$.

Study space of lifts $\tilde{\varphi}$

Claim: $\alpha, \beta: S \to B$ both lift φ, then $h = (\alpha - \beta): S \to M$ is a k-linear derivation:
$h(fg) = h(f)g(0) + h(g)f(0)$, hence,
$h(c_0 + \sum c_i x_i + \text{h.o.t.}) = \sum c_i h(x_i)$

Conclusion: $\{\tilde{\varphi}\} \simeq T_1 \otimes_k M$

$h(n^2) = 0 \Rightarrow h(J) = 0$, hence $\tilde{\varphi}|_J$ is independent of $\tilde{\varphi}$

Construction of ob: $\{R \to A\} \rightsquigarrow \{J \to M\}$
$\varphi \longmapsto \tilde{\varphi}|_J$ $\quad \tilde{\varphi}|_J$ kills nJ as $\tilde{\varphi}(nJ) \subseteq m_B \cdot M = 0$
hence $ob: \{R \to A\} \rightsquigarrow \{J/nJ \to M\}$.

Defn: A deformation functor admits a tangent-obstruction theory if $\exists$ f. dim. vector space/k; T_1, T_2, s.t.
① $\forall$ small extn $0 \to M \to B \to A \to 0$ — only exact at these two places.
$\exists$ exact sequence $T_1 \otimes_k M \subset D(B) \to D(A) \to T_2 \otimes_k M$.
② This is left exact for $A = k$.
③ functoriality.

Prop. T_1 is unique up to canonical isom.:
$0 \to T_1 \xrightarrow{\sim} D(k[\varepsilon]/(\varepsilon^2)) \longrightarrow * \xrightarrow{ob} T_2$
$T_1 \simeq D(k[\varepsilon]/(\varepsilon^2)) \longrightarrow D(k)$ $\qquad D(k[\varepsilon]/\varepsilon^2) \times D(k[\varepsilon]/\varepsilon^2) \longrightarrow D(k[\varepsilon]/\varepsilon^2)$
$T_1 \simeq D(k[x,y]/(x,y)^2) \longrightarrow D(k[y]/y^2)$ $\qquad D(k[x,y]/(x,y)^2)$
$\cdot\lambda$ is easy to define.

pro-representable functors.

Defn. A deformation functor D is pro-representable if it's isomorphic to h_R for $R \in Loc/k$.

Last time: $R \cong k[\![x_1, \dots, x_n]\!]/J$, $J \subseteq n^2$, $n = (x_1, \dots, x_n)$ h_R has tangent-obstruction theory $T_1(R) \cong (m_R/m_R^2)^\vee$ $\quad T_2(R) = (J/nJ)^\vee$

Thm: If T_1, T_2 is a tangent-obstruction theory for h_R, then there exists a canonical isomorphism $T_1(R) \xrightarrow{\simeq} T_1$ and injection $T_2(R) \hookrightarrow T_2$

Idea: suppose R is Artinian
$0 \to J/nJ \to S/nJ \to R \to 0$ small extension.
$h_R(R) \longrightarrow (J/nJ) \otimes T_2 \simeq \mathrm{Hom}(T_2(R), T_2)$.
$id \longmapsto \varphi$

pf. By Artin-Rees, $n^i \cap J \subseteq nJ$ for sufficiently large i.
$0 \to (J+n^i)/(nJ+n^i) \to S/(nJ+n^i) \to S/(J+n^i) \to 0$
$(J+n^i)/(nJ+n^i) \simeq J/(nJ + (n^i \cap J)) = J/nJ$
ob: $h_R(S/(J+n^i)) \to (J/nJ) \otimes T_2 \cong \mathrm{Hom}(T_2(R), T_2)$.
$proj \longmapsto \varphi$.

Claim: TFAE
1. $T_2(R) \xrightarrow{\varphi} T_2$ is injective
2. $\forall$ ideal V of S w/ $nJ + n^i \subseteq V \subsetneq J + n^i$, the image of $proj$ under $ob: h_R(S/(J+n^i)) \to ((J+n^i)/V) \otimes T_2$ is nonzero.

by T-O theory

3. $\forall$ ideal V of S w/ $nJ + n^i \subseteq V \subsetneq J + n^i$, there is no

$$\begin{tikzcd} S/J \arrow[r, two heads] \arrow[d, dashed] & S/(J+n^i) \\ S/V \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[r] \arrow[d, dashed, "\psi\ \varphi"'] & S/J \arrow[r] \arrow[d, dashed] & S/(J+n^i) \\ S \arrow[r] & S/V \arrow[ur] & \end{tikzcd}$$

$\psi(s) - s \in J + n^i \quad \forall s \in S$

$\psi(J) \subseteq V$

if $j \in J \subseteq n^2$, $\cancel{\varphi}\psi(j) - j \in n(J+n^i)$

So $J + n^i \subseteq \cancel{\varphi(J}\psi(J) + n(J+n^i) + n^i \subseteq V + (nJ + n^i) = V$

But $J + n^i \not\supseteq V$

Cor. Let $D = h_R$, and let T_1, T_2 be $\overset{a}{\cancel{an}}$ T-O theory for D.
Let $d = \dim T_1$, $r = \dim T_2$, then $d \geq \dim(R) \geq d - r$.
If $\dim(R) = d - r$, then R is a complete intersection.
If $r = 0$, then $R \cong k[[x_1, \cdots, x_d]]$

pf. $R = k[[x_1, \cdots, x_d]]/J$, $J \subseteq n^2$, $\dim(J/nJ) \leq r$.
So J has $\leq r$ generators.

Defn. Let $\alpha : F \to G$ be a morphism of deformation functors, we say α is smooth if for all small extns $0 \to M \to B \to A \to 0$, the induced map $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective.

Defn. Let F be a deformation functor, and let $R \in \mathrm{loc}/k$. We say that $\alpha : h_R \to F$ is a pro-representable hull if α is smooth and the induced map $\alpha_* : h_R(k[\varepsilon]/\varepsilon^2) \xrightarrow{\sim} F(k[\varepsilon]/\varepsilon^2)$.

Lemma If F is a deformation functor, T_1, T_2 a tangent-obstruction theory for F. $\alpha : h_R \to F$ pro-rep hull, then T_1, T_2 also form a tangent-obstruction theory for h_R.

pf. For T_1, take $0 \to (\varepsilon)/(\varepsilon^2) \to k[\varepsilon]/\varepsilon^2 \to k \to 0$.

$$\begin{tikzcd} 0 \arrow[r] & T_1(R) \arrow[r] \arrow[d, "\wr"] & T_1(R) \arrow[r] \arrow[d, "\wr"] & pt \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T_1(R) \arrow[r] & T_1(F) \arrow[r] & pt \arrow[r] & 0 \end{tikzcd}$$

For T_2: choose $0 \to M \to B \to A \to 0$ small extn.

$$\begin{tikzcd} T_1(R) \otimes_k M \arrow[r] \arrow[d, "\wr"] & h_R(B) \arrow[r] \arrow[d] & h_R(A) \arrow[d] \arrow[dr] & \\ T_1 \otimes_k M \arrow[r] & F(B) \arrow[r] & F(A) \arrow[r] & T_2 \otimes_k M \end{tikzcd}$$

smooth implies exactness here.

Thm (Schlessinger) Let D be a deformation functor that has a tangent-obstruction theory. Then D has a pro-rep. hull.

Cor. A deform functor F w/ t-o theory. T_1, T_2 is pro-rep iff $\forall$ every $0 \to M \to B \to A \to 0$ small.
$0 \to T_1 \otimes_k M \to F(B) \to F(A) \to T_2 \otimes_k M$ is exact.

pf. Let $h_R \to F$ be a hull. Let B be the lowest dim'l
$\{B \mid h_R(B) \not\cong F(B)\}$

then

$$\begin{tikzcd} 0 \arrow[r] & T_1 \otimes M \arrow[r] \arrow[d, "\wr"] & h_R(B) \arrow[r] \arrow[d] & h_R(A) \arrow[d, "\wr"] \\ 0 \arrow[r] & T_1 \otimes M \arrow[r] & F(B) \arrow[r] & F(A) \end{tikzcd}$$

Prop Any two pro-rep. hull of F are isom.

$$\begin{tikzcd} h_R(R_n') \arrow[r] \arrow[d] & h_R(R_{n-1}') \arrow[d] \\ F(R_n') \arrow[r] & F(R_{n-1}') \\ h_{R'}(R_n') \arrow[r] \arrow[u] & h_{R'}(R_{n-1}') \arrow[u] \end{tikzcd}$$

gives $R \twoheadrightarrow R'$

conversely $R' \twoheadrightarrow R$.

Lemma auto-surj: $R \to R$ of a Noeth. ring is an isom.

Schlessinger's paper   /k field   Art/k

Defn. $F \longrightarrow G$ of functors is smooth if $\forall\ B \twoheadrightarrow A$ in Art/k, $F(B) \longrightarrow F(A) \times_{G(A)} G(B)$ is surjective.

Prop.
(1) $F \longrightarrow G$ smooth, then $F(A) \twoheadrightarrow G(A)$
(2) $R \longrightarrow S$, $h_R \longrightarrow h_S$ is smooth iff S is $\simeq R[[X_1, \cdots, X_n]]$.
(3) $F \longrightarrow G \longrightarrow H$, both smooth $\Rightarrow$ $F \longrightarrow H$ is smooth
(4) $u: F \to G$ surj., $v: G \longrightarrow H$, vu smooth $\Rightarrow$ v smooth.
(5) $F \to G$ smooth, then $F \times_G H \longrightarrow H$ is smooth.

pf of (2): if $h_S \to h_R$ smooth, $r\ (s)$ be m'x'l ideal in $R\ (S)$, $x_1, \cdots, x_n \in S$ be basis of $t^*_{S/R} = S/(s^2 + rS)$. Let $T = R[[X_1, \cdots, X_n]]$. We get
$S \longrightarrow T/(t^2 + rT)$, then lift by smoothness to $S \longrightarrow T$
induce $t^*_{S/R} \simeq t^*_{T/R}$, hence surjective. We ~~have~~ ~~may find~~ have $T \longrightarrow S$, induce $t^*_{T/R} \simeq t^*_{S/R}$ ...

Defn. (R, ξ) for F is a hull if $h_R \longrightarrow F$ is smooth, induce $t_R \xrightarrow{\sim} t_F$.

Rmk if (R, ξ) is a hull of F, $h_R \longrightarrow h_k$, $R \cong k[[X_1, \cdots, X_n]] \Leftrightarrow$ $F \longrightarrow h_k$ is smooth $\Leftrightarrow$ $F(B) \twoheadrightarrow F(A)$ for all $B \twoheadrightarrow A$.

$$\begin{array}{ccc} A' \times_A A'' = \{(a', a'') \mid \varphi(a') = \psi(a'')\} & \longrightarrow & A'' \\ \downarrow & & \downarrow \psi \\ A' & \xrightarrow{\varphi} & A \end{array}$$

Thm (Schlessinger) Let F be a deformation functor.
$(*)\ F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A'')$
Then F has a hull iff H_1: $*$ is surj. if $A'' \to A$ ~~small ext'n~~.
H_2: $*$ is bij. if $A'' = k[\varepsilon]/(\varepsilon^2)$, $A = k$.
H_3: $\dim_k(t_F) < \infty$.
pro-represented iff (H_1)-(H_3) + H_4: $F(A' \times_A A') \simeq F(A') \times_{F(A)} F(A')$ for all small ext'n $A' \to A$.

Rmk ① T-O theory for $F \Rightarrow (H_1)$-(H_3):
$$\begin{array}{ccccccccc} 0 \to & M & \to & A'' & \to & A & \to 0 \\ & \uparrow \wr & & \uparrow & & \uparrow & \\ 0 \to & (0, M) & \to & A' \times_A A'' & \to & A' & \to 0 \end{array}$$
gives
$$\begin{array}{ccccccc} T_1 \otimes M & \longrightarrow & F(A' \times_A A'') & \longrightarrow & F(A') & \longrightarrow & T_2 \otimes M \\ \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \wr \\ T_1 \otimes M & \longrightarrow & F(A'') & \longrightarrow & F(A) & \longrightarrow & T_2 \otimes M \end{array}$$

② H_4 may be viewed in another way: $0 \to I \to A' \xrightarrow{\varphi} A \to 0$ small ext'n.
$A' \times_A A' \xrightarrow{\sim} A' \times_k k[I]$. $(k[I] = k \oplus I,\ I^2 = 0)$.
$(x, y) \longmapsto (x, x_0 + y - x)$.
So we get $F(A') \times (t_F \otimes I) \xrightarrow[\sim]{H_2} F(A' \times_k k[I]) \longrightarrow F(A') \times_{F(A)} F(A')$.
H_4 simply says $t_F \otimes I \circlearrowright F(A') \to F(A)$ is exact.
H_1 says $t_F \otimes I$ action is transitive on fiber of $F(\varphi)$.

pf of thm Let $t_1, \cdots, t_r$ be a dual basis of t_F, let $S = k[[t_1, \cdots, t_r]]$, $n = (t_1, \cdots, t_r)$.
Let $R_2 = S/n^2 \cong k[\varepsilon] \times_k \cdots \times_k k[\varepsilon]$ (r times). $\exists\ \xi_2 \in F(R_2)$ induces $t_{R_2}\ (\cong \mathrm{Hom}(R_2, k[\varepsilon])) \simeq t_F$. Suppose we have found (R_q, ξ_q) where $R_q = S/J_q$. Let $\mathcal{J}_{q+1} = \{J \text{ ideal in } S \mid nJ_q \subseteq J \subseteq J_q,\ \xi_q \text{ lifts to } S/J\} \neq \emptyset$.
Claim: $\forall\ J, K \in \mathcal{J}_{q+1}$, $J \cap K \in \mathcal{J}_{q+1}$
we may enlarge J, s.t. $J + K = J_q$ without changing $J \cap K$.
Then $S/J \times_{S/J_q} S/K \simeq S/(J \cap K)$, hence the claim.
Let J_{q+1} be the ideal minimal among $\mathcal{J}_{q+1}$. Let $R_{q+1} = S/J_{q+1}$, pick $\xi_{q+1} \in F(R_{q+1})$ projects onto ξ_q.
Let $(R = \varprojlim R_q,\ \xi = \varprojlim \xi_q)$, claim: $h_R \to F$ is smooth.
Let $p: (A', \eta') \longrightarrow (A, \eta)$, small ext'n w/ 1-dim'l kernel, $A = A'/I$, and $u: (R, \xi) \longrightarrow (A, \eta)$ be given. It suffices to lift u to $u': R \to A'$.

as $t_F \otimes I$ act transitively on $F(p)^{-1}(\eta)$, so given such u', we may twist it.

Suppose u factors $R \to R_q \to A$, it suffices to ~~lift~~ complete $R_{q+1} \to A'$ or equivalently:

$$\begin{tikzcd} R_{q+1} \arrow[r, dashed] \arrow[d] & A' \arrow[d, "p"] \\ R_q \arrow[r] & A \end{tikzcd}$$

where w was chosen to make it commute.

$$\begin{tikzcd} S \arrow[r, "w"] \arrow[d] & R_q \times_A A' \arrow[d, "pr_1"] \\ R_{q+1} \arrow[ur, dashed, "v"] \arrow[r] & R_q \end{tikzcd}$$

pr_1 is either essential or has a section

2nd case, $\ker w \supseteq J_{q+1}$, so v exists.

Defn. $B \xrightarrow{p} A$ in Art/k, p is essential if $\forall\, q: C \to B$, pq surj $\Leftrightarrow$ q is surj.

Lemma. $p: B \to A$, then (1) p is essential iff $p_*: t_B^* \to t_A^*$ is an isom.

(2) if p is a small extn, then p is not essential iff p has a section.
w/ 1-dim'l kernel.

e.g. $k[[x,y]]/(xy)$ deform $/k$, a hull: $k[[t,x,y]]/(xy-t)$.
which doesn't pro-represent it.

Example of T-O theory

X/k, $Z \subseteq X$, $\mathcal{I}_Z$. $H_{X,Z}(A) = \{Z_A \subseteq X \times_k A,\ A\text{-flat, w/ } Z_A \times_A k \simeq Z\}$

which has T-O theory, $T_1 \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$, $T_2 \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$

Setup. X/k, $F \in Coh(X)$. $0 \to M \to B \to A \to 0$, $M^2 = 0$ in (Art/k).

Defn. assume (e) $0 \to S \to F \otimes_k A \to Q \to 0$, $Q \in Coh(X_A)$, flat $/A$. we say
(e') $0 \to S' \to F \otimes_k B \to Q' \to 0$ $Q' \in Coh(X_B)$, flat $/B$ is an extn of
(e) if $(e') \otimes_B A = (e)$ as $F \otimes_k B \otimes_B A = F \otimes_k A$

Lemma $Q' \in Coh(X_B)$, then Q' is flat$/B$ $\Leftrightarrow$ $Q' \otimes_B M \hookrightarrow Q'$ & $Q' \otimes_B A$ is A-flat.

Thm. (a) (e) & $B \xrightarrow{\text{surj}} A \to 0$ $\rightsquigarrow$ $ob(e) \in \mathrm{Ext}^1_{X_A}(S, Q \otimes_A M)$.

(b) $\exists\, (e') \Leftrightarrow ob(e) = 0$, if so $\{(e')\}$ is a torsor under $\mathrm{Hom}_{X_A}(S, Q \otimes_A M)$

pf (a)

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d, dashed] & 0 \arrow[d] & \\ 0 \arrow[r, dashed] & S \otimes_A M \arrow[r, dashed] \arrow[d] \arrow[dr, "\alpha"] & S' \arrow[r, dashed] \arrow[d, dashed] & S \arrow[r, dashed] \arrow[d] & 0 \\ 0 \arrow[r] & F \otimes_k M \arrow[r] \arrow[d] & F \otimes_k B \arrow[r] \arrow[d, dashed] \arrow[dr, "\beta"] & F \otimes_k A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r, dashed] & Q \otimes_A M \arrow[r, dashed] \arrow[d] & Q' \arrow[r, dashed] \arrow[d, dashed] & Q \arrow[r, dashed] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

$\ker\beta \supseteq \mathrm{Im}\,\alpha$.

$\tilde{F} = \ker\beta / \mathrm{Im}\,\alpha$

$0 \to Q \otimes_A M \to \tilde{F} \to S \to 0$

over X_A.

~~(b)~~ so we get $ob(e) \in \mathrm{Ext}^1_{X_A}(S, Q \otimes_A M)$.

(b) $ob(e) = 0 \Leftrightarrow$ $0 \to Q \otimes_A M \to \tilde{F} \underset{\varepsilon}{\leftrightarrows} S \to 0$.

Let S' = preimage in $F \otimes_k B$ of $\varepsilon(S)$.

sections $\simeq \mathrm{Hom}_{X_A}(S, Q \otimes_A M)$.

S is A-flat, so $S \cong S_0 \oplus m_A \otimes_k S_0$ as it's somehow only k-map.

Now $\mathrm{Hom}_{X_A}(S, Q \otimes_A M) \xrightarrow{\sim} \mathrm{Hom}_X(S \otimes_A k, Q \otimes_A M) \xrightarrow{\sim} \mathrm{Hom}_X(S|_X, Q|_X) \otimes_k M$

$\mathrm{Ext}^1_{X_A}(S, Q \otimes_A M) \xrightarrow{\sim} \mathrm{Ext}^1_X(S \otimes_A k, Q \otimes_A M) \xrightarrow{\sim} \mathrm{Ext}^1_X(S|_X, Q|_X) \otimes_k M$

for Ext's, we may run spectral sequences, and as Q is A-flat, we win!

Def. X scheme$/k$, $F_0 \in Coh(X)$, $S_0 \subseteq F_0$, $Q_0 = F_0/S_0$.

$D_{F_0, S_0}(A) = \{ S \in Coh(X_A),\ S \hookrightarrow F_0 \otimes_k A \twoheadrightarrow Q \to 0$, s.t. Q flat$/A$, and $S \otimes_A k = S_0$ thru $F_0 \otimes_k A \otimes_A k \cong F_0 \}$.

which has T-O theory w/ $T_1 = \mathrm{Hom}_{\mathcal{O}_X}(S_0, Q_0)$, $T_2 = \mathrm{Ext}^1_X(S_0, Q_0)$.

Thm If $Z \subseteq X$, local complete Intersection, then $H_{Z,X}$ has a T-O theory, w/ $T_i \cong H^{i-1}(Z, N_{Z/X})$ where $N_{Z/X} \cong (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee$.

Several Picard Functors

$X \xrightarrow{f} S$ finite type, separated $\quad Pic(X) = \{$invertible sheaf on $X\}$

Absolute Picard Functor $Pic_X : \{S\text{-schemes}\} \longrightarrow Ab$

$T \longmapsto Pic(X_T)$

Thm (Grothendieck) Let $f: X \to S$ proj., flat, w/ geom. integral fibers. Then sheafified Pic_X in the étale topology is representable by a Scheme $\underline{Pic}_{X/S}$ which is separated, locally finite type over S.

Let F be a presheaf on $\{S\text{-schemes}\}$.

Assume $F(\coprod T_i) = \prod_i F(T_i)$ (e.g. Pic_X).

Let $T' \to T$ a covering, $H^0(T'/T, F) = \{ s \in F(T') \mid$ exists $T'' \xrightarrow{p} T' \times_T T'$ s.t. $p^* pr_1^* s = p^* pr_2^* s \}$.

$F^a(T) = \operatorname{colim}_{T' \xrightarrow{\text{cover}} T} H^0(T'/T, F)$.

Pic_X is not a sheaf under Zariski topology, worse, it's not separated. as $f^*(\mathcal{L})$ is not trivial but locally trivial.

Defn So we define relative Picard Functor $Pic_{X/S} : \{S\text{-schemes}\} \longrightarrow Ab$

Let $Pic_{X/S\,(Zar,\, ét,\, fppf)}$ be the corresponding sheafified sheaf $T \longmapsto Pic(X_T)/f_T^* Pic(S)$

Prop. They are also sheafification of Pic_X.

pf. because differ by a line bdle downstair won't matter after pulling back.

If A is a local ring, w/ $\mathrm{Spec}A \to S$.

Then $\mathrm{Pic}_X(\mathrm{Spec}A)=\mathrm{Pic}_{X/S}(\mathrm{Spec}A)$ as $\mathrm{Pic}(\mathrm{Spec}A)=0$.

and $\mathrm{Pic}_{X/S}(\mathrm{Spec}A)=\mathrm{Pic}_{X/S(Zar)}(\mathrm{Spec}A)$ as every Zar cover of $\mathrm{Spec}\,A$ can be refined by identity.

Example. $\mathrm{Pic}_{X/S(Zar)} \longrightarrow \mathrm{Pic}_{X/S(ét)}$ is not surj. in general

$X=\mathrm{Proj}\,\mathbb{R}[u,v,w]/(u^2+v^2+w^2)$ $\quad$ $\mathrm{Pic}_{X/S(ét)}(\mathrm{Spec}\mathbb{R})=\mathbb{Z}$

$S=\mathrm{Spec}(\mathbb{R}) \longleftarrow \mathrm{Spec}(\mathbb{C})$ $\quad$ $\mathcal{O}(1)$ is not in the image!

Lemma. assume $\mathcal{O}_S \simeq f_*\mathcal{O}_X$ then f^*: {locally free ~~fin~~ coh. sheaves on S} $\longrightarrow$ {locally free coh. sheaves on X} $F \longmapsto f^*F$ is fully faithful. essential image is M, s.t. ① f_*M is locally free ~~finite~~ coh. on S
② $f^*f_*M \longrightarrow M$ is an iso.

Thm (Comparison) Assume $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally:

① Then the natural maps are inj.

$\mathrm{Pic}_{X/S} \hookrightarrow \mathrm{Pic}_{X/S(Zar)} \hookrightarrow \mathrm{Pic}_{X/S(ét)} \hookrightarrow \mathrm{Pic}_{X/S(fppf)}$

② all 3 are iso. if f has a section, latter 2 are iso. if f has a section Zar. locally, last map are isom. if f has a section fppf locally

Relative Effective Divisors (RED)

Defn. $D\subseteq X$ is a closed subscheme, X/S is a rel. eff div. if I_D is a line bundle + $\mathcal{O}_X/I_D$ flat $/S$.

Lemma. $D\subseteq X$ is a closed subscheme, $x\in D$, $s=f(x)\in S$, TFAE

(1) D is a RED at x.

(2) X and D are S-flat at x, and D_s is an eff. divisor on X_s at x.

(3) X is S-flat at x, D is cut out at x by a nonzerodivisor on X_s, i.e. $(f)\cap(m\mathcal{O}_{X,x})=0$ where $m\subseteq\mathcal{O}_{S,s}$ the max'l ideal.

Defn. $\mathrm{Div}_{X/S}(T) := \{$ RED's $D\subseteq X_T/T\}$.

X/S flat, proj.

Thm. $\mathrm{Div}_{X/S}$ is represented by an open subscheme of $\mathrm{Hilb}_{X/S}$.

Thm $f: X\to S$ proper, $F\in \mathrm{Coh}(X)$, flat$/S$, then $\exists\, Q\in \mathrm{Coh}(S)$, $q: \mathrm{Hom}(Q,N) \xrightarrow{\sim} f_*(F\otimes f^*N)$, (Q,q) is unique up to unique isom., commutes w/ base change $T\to S$, $s\in S$, TFAE:

(1) Q locally free (equivalently, projective)

(2) $N \longmapsto f_*(F\otimes f^*N)$ is right exact.

(3) $\forall N$, $f_*(F)\otimes N \xrightarrow{\sim} f_*(F\otimes f^*N)$

(4) $H^0(X,F)\otimes k_s \longrightarrow H^0(X_s,F_s)$.

They are all implied by the following:

(5) $H^1(X_s,F_s)=0$.

Lemma $f: X \to S$ proper flat, geom. fibers are reduced and connected, then $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally.

Defn. Let $\mathcal{L}$ be invertible on X, $LinSys_{\mathcal{L}/X/S}(T) = \{$RED's D on X_T/T s.t. $\mathcal{O}_{X_T}(D) \simeq \mathcal{L}_T \otimes f_T^* N$ for some invertible N on $T\}$.

Thm Assume X/S proper flat, geom. fibers are integral. Let $\mathcal{L}$ be inv. on X, Q associated to $F := \mathcal{L}$. Then $\mathbb{P}(Q)$ represents $LinSys_{\mathcal{L}/X/S}$.

sketch pf: D is determined by $\sigma \in H^0(X_T, \mathcal{L}_T \otimes f_T^* N)$
$\simeq \mathrm{Hom}(Q_T, N)$.
D is a RED $\iff$ corresponding $u: Q_T \to N$ is surjective everywhere by a local argument.
(and the fact Q base changes).

Constructing Picard Schemes

Thm. S locally Noetherian, $X \to S$ flat locally proj. (embedded in $\mathbb{P}(V)$), f has geometrically integral fibers. Then $P = \underline{\mathrm{Pic}}_{X/S,\acute{e}t}$ is represented by a scheme $\mathrm{Pic}_{X/S}$ which is separated and locally finite type.

Strategy: $\mathrm{Pic}_{X/S}$ is disjoint union of open subschemes $\mathrm{Pic}^{\phi}_{X/S}$, each $\mathrm{Pic}^{\phi}_{X/S}$ will be a union of quasi-proj. open subset $\underline{\mathrm{Pic}}^{\phi}_{X/S,m}$.

Lemma: Let $F: (S\text{-}Sch^{op}, \acute{e}t) \to Sets$, F_i be a collection of subfunctors, s.t.
1. F is a sheaf in Zariski topology
2. each F_i is representable by X_i
3. each F_i is an open subfunctor of F.
4. F_i's cover F: given $\xi \in F(T)$, $\exists$ étale refinement $\amalg V_i \to T$, s.t. $\xi|_{V_i} \in F_i(V_i)$

Then F is rep. by S-Sch. X, X_i's form an open cover.

Lemma $\underline{P}^{\phi}(T) := \{\mathcal{L} \text{ on } X_T \mid \chi(\mathcal{L}_t) = \phi, \forall t \in T\} / f_T^* N$ is open in $\underline{P}$.

Defn. $\underline{P}^{\phi}_m$ is the étale subsheaf associated to presheaf where T points are $\mathcal{L} \in \underline{P}^{\phi}(T)$ s.t. $R^i f_{T,*}\mathcal{L}(n) = 0$, $\forall i \geq 1$, $n \geq m$.

Lemma $R^i f_* \mathcal{L}(n) = 0$ $\forall i \geq 1, n \geq m \iff H^i(X_t, \mathcal{L}_t(n)) = 0$ $\forall t, i \geq 1, n \geq m$.

Lemma Let A be a noeth. local ring, m be max'l ideal. $T: Mod_A \to Ab$, s.t.

1. $0 \to M' \to M \to M'' \to 0 \Rightarrow T(M') \to T(M) \to T(M'')$ exact
2. commute w/ inductive limit
3. $T(M)$ finite gen. module $\longrightarrow$ f.g. abel. gp.
4. $\widehat{T(M)} \cong \varprojlim T(M_n)$ $\quad M_n = M/m^n M$.
5. $T(k)=0$

Then $T(M)=0 \quad \forall M \in Mod_A$

Now Take $T = R^i f_*(\ )_t$.

Lemma. P_m^ϕ is open in P^ϕ as it's cut out by vanishing of $R^i f_*(\mathcal{L}(n))$, by Serre's vanishing there are only finitely many of them & they are coherent.

$\phi_n(m) = \phi(m+n)$, $\quad P_m^\phi = P_0^{\phi_m}$. Show P_0^ϕ is representable by realizing it as $Z \longrightarrow P_0^\phi$.

consider Abel map: $Div_{X/S} \longrightarrow P$, Take $Z = P_0^\phi \times_P Div_{X/S}$.

$\alpha: Z \longrightarrow P_0^\phi$ is given by universal lb $\mathcal{L}$ on X_Z.

Thm. Let $Z \longrightarrow P$ be a map of étale sheaves, s.t.

(1) α is surj. as étale sheaves

(2) Z is represented by q-proj. S-scheme

(3) $Z \times_P Z = R$ is represented by S-scheme.

(4) $p_1: R \to Z$ is smooth flat.

Then P is represented by a q-proj S-scheme & α is smooth.

pf. by (3)

$$\begin{tikzcd} R=(g\times 1)^{-1}W \arrow[r] \arrow[d] & W \subseteq Hilb_Z \times Z \arrow[d] \\ Z \arrow[r, "g"] & Hilb_Z \end{tikzcd}$$

$\Gamma_g \subseteq W \subseteq Hilb_Z \times Z$.

$S' \to S \quad H(S) \to H(S') \rightrightarrows H(S' \times_S S')$ $\quad H(S)$ = closed subsch. of X_S.

let $S' = W$, $S = Hilb$

check $\Gamma_g \times_H W = W \times_H \Gamma_g$. So we represent P by subsch. of H.

check α satisfies (1)-(4).

$$\begin{tikzcd} \alpha: Z \arrow[r] & P_0^\phi \\ Z \times_{P_0^\phi} T \arrow[r] \arrow[u] & T \arrow[u] \\ Z \times_{P_0^\phi} T' \arrow[r, dashed] \arrow[u] & T' \arrow[u, "sm."] \end{tikzcd}$$

$Z \times_{P_0^\phi} T' = LinSys_{\mathcal{L}'/X'/T'} = \mathbb{P}(Q)$

hence étale locally has section,

(1) ✓

(2) is automatic

if $Z' = T$ we checked

(3) & (4).

General Grothendieck topos theory:

$F \longrightarrow G$ morphism of sheaves.

$F \longrightarrow G$ is surj. $\Leftrightarrow$ G is a coeq. of $F \times_G F \rightrightarrows F$.