1. Breuil–Kisin modules.

Setup: $K/\mathbb{Q}_p$ finite extn. $\mathcal{O} = \mathcal{O}_K$. $\pi \in \mathcal{O}$, uniformizer.

$$\tilde{\theta} : \mathfrak{S} = W(k)[[T]] \longrightarrow \mathcal{O}$$
$$T \longmapsto \pi.$$

$\ker(\tilde{\theta}) = (E(T))$, where $E(T) \in W(k)[[T]]$ is an Eisenstein polynomial.

$\varphi : \mathfrak{S} \longrightarrow \mathfrak{S}$, lifting Frobenius on $W(k)$, which sends

$$T \longmapsto T^p.$$

Defn. A Breuil–Kisin module (B–K mod.) is a f.g. $\mathfrak{S}$-module M equipped w/ an isom. $\varphi_M : M \otimes_{\mathfrak{S},\varphi} \mathfrak{S}[\frac{1}{E}] \cong M[\frac{1}{E}]$.

Example (Tate twist).

Lemma: $E_r := E \cdot \varphi(E) \cdots \varphi^{r-1}(E)$ (so $E_1 = E$). Then

$$\frac{E_r}{E_s} \equiv p^{r-s} \cdot \text{unit} \text{ modulo } E_s \quad \text{for } r > s.$$

pf. Since E_s divides E_r for $r>s$, it suffices to prove

$$\frac{E_r}{E_{r-1}} \equiv \cancel{p^{r-s}}\, p \cdot \text{unit} \mod E_{r-1}.$$

Now $E_r / E_{r-1} = \varphi^{r-1}(E)$

E is of the form $E = T^e + p \cdot u_0$ ← unit in $\mathfrak{S}$.

Then $E_{r-1} = T^{e(1+p+\cdots+p^{r-2})} + p \cdot u_1$

hence $\varphi^{r-1}(E) = T^{ep^{r-1}} + p \cdot u_2 \equiv (-p \cdot u_1)^{p-1} \cdot T + p \cdot u_2 = p \cdot \text{unit} \mod E_{r-1}$. □

Now set $(\mathfrak{S}/E_r)\{1\} := E_r\mathfrak{S} / E_r^2 \mathfrak{S}$

For $r > s$, we see that $\exists!$ $\underline{\mathfrak{S}/E_r\{1\}} \longrightarrow \underline{\mathfrak{S}/E_s\{1\}}$

by ~~$E_r f$~~ $\quad E_r \cdot f \longmapsto (E_r \cdot f / p^{r-s}) \in \overline{(E_s)}$.

i.e. the canonical proj.: $E_r/E_r^2 \longrightarrow E_s/E_s^2$ has image

$p^{r-s} E_s/E_s^2$, dividing by p^{r-s} gives the map above.

Then we define $\mathfrak{S}\{1\} := \lim_r (\mathfrak{S}/E_r)\{1\}$. which is a free $\mathfrak{S}$-mod of rk 1.

Concretely: $\mathfrak{S}\{1\} = \{(a_1E_1, a_2E_2, \cdots) \mid a_{i+1}E_{i+1} \equiv p a_i E_i \mod E_i^2\}$.

The $\varphi_{\mathfrak{S}\{1\}}: (a_1E_1, a_2E_2, \cdots) \longmapsto (?, E\varphi(a_1)\varphi(E_1), E\varphi(a_2)\varphi(E_2), \cdots)$

$= (?, \varphi(a_1)E_2, \varphi(a_2)E_3, \cdots)$.

where $? =$ unique elt $\widehat{aE}$, in E/E^2, s.t. $paE \equiv \varphi(a_1)E_2 \mod E^2$.

It's easy to see that $\varphi_{\mathfrak{S}\{1\}}$ is inj., and up to inverting E, the image is surjective, and it's φ-semilinear.

Prop. (M, φ_M) B-K mod., $\exists$ a canonical sequence of B-K mod.

$$0 \longrightarrow (M_{tor}, \varphi_{M_{tor}}) \longrightarrow (M, \varphi_M) \longrightarrow (M_{free}, \varphi_{M_{free}}) \longrightarrow (\bar{M}, \varphi_{\bar{M}}) \longrightarrow 0$$

where (1) M_{tor} is the torsion submodule in M, killed by a power of p.

(2) M_{free} is a finite free $\mathfrak{S}$-mod.

(3) $\bar{M}$ is killed by a power of (p, T).

In particular, $M[\frac{1}{p}] \cong M_{free}[\frac{1}{p}]$ is a finite free $\mathfrak{S}[\frac{1}{p}]$-mod.

pf. $\mathfrak{S}$ is a regular 2-dim'l local ring. it's like $\hat{\mathbb{A}}^2_{(0,0)}$.

View M as a coherent sheaf on $\operatorname{Spec}(\mathfrak{S})$. Denote the punctured Spec by U

Then we have $0 \longrightarrow \mathcal{F}_{tor} \longrightarrow \mathcal{F} \longrightarrow j_*j^*(\mathcal{F}/\mathcal{F}_{tor}) \longrightarrow \operatorname{Coker} \longrightarrow 0$

functorially assigned to $\mathcal{F}$.

- $\mathcal{F}/\mathcal{F}_{tor}$ defines a vector bundle on U (1 dim'l Noeth. reg scheme).

Functoriality assures that these ~~terms~~ still have B-K mod. structure

It's clear that Coker is supported on the closed pt, hence killed by $(p, T)^N$

We just need to show $\mathrm{Ann}(M_{tor}) \ni p$. Call $I := \mathrm{Ann}(M_{tor})$. Since M_{tor} is a B-K mod., $I \otimes_{\mathfrak{S},\varphi} \mathfrak{S}[\frac{1}{E}] = I[\frac{1}{E}]$,

(*). hence $(\mathfrak{S}/I)[\frac{1}{E}] = (\mathfrak{S}/\varphi^* I)[\frac{1}{E}]$ as quotients of $\mathfrak{S}[\frac{1}{E}]$.

On the other hand, by Weierstraß preparation & Weierstraß division,

$$A := (\mathfrak{S}/I)[\tfrac{1}{p}] \cong \prod_{i=1}^{n} K_0[T]/(f_i(T)^{n_i}), \text{ where}$$

$f_i \in W(k)[T]$ is a monic irred. poly. congruent to T^{d_i} mod p, $n_i \geq 1$ & $f_i \neq f_j$ for $i \neq j$.

We need to show $A = 0$

Look at the $\mathbb{C}$-pts of A, $\underset{=\{z_i \in \mathbb{C}\}}{Z}$. The description of $f_i \Rightarrow |z_i| < 1$.

$(*) \Rightarrow Z' = \{z \mid z^p \in Z\}$, then $Z' \cap U = Z \cap U$, where

$U = m_{\mathbb{C}} - \{\pi_1, \cdots, \pi_e\}$, π_i's are roots of E in $\mathbb{C}$.

If $Z \neq \emptyset$, choose $z_i \in Z$ w/ $|z_i|$ max'l, $\exists\, z \in Z'$ w/ $z^p = z_i$.

If $|z_i| \geq |\pi|$, then $|z^p| \geq |\pi|$, hence $|z| > |\pi|$, hence $z \in \underset{\overset{\shortparallel}{Z \cap U}}{Z' \cap U}$,

but $|z| > |z^p| = |z_i| \Rightarrow z \notin Z$ by max'lity of $|z_i|$. $\times$.

Hence $|z_i| < |\pi|$ $\forall z_i \in Z$. But then $Z \cap U = Z$, hence $z^{p^n} \in Z$ $\forall n \geq 0$.

Since Z is finite, we must have $Z = \{0\}$.

Hence $A = (\mathfrak{S}/I)[\frac{1}{p}] \cong K_0[T]/T^{d_i}$

Then E is invertible in A. Compare Length of both sides of $(*)$, $\times$.

Thm. (Kisin) $\exists$ a natural fully faithful tensor functor $T \mapsto M(T)$ from $\mathbb{Z}_p$-lattices T in crystalline G_K-repns V to finite free B-K modules. Moreover, given T, $M(T)$ is characterized by the existence of a φ, G_{K_∞}-equiv. identification

$$M(T) \otimes_{\mathfrak{S}} W(C^b) \cong T \otimes_{\mathbb{Z}_p} W(C^b).$$

Here $K_\infty = K(\pi^{1/p^\infty}) \subseteq \overline{K}$, hence we have $\pi^\flat = (\pi, \pi^{1/p}, \cdots) \in \mathcal{O}^\flat$.

We have the following diagram, equivariant under G_{K_∞}.

$$\begin{tikzcd} \mathfrak{S} \arrow[r, "\tilde{\theta}"] \arrow[d] & \mathcal{O} \arrow[d] \\ A_{\inf} \arrow[r, "\tilde{\theta} = \theta\circ\varphi^{-1}"] & \mathcal{O}_C \end{tikzcd}$$

$T \longmapsto \pi$,

$T \longmapsto [\pi^\flat]^p \longmapsto \theta\circ\varphi^{-1}([\pi^\flat]^p) = \theta([\pi^\flat]) = \pi$.

Warning: $T \longmapsto M(T)$ is not an exact functor!!!

We will check later, that $\mathbb{Z}_p(1) \leftrightsquigarrow \mathfrak{S}\{1\}$.

2. More on $\mathbf{A}_{\inf}$.

Recall that Witt ring is a system of rings W_r equipped w/

$$R: W_r \longrightarrow W_{r-1}, \quad V: W_{r-1} \longrightarrow W_r \quad \text{and} \quad F: W_r \longrightarrow W_{r-1}.$$

Fact: Let S be a p-adically complete comm. ring. Then:

$$\begin{tikzcd} & & \varprojlim_F W_r(S) \arrow[d] \\ A_{\inf}(S) := \mathbf{W}(S^\flat) = \varprojlim_R W_r(S^\flat) & \varprojlim_F W_r(S^\flat) \arrow[l, "\varphi^\infty"'] \arrow[r] & \varprojlim_F W_r(S/pS) \end{tikzcd}$$

is a sequence of isomorphisms.

Here φ^∞ is induced by

$$\begin{tikzcd} W_r \arrow[d, "R"'] & W_r \arrow[l, "\varphi^r"'] \arrow[d, "F"] \\ W_{r-1} & W_{r-1} \arrow[l, "\varphi^{r-1}"] \end{tikzcd}$$

Call $\tilde{\theta}_r : A_{\inf}(S) \longrightarrow \varprojlim_F W_r(S) \longrightarrow W_r(S)$ the composition.

$\theta_r := \tilde{\theta}_r \circ \varphi^r$.

Then $\forall\ x = (x_0, x_1, \cdots) \in S^\flat$, $\tilde{\theta}_r([x]) = [x_r]$, $\theta_r([x]) = [x_0]$.

~~Moreover~~ Moreover, R on $\varprojlim_F W_r(S)$ corresponds to

φ^{-1} on $\varprojlim_R W_r(S^\flat) = A_{\inf}(S)$.

More diagrams: let S be as before, we have the following comm. diagrams.

$$\begin{tikzcd} A_{inf}(S) \arrow[r, "\theta_{r+1}"] \arrow[d, "id"'] & W_{r+1}(S) \arrow[d, "R"] \\ A_{inf}(S) \arrow[r, "\theta_r"] & W_r(S) \end{tikzcd} \quad \begin{tikzcd} A_{inf}(S) \arrow[r, "\theta_{r+1}"] \arrow[d, "\varphi"] & W_{r+1}(S) \arrow[d, "F"] \\ A_{inf}(S) \arrow[r, "\theta_r"] & W_r(S) \end{tikzcd} \quad \begin{tikzcd} A_{inf}(S) \arrow[r, "\theta_{r+1}"] & W_{r+1}(S) \\ A_{inf}(S) \arrow[r, "\theta_r"] \arrow[u, "\cdot\lambda_{r+1}\varphi^{-1}"] & W_r(S) \arrow[u, "V"'] \end{tikzcd}$$

$$\begin{tikzcd} A_{inf}(S) \arrow[r, "\tilde{\theta}_{r+1}"] \arrow[d, "\varphi^{-1}"'] & W_{r+1}(S) \arrow[d, "R"] \\ A_{inf}(S) \arrow[r, "\tilde{\theta}_r"] & W_r(S) \end{tikzcd} \quad \begin{tikzcd} A_{inf}(S) \arrow[r, "\tilde{\theta}_{r+1}"] \arrow[d, "id"'] & W_{r+1}(S) \arrow[d, "F"] \\ A_{inf}(S) \arrow[r, "\tilde{\theta}_r"] & W_r(S) \end{tikzcd} \quad \begin{tikzcd} A_{inf}(S) \arrow[r, "\tilde{\theta}_{r+1}"] & W_{r+1}(S) \\ A_{inf}(S) \arrow[r, "\tilde{\theta}_r"] \arrow[u, "\cdot\tilde{\lambda}_{r+1}"] & W_r(S) \arrow[u, "V"'] \end{tikzcd}$$

where $\lambda_{r+1} \in A_{inf}(S)$, satisfies $\theta_{r+1}(\lambda_{r+1}) = V(1)$ (if exists).

$\tilde{\lambda}_{r+1} = \varphi^{r+1}(\lambda_{r+1})$ satisfies $\tilde{\theta}_{r+1}(\tilde{\lambda}_{r+1}) = V(1)$.

From now on, let $S = \mathcal{O}_C$

Lemma. Let $\xi \in A_{inf}$ be a generator of $\ker(\theta = \theta_1)$, then $\ker(\theta_r)$ is generated by

$$\xi_r := \xi \cdot \varphi^{-1}(\xi) \cdot \varphi^{-2}(\xi) \cdots \varphi^{-(r-1)}(\xi).$$

Equivalently, $\ker(\tilde{\theta}_r)$ is generated by $\tilde{\xi}_r = \varphi^r(\xi_r) = \varphi^r(\xi)\,\varphi^{r-1}(\xi) \cdots \varphi(\xi)$.

pf.

$$\begin{tikzcd} 0 \arrow[r] & A_{inf} \arrow[r, "\xi\cdot\varphi^{-1}"] \arrow[d, "\theta_r"'] & A_{inf} \arrow[r, "\theta"] \arrow[d] & \mathcal{O}_C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & W_r(\mathcal{O}_C) \arrow[r, "V"] & W_{r+1}(\mathcal{O}_C) \arrow[r, "R^r"] & \mathcal{O}_C \arrow[r] & 0 \end{tikzcd}$$

Fix a system of p-power roots of unity $(\zeta_{p^r}) \in \mathcal{O}_C$.

$$\varepsilon := (1, \zeta_p, \zeta_{p^2}, \cdots) \in \mathcal{O}_C^\flat \qquad \mu := [\varepsilon] - 1$$

Let $\xi := 1 + [\varepsilon^{1/p}] + [\varepsilon^{1/p}]^2 + \cdots + [\varepsilon^{1/p}]^{p-1} = \dfrac{[\varepsilon]-1}{[\varepsilon]^{1/p}-1} = \dfrac{\mu}{\varphi^{-1}(\mu)} \in A_{inf}$

Lemma: $\theta_r(\xi) = V(1)$. $\forall r > 0$.

pf. $\mathcal{O}_C$ is p-torsion free, hence to check $\theta_r(\xi) = V(1)$ it suffices to check their images under ghost maps $gh: W_r(\mathcal{O}_C) \longrightarrow \mathcal{O}_C^{\oplus r}$ are the same

Now since $\theta_r([\underline{x}]) = [x_0]$, $\forall\ \underline{x} = (x_0, x_1, \cdots) \in \mathcal{O}_C^\flat$, we have $gh \circ \theta_r = (\theta, \theta\circ\varphi, \theta\circ\varphi^2, \cdots)$.

So $gh\circ\theta_r(\xi) = (\theta(\xi), \theta\circ\varphi(\xi), \theta\circ\varphi^2(\xi), \cdots)$

$$\theta(\xi) = \sum_{i=0}^{p-1} \zeta_p^i = 0.$$

$$\theta\circ\varphi^i(\xi) = \sum_{i=0}^{p-1} 1 = p \quad \forall i \geq 1.$$

Also $gh(V(1)) = gh((0,1,0,\cdots)) = (0, p, p, \cdots)$. ✓

Fact: $\{\theta_r\}_r$ gives $\theta_\infty : A_{inf} \longrightarrow W(\mathcal{O})$, $\ker(\theta_\infty) = (\mu)$,

coker is $\begin{cases} \text{killed by } W(m^b). \\ 0 \quad \text{if } K \text{ is spherically complete.} \end{cases}$

Recall that $\tilde{\theta}_r : A_{inf} \longrightarrow W_r(\mathcal{O})$ has kernel generated by

$$\tilde{\xi}_r = \varphi(\xi)\cdot\varphi^2(\xi)\cdot\cdots\cdot\varphi^r(\xi)$$

Lemma. The obvious map $\dfrac{\tilde{\xi}_r A_{inf}}{\tilde{\xi}_r^2 A_{inf}} \longrightarrow \dfrac{\tilde{\xi}_s A_{inf}}{\tilde{\xi}_s^2 A_{inf}}$ has image $\dfrac{p^{r-s}\cdot\tilde{\xi}_s A_{inf}}{\tilde{\xi}_s^2 A_{inf}}$. $\forall r > s$

pf: Same as before, it suffices to show that

$$\frac{\tilde{\xi}_r}{\tilde{\xi}_{r-1}} \equiv p\cdot\text{unit} \mod \tilde{\xi}_{r-1}.$$

But $A_{inf}/\tilde{\xi}_{r-1} \xrightarrow{\tilde{\theta}_{r-1}} W_{r-1}(\mathcal{O}_C)$ and $\dfrac{\tilde{\xi}_r}{\tilde{\xi}_{r-1}} = \varphi^r(\xi)$.

So we need to show $\tilde{\theta}_{r-1}(\varphi^r(\xi)) = p\cdot\text{unit}$.

Now $\tilde{\theta}_{r-1}(\varphi^r(\xi)) = \theta_{r-1}(\varphi(\xi)) = F(\theta_r(\xi)) = F(V(1)) = p$!

Recall $\xi \in A_{inf}$, $\tilde{\xi} = \varphi(\xi)$.

3. Breuil–Kisin–Fargues modules (BKF mod.).

Defn. A BKF module is a **f.p.** A_{inf}-module M w/ an isom:

$$\varphi_M : M \otimes_{A_{inf},\varphi} A_{inf}\left[\tfrac{1}{\xi}\right] \xrightarrow{\simeq} M\left[\tfrac{1}{\tilde{\xi}}\right].$$

s.t. $M[\frac{1}{p}]$ is a finite free $A_{inf}[\frac{1}{p}]$-module.

Rmk: (1) Any finite proj. $A_{inf}[\frac{1}{p}]$-module is free!

(2) The result above follows from the fact that

- Any f.g. p-torsion-free A_{inf}-mod. s.t. inverting p is a vector bundle gives rise to a vector bundle on $\mathcal{U} = \mathrm{Spec}(A_{inf}) \setminus \{s\}$, where s is the unique closed pt.
- Restriction to $\mathcal{U}$ is an equivalence between categories of vector bundles on $\mathrm{Spec}(A_{inf})$ & $\mathcal{U}$.

Non-example: $\mu = [\varepsilon]-1$, $\frac{\mu}{\varphi^{-1}(\mu)} = \xi$, $\tilde{\xi} = \varphi(\xi) = \frac{\varphi(\mu)}{\mu}$.

Let $M = A_{inf}/(\mu)$ which is certainly f.p. φ on A_{inf} induces:

$$\varphi_M : A_{inf}/(\varphi(\mu)) \longrightarrow A_{inf}/(\mu) \quad \left((\varphi(\mu)) \subseteq (\mu) \right).$$

which is an isom after inverting $\tilde{\xi}$.

$M[\frac{1}{p}]$ is clearly <u>NOT</u> finite free as A_{inf} is a domain, hence μ is non-zero in $A_{inf}[\frac{1}{p}]$.

Example (Tate twist).

Recall that last time we showed:

$$\tilde{\xi}_r A_{inf} / \tilde{\xi}_r^2 A_{inf} \longrightarrow \frac{\tilde{\xi}_s A_{inf}}{\tilde{\xi}_s^2 A_{inf}} \quad \text{has image } \frac{p^{r-s}\tilde{\xi}_s}{\tilde{\xi}_s^2}.$$

Moreover, the image of $\tilde{\xi}_r = \varphi(\xi) \cdots \varphi^r(\xi)$ where $\xi = \frac{\mu}{\varphi^{-1}(\mu)}$ is exactly $p^{r-s}\tilde{\xi}_s$.

Where $\tilde{\xi}_r$ is the kernel of $\tilde{\theta}_r : A_{inf} \longrightarrow W_r(\mathcal{O}_C)$.

Lemma: $W_r(\mathcal{O})\{1\} := \hat{\mathbb{L}}_{W_r(\mathcal{O})/\mathbb{Z}_p}[-1] \overset{①}{=} \mathbb{L}_{W_r(\mathcal{O})/A_{inf}}[-1] \overset{②}{=} \tilde{\xi}_r A_{inf} / \tilde{\xi}_r^2 A_{inf}$.

pf. ②: $W_r(\mathcal{O}_C) = \frac{A_{inf}}{(\tilde{\xi}_r)}$ where $\tilde{\xi}_r$ is a non-zero divisor.

hence $\mathbb{L}_{W_r(\mathcal{O})/A_{inf}}[-1] = \frac{(\tilde{\xi}_r)}{(\tilde{\xi}_r^2)}$ which is already p-adically cplt.

①: Suffices to show $\left(\mathbb{L}_{A_{inf}/\mathbb{Z}_p} \overset{L}{\underset{A_{inf},\tilde{\theta}_r}{\otimes}} W_r(\mathcal{O}_C)\right)^\wedge = 0$. ← p-adic derived cpltion.

$\cancel{\hat{\mathbb{L}}_{A_{inf}/\mathbb{Z}_p} = R\lim\left(\mathbb{L}_{A_{inf}/\mathbb{Z}_p} \overset{L}{\underset{\mathbb{Z}_p}{\otimes}} \mathbb{Z}/p^n\right) = R\lim\left(\mathrm{Cone}\left(\mathbb{L}_{A_{inf}/\mathbb{Z}_p} \xrightarrow{p^n}\right.\right.}$

First: $\mathrm{Cone}\left(\mathbb{L}_{A_{inf}/\mathbb{Z}_p} \xrightarrow{\cdot p} \mathbb{L}_{A_{inf}/\mathbb{Z}_p}\right) = \mathbb{L}_{\frac{A_{inf}/p}{\mathbb{Z}_p/p}}$

$= \mathbb{L}_{\mathcal{O}_C^\flat/\mathbb{F}_p} = 0$

Second: $\mathrm{Cone}\left(\mathbb{L}_{A_{inf}/\mathbb{Z}_p} \xrightarrow{p^n} \mathbb{L}_{A_{inf}/\mathbb{Z}_p}\right) = 0$.

Third: $= R\lim\left(\mathrm{Cone}\left(\mathbb{L}_{A_{inf}/\mathbb{Z}_p} \overset{L}{\underset{A_{inf},\tilde{\theta}_r}{\otimes}} W_r(\mathcal{O}) \xrightarrow{p^n} \quad \right)\right)$

$= R\lim\left(\underbrace{\mathrm{Cone}\left(\mathbb{L}_{A_{inf}/\mathbb{Z}_p} \xrightarrow{p^n} \mathbb{L}_{A_{inf}/\mathbb{Z}_p}\right)}_{0} \overset{L}{\underset{A_{inf},\tilde{\theta}_r}{\otimes}} W_r(\mathcal{O}_C)\right)$

$= R\lim 0 = 0$

Defn. $A_{inf}\{1\} = \lim_r W_r(\mathcal{O})\{1\}$, where transition maps are the natural proj. $\tilde{\xi}_r/\tilde{\xi}_r^2 \longrightarrow \tilde{\xi}_s/\tilde{\xi}_s^2$ followed by (dividing by p^{r-s}).

$$A_{\inf}\{1\} = \left\{ (a_r \tilde{\xi}_r)_r \;\middle|\; a_i \equiv a_{i-1} \bmod \tilde{\xi}_{i-1} \right\}$$

and $\varphi_{A_{\inf}\{1\}}\left((a_r \tilde{\xi}_r) \right) = \left(?, \varphi(a_1)\varphi(\tilde{\xi}_1), \varphi(a_2)\varphi(\tilde{\xi}_2), \cdots \right)$

We easily compute: $\varphi_{A_{\inf}\{1\}}(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \cdots) = (?, \varphi(\tilde{\xi}_1), \varphi(\tilde{\xi}_2), \cdots)$

$$= \left(?, \frac{1}{\tilde{\xi}}\tilde{\xi}_2, \frac{1}{\tilde{\xi}_1}\tilde{\xi}_3, \cdots\right) = \frac{1}{\tilde{\xi}}(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \cdots)$$

If we let $e = (\tilde{\xi}_1, \tilde{\xi}_2, \cdots)$ ~~to be~~ a to be a basis of $A_{\inf}\{1\}$,

then $\varphi_{A_{\inf}\{1\}}(e) = \frac{1}{\tilde{\xi}} \cdot e$

Fact: The canonical map $\mathbb{L}_{W^r(\mathcal{O})/\mathbb{Z}_p} \longrightarrow \Omega^1_{W^r(\mathcal{O})/\mathbb{Z}_p}$ induces isom

$$\hat{\mathbb{L}}_{W_r(\mathcal{O})/\mathbb{Z}_p} \xrightarrow{\sim} \left(\Omega^1_{W_r(\mathcal{O})/\mathbb{Z}_p}\right)^\wedge = T_p\left(\Omega^1_{W_r(\mathcal{O})/\mathbb{Z}_p}\right)[-1].$$

Under this identification $e \longmapsto \frac{1}{\mu} \cdot \left(d\log(\zeta_{p^s})_\bullet\right)_s$

$$\begin{array}{c} \Uparrow \\ \hat{\mathbb{L}}_{W^r(\mathcal{O})/\mathbb{Z}_p}[1] \longrightarrow T_p\left(\Omega^1_{W_r(\mathcal{O})/\mathbb{Z}_p}\right) \end{array}$$

where $d\log : W_r(\mathcal{O})^\times \longrightarrow \Omega^1_{W_r(\mathcal{O})/\mathbb{Z}_p}$ induces map

$$d\log : T_p W_r(\mathcal{O})^\times = \mathbb{Z}_p(1) \longrightarrow T_p\left(\Omega^1_{W_r(\mathcal{O})/\mathbb{Z}_p}\right).$$

From this fact: $\varphi_{A_{\inf}\{1\}}\left((d\log(\zeta_{p^s}))_s \right) = \varphi_{A_{\inf}\{1\}}(\mu e)$

$$= \varphi(\mu) \cdot \varphi_{A_{\inf}\{1\}}(e) = \varphi(\mu) \cdot \frac{1}{\tilde{\xi}} \cdot e = \frac{\varphi(\mu)}{\varphi(\mu)} \mu \cdot e = \mu e.$$

By Fundamental sequence in p-adic Hodge Theory:

$$\left(A_{\inf}\{1\}\right)^{\varphi=1} = \mathbb{Z}_p(1) = \mathbb{Z}_p \cdot \left(d\log(\zeta_{p^s})\right)_s$$

Prop. Let M be a f.p. A_{inf}-mod s.t. $M[\frac{1}{p}]$ is finitely proj. (equivalently, free) over $A_{inf}[\frac{1}{p}]$. Then $\exists$ functorial exact sequence

$$0 \longrightarrow M_{tor} \longrightarrow M \longrightarrow M_{free} \longrightarrow \overline{M} \longrightarrow 0 .$$

satisfying:

(1) M_{tor} is f.p. & perfect as an A_{inf}-mod., and is killed by p^N.

(2) M_{free} is a finite free A_{inf}-module

(3) $\overline{M}$ is f.p. & perfect as an A_{inf}-mod., supported at $s \in \mathrm{Spec}(A_{inf})$, i.e. $\overline{M}$ is killed by $([\varpi], p)^{N'}$.

Moreover, TFAE:

(1) $M \otimes_{A_{inf}} W(k)$ is p-torsion-free,

(2) $M \otimes_{A_{inf}} \mathcal{O}_C$ is p-torsion-free,

(3) M is finite free.

Lemma. M is a BKF-mod., then

$$T = \left(M \otimes_{A_{inf}} W(C^\flat)\right)^{\varphi_M = 1}$$

is a f.g. $\mathbb{Z}_p$-mod w/

$$M \otimes_{A_{inf}} W(C^\flat) = T \otimes_{\mathbb{Z}_p} W(C^\flat).$$

Moreover, one has $M \otimes_{A_{inf}} A_{inf}[\frac{1}{\mu}] = T \otimes_{\mathbb{Z}_p} A_{inf}[\frac{1}{\mu}]$.

as submodules of

pf. as $\tilde{\xi} \equiv \underbrace{\sum_{i=0}^{p-1} \varepsilon^i}_{\text{mod } p}$ is invertible in $C^\flat$, we see that $\tilde{\xi}$ is invertible in $W(C^\flat)$.

Therefore $M \otimes_{A_{inf}} W(C^\flat)$ along w/ φ_M is ~~equivalent~~

given by $\left(M \otimes_{A_{inf}} W(C^\flat)\right)^{\varphi_M = 1} \otimes_{\mathbb{Z}_p} W(C^\flat)$ w/ $\mathrm{id} \otimes \varphi$.

(Dieudonné – Manin).

The moreover part requires one to use the Prop. above.
Indeed, Prop. implies that we only have to consider the case when M is finite free $/A_{inf}$. $\square$

(Fargues–Fontain)
Lemma. M – BKF mod., then $M \otimes_{A_{inf}} W(k) =: \bar{M}$ is a f.g. $W(k)$-mod. equipped w/ a Frob. automorphism (after inverting p).

Fixing a section $k \longrightarrow \mathcal{O}/p$ which induces $W(k) \longrightarrow A_{inf}$.

Then $\exists$ a (noncanonical) φ-eq. isom.

$$M \otimes_{A_{inf}} B^+_{crys} \cong \bar{M} \otimes_{W(k)} B^+_{crys}$$ reducing to the identity over $W(k)[\frac{1}{p}]$.

Thm (Fargues) The cat. of finite free BKF-mod's is equivalent to the cat. of pairs (T, Ξ) where T is a finite free $\mathbb{Z}_p$-mod and Ξ is a B^+_{dR}-lattice in $T \otimes_{\mathbb{Z}_p} B_{dR}$.

The functor is given by $(M, \varphi_M) \longmapsto T = (M \otimes_{A_{inf}} W(C^\flat))^{\varphi_M = 1}$

and $$\Xi = M \otimes_{A_{inf}} B^+_{dR} \subseteq M \otimes_{A_{inf}} B_{dR} = M \otimes_{A_{inf}} A_{inf}[\tfrac{1}{\mu}] \otimes_{A_{inf}[\frac{1}{\mu}]} B_{dR}$$

$$\shortparallel$$

$$T \otimes_{\mathbb{Z}_p} B_{dR} \cong T \otimes_{A_{inf}} A_{inf}[\tfrac{1}{\mu}] \otimes_{A_{inf}[\frac{1}{\mu}]} B_{dR}.$$

Example: $A_{inf}\{1\} \longmapsto T = \mathbb{Z}_p(1)$. and $\Xi = \frac{1}{\mu} B^+_{dR}(1) = \xi^{-1}(T \otimes_{\mathbb{Z}_p} B^+_{dR}) \subseteq B^{(1)}_{dR}$.

Indeed $\frac{\xi}{\mu} = \frac{1}{\varphi^{-1}(\mu)}$ is invertible in B^+_{dR} as $\theta(\varphi^{-1}(\mu)) = \zeta_p - 1 \in C$ which is invertible.

4. Relating BK & BKF modules.

Recall: $\mathfrak{S} =$

$$\begin{tikzcd}
T \arrow[rr, mapsto] \arrow[ddd, mapsto, bend right] & & {[\pi^\flat]^p} \arrow[ddd, mapsto] \\
W(k)[\![T]\!] \arrow[r, "f"] \arrow[d, "\tilde\theta"'] & A_{inf} \arrow[d, "\tilde\theta"] & \\
\mathcal{O}_K \arrow[r] & \mathcal{O}_C & \\
\pi \arrow[rr, equal] & & \pi
\end{tikzcd}$$

Lemma: $\mathfrak{S} \xrightarrow{f} A_{inf}$ above is flat.

pf. $\mathfrak{S}$ is Noeth., both $\mathfrak{S}$ & A_{inf} are p-adically cplt. and flat over $\mathbb{Z}_p$. Therefore it suffices to look at $\mathfrak{S}/p \longrightarrow A_{inf}/p$, which is flat.

$$\begin{array}{ccc} \mathfrak{S}/p & \longrightarrow & A_{inf}/p \\ \| & & \| \\ \mathbb{F}_p[\![T]\!] & \longrightarrow & \mathcal{O}_C^\flat \end{array}$$

Prop. $M \longmapsto M \otimes_{\mathfrak{S}} A_{inf}$ defines an exact tensor functor from $\{BK\}$ to $\{BKF\}$.

pf. check
- $f(E)$ is a generator of $\ker(\tilde\theta)$. (E is Eisenstein).
- $M[\frac{1}{p}]$ is free by previous lecture.
- exactness follows from flatness of f.
- it's clearly a tensor functor.

Cor. $\mathbb{Z}_p(1)$ is sent to $\mathfrak{S}\{1\}$ under the Thm of last time.

pf. Clearly $\mathfrak{S}\{1\} \otimes_{\mathfrak{S}} A_{inf} \cong A_{inf}\{1\}$ by the identification of both sides w/ cotangent cplxes, this is cptible w/ G_{K_∞}-action.

$A_{inf}\{1\} = \frac{1}{\mu}(\mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} A_{inf})$, hence we have φ, G_{K_∞}-eq. isom.

$$\mathfrak{S}\{1\} \otimes_{\mathfrak{S}} W(C^\flat) \cong A_{inf}\{1\} \otimes_{A_{inf}} W(C^\flat) \cong \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} W(C^\flat)$$

by the Thm, this means that $\mathbb{Z}_p(1) \leftrightsquigarrow \mathfrak{S}\{1\}$.